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On almost periodic-in-time solutions to Navier-Stokes equations in unbounded domains

Yasushi Taniuchi
Department of Mathematical Sciences,
Shinshu University,
Matsumoto 390-8621, Japan

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1 Introduction

This note is a survey of the works [9, 10] jointly with R. Farwig. We consider a viscous incompressible fluid in 3-dimensional unbounded domains Ω. The motion of such a fluid is governed by the Navier-Stokes equations:

$$\text{(N-S)}\begin{cases} \partial_t u - \Delta u + u\cdot\nabla u + \nabla p = f, & t\in\mathbb{R},\quad x\in\Omega,\\ \operatorname{div} u = 0, & t\in\mathbb{R},\quad x\in\Omega,\\ u|_{\partial\Omega} = 0, & t\in\mathbb{R},\end{cases}$$

where $u=(u^1(x,t),u^2(x,t),u^3(x,t))$ and $p=p(x,t)$ denote the velocity vector and the pressure, respectively, of the fluid at the point $(x,t)\in\Omega\times\mathbb{R}$. Here f is a given external force. It is known that if f is almost periodic-in-time and small in some sense, then there exists a small almost periodic-in-time solution to (N-S). In [9, 10], we consider the uniqueness of almost and backward asymptotically almost periodic-in-time solutions to (N-S).

In case where the domain Ω is bounded, the problem of existence of time-periodic solutions was considered by several authors [34, 43, 16, 37, 32, 31, 40]. Maremonti [27] was the first to prove the existence of time-periodic regular solutions to (N-S) in *unbounded* domains. He showed that if $\Omega=\mathbb{R}^3$ and if $f(t)$ is time periodic and small in some sense, then there exists a unique time-periodic solution u to (N-S) in the class

$$\{u \in C(\mathbb{R}; L^3_\sigma);\ \sup_t \|u(t)\|_3 < \gamma,\ \ \sup_t \|\nabla u(t)\|_2 < \infty\}, \tag{1.1}$$

where γ is a small number. The same problem in $\mathbb{R}^3_+$ is considered in [28]. Kozono-Nakao [19] showed that if $\Omega = \mathbb{R}^n, \mathbb{R}^n_+$, $n \geq 3$, or $\Omega \subset \mathbb{R}^n$, $n \geq 4$, is an exterior domain, and if $f(t)$ is time periodic and small in some sense, then there exists a unique time-periodic solution u to (N-S) in the class $\{u \in C(\mathbb{R}; L^n_\sigma);\ \sup_t \|u(t)\|_r + \sup_t \|\nabla u(t)\|_q < \gamma\}$ $(2 < r < n,$ $\frac{n}{2} < q < n)$, where γ is a small number depending on Ω, r and q. Kozono-Nakao used the following integral equation

$$u(t) = \int_{-\infty}^{t} e^{-(t-s)A} Pf(s)ds - \int_{-\infty}^{t} e^{-(t-s)A} P(u \cdot \nabla u)(s)ds.$$

In [38], the present author proved the stability of Kozono-Nakao's periodic solutions. Kubo [24] proved the same result as [19] in the case where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a perturbed half space or an aperture domain. While he assumed a null flux condition in case of an aperture domain, Crispo-Maremonti [4] proved existence of unique time-periodic solutions for given time-periodic fluxes.

With respect to 3-dimensional exterior domains, we mention the results given by Maremonti-Padula [29], Salvi [33], Yamazaki [42] and Galdi-Sohr [12]. Maremonti-Padula [29] showed that for any $\Omega \subset \mathbb{R}^3$, if $f(t)$ is time-periodic and can be expressed as $f = \nabla \cdot F$, where $f, F \in C(\mathbb{R}; L^2)$, then there exists at least one time-periodic weak solution u to (N-S) in the class $\nabla u \in L^2_{loc}(\mathbb{R}; L^2)$. Moreover, they showed under some symmetry assumptions on Ω and on f that there exists a unique time-periodic solution u to (N-S) in the class defined in (1.1). In the case where Ω is an exterior domain with a periodically moving boundary, Salvi [33] proved the existence of weak time-periodic solutions and of a strong periodic solution. In the case where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is an exterior domain, $\mathbb{R}^n$, or $\mathbb{R}^n_+$, Yamazaki [42] showed that if $f = \nabla \cdot F$, $F \in BUC(\mathbb{R}; L^{n/2,\infty})$ and $\sup_t \|F(t)\|_{L^{n/2,\infty}}$ is small, then there exists a unique mild solution u to (N-S) in the class

$$\{u \in C(\mathbb{R}; L^{n,\infty});\ \sup_t \|u(t)\|_{L^{n,\infty}} < \gamma\},$$

where $\gamma = \gamma(\Omega)$ is sufficiently small. In particular, he shows that if f is time-periodic or almost periodic-in-time, then the mild solution is time-periodic or almost periodic-in-time. In the case of a 3-dimensional exterior domain, Galdi-Sohr [12] proved the existence of a small periodic strong solution u in $C(\mathbb{R}; L^r(\Omega))$, $r > 3$, satisfying the condition that

$\sup_{x,t}(1+|x|)|u(x,t)|$ is small, under the assumption that $f = \text{div } F$ is periodic and small in some function spaces. Moreover, they proved the uniqueness of such solutions in the larger class of all periodic weak solutions v with $\nabla v \in L^2(0,T;L^2)$, satisfying the energy inequality $\int_0^T \|\nabla v\|_2^2\, d\tau \le -\int_0^T (F, \nabla v)\, d\tau$ and mild integrability conditions on the corresponding pressure; here T is a period of f. Another type of uniqueness theorem for time-periodic L^3_w-solution was given in [39] without assuming the energy inequality. In the case of an exterior domain $\Omega \subset \mathbb{R}^3$, the whole space $\mathbb{R}^3$, the halfspace $\mathbb{R}^3_+$, a perturbed halfspace, or an aperture domain, it was shown in [39] that if u and v are time-periodic L^3_w-solutions in $L^2_{loc}(\mathbb{R}; L^{6,2})$ for the same force f, and if *one of them* is small, then $u = v$.

On the other hand, thus far, *uniqueness of almost periodic-in-time solutions* in unbounded domains is only known for a small almost periodic-in-time L^3_w-solution within the class of solutions which have sufficiently small $L^\infty(L^3_w)$-norm; i.e., if u and v are L^3_w-solutions for the same force f, and if *both of them* are small, then $u = v$, see [42]. In [9], we establish a new uniqueness theorem for almost periodic-in-time solutions. We show that if u and v are almost periodic-in-time solutions in

$$C(\mathbb{R}; L^3_w) \cap L^2_{loc}(\mathbb{R}; L^{6,2})$$

for the same force f, and if *one of them* is small, then $u = v$. Moreover, in [10] we show a similar uniqueness theorem for backward almost periodic solutions.

2 Preliminaries and Results

Throughout this paper we impose the following assumption on the domain.

Assumption 1 *$\Omega \subset \mathbb{R}^3$ is an exterior domain, the half-space $\mathbb{R}^3_+$, the whole space $\mathbb{R}^3$, a perturbed half-space, or an aperture domain with $\partial\Omega \in C^\infty$.*

For the definitions of perturbed half-spaces and aperture domains, see Kubo-Shibata [25] and Farwig-Sohr [6, 7].

Before stating our results, we introduce some notation and function spaces. Let $C^\infty_{0,\sigma}(\Omega) = C^\infty_{0,\sigma}$ denote the set of all C^∞-real vector functions $\phi = (\phi^1, \cdots, \phi^n)$ with compact support in Ω such that $\text{div } \phi = 0$. Similarly $C^m_{0,\sigma}$ is defined. Then L^r_σ is the closure of $C^\infty_{0,\sigma}$ with respect to the L^r-norm $\|\cdot\|_r$. The symbol $(\cdot,\cdot)$ denotes the L^2- inner product and the duality pairing between L^r and $L^{r'}$, where $1/r + 1/r' = 1$. Concerning Sobolev spaces we use the notations $W^{k,p}(\Omega)$ and $W^{k,p}_0(\Omega)$, $k \in \mathbb{N}$, $1 \le p, q \le \infty$.

Note that very often we will simply write L^r and $W^{k,p}$ instead of $L^r(\Omega)$ and $W^{k,p}(\Omega)$, respectively. Let $L^{p,q}(\Omega)$, $1 \le p, q \le \infty$, denote the Lorentz spaces and $\|\cdot\|_{p,q}$ denote the norm of $L^{p,q}(\Omega)$. We note that $L^{p,\infty}$ is equivalent to the weak-L^p space (L^p_w) and $L^{p,p}$ is equivalent to L^p. Finally,

$$L^2_{uloc}(\mathbb{R}; L^{6,2}) = \{g \in L^2_{loc}(\mathbb{R}; L^{6,2}) \ ; \ \sup_t \|g\|_{L^2(t,t+1;L^{6,2})} < \infty\}$$

denotes the space of uniformly locally integrable L^2-function on $\mathbb{R}$ with values in $L^{6,2}(\Omega)$.

For a Banach space B, let B^* be the dual space of B. Let X be a Banach space of functions on Ω such that $L^2_\sigma \cap X$ is dense in X; if $g \in L^2_\sigma \cap X^*$ and ${}_{X^*}\!< g, \phi >_X = (g, \phi)$ for all $\phi \in L^2_\sigma \cap X$, then we denote ${}_{X^*}\!< \cdot, \cdot >_X$ by $(\cdot, \cdot)$ for simplicity.

In this paper, we denote by C various constants. In particular, $C = C(*, \cdots, *)$ denotes a constant depending only on the quantities appearing in the parentheses.

Let us recall the Helmholtz decomposition: $L^r(\Omega) = L^r_\sigma \oplus G_r$ $(1 < r < \infty)$, where $G_r = \{\nabla p \in L^r; p \in L^r_{loc}(\overline{\Omega})\}$, see Fujiwara-Morimoto [11], Miyakawa [30], Simader-Sohr [35], Borchers-Miyakawa [2], and Farwig-Sohr [6, 8]; P_r denotes the projection operator from L^r onto L^r_σ along G_r. The Stokes operator A_r on L^r_σ is defined by $A_r = -P_r\Delta$ with domain $D(A_r) = W^{2,r} \cap W^{1,r}_0 \cap L^r_\sigma$. It is known that

$$(L^r_\sigma)^* \text{ (the dual space of } L^r_\sigma) = L^{r'}_\sigma, \quad A^*_r \text{ (the adjoint operator of } A_r) = A_{r'},$$

where $1/r + 1/r' = 1$. It is shown by Giga [13], Giga-Sohr [14], Borchers-Miyakawa [2] and Farwig-Sohr [6, 8] that $-A_r$ generates a uniformly bounded holomorphic semigroup $\{e^{-tA_r}; t \ge 0\}$ of class C_0 in L^r_σ. Moreover, it is found that

$$\|u\|_{W^{2,r}} \le C\|(1 + A_r)u\|_r \quad \text{for all } u \in D(A_r) \tag{2.1}$$

with a constant $C = C(r, n, \Omega)$; see e.g. [15, Lemma 2.8].

In this paper, $\dot{W}^{1,r}_{0,\sigma}$ denotes the closure of $D(A_r)$ with respect to the norm $\|\phi\|_{\dot{W}^{1,r}} = \|\nabla\phi\|_r$, where $\nabla\phi = (\partial\phi^i/\partial x_j)_{i,j=1,\cdots,n}$. Its dual space $(\dot{W}^{1,2}_{0,\sigma})^*$ is equipped with the norm $\|\phi\|_{(\dot{W}^{1,2}_{0,\sigma})^*} = \sup\{\frac{|<\phi,\theta>|}{\|\nabla\theta\|_2} \ ; \ \theta \in \dot{W}^{1,2}_{0,\sigma}\}$.

Since $P_r u = P_q u$ for all $u \in L^r \cap L^q$ $(1 < r, q < \infty)$ and since $A_r u = A_q u$ for all $u \in D(A_r) \cap D(A_q)$, for simplicity, we shall abbreviate $P_r u, P_q u$ as Pu for $u \in L^r \cap L^q$ and $A_r u, A_q u$ as Au for $u \in D(A_r) \cap D(A_q)$, respectively. Finally $L^{q,\infty}_\sigma$ denotes the space $PL^{q,\infty}(\Omega)$.

Following Kozono-Ogawa [20], we define mild $L^{3,\infty}$-solutions to (N-S).

Definition 1. *Let $T \in (-\infty, \infty]$ and $f \in L^1_{loc}(-\infty, T; D(A_p)^* + D(A_q)^*)$ for some $1 < p, q < \infty$. A function $v \in C_w((-\infty, T); L^{3,\infty}_\sigma)$ is called a mild $L^{3,\infty}$-solution to (N-S) on $(-\infty, T)$ if v satisfies*

$$(2.2)\quad (v(t), \psi) = (e^{-(t-s)A}v(s), \psi) + \int_s^t \left((v \cdot \nabla e^{-(t-\tau)A}\psi, v)(\tau) + < f(\tau), e^{-(t-\tau)A}\psi > \right) d\tau$$

for all $\psi \in L^{3/2,1}_\sigma$ and all $-\infty < t < s < T$.

Next, we introduce the definitions of almost and *backward asymptotically almost periodic* functions with values in a Banach space B; see e.g. [5, Ch. VI], [1, Sect. 4.7].

Definition 2. *(i) A function $f \in BUC(\mathbb{R}; B)$ is called an almost periodic function in B on $\mathbb{R}$ if for all $\epsilon > 0$ there exists $L = L(\epsilon) > 0$ with the following property: For all $a \in \mathbb{R}$, there exists $\tau \in [a, a + L]$ such that*

$$\sup_{t\in\mathbb{R}} \|f(t + \tau) - f(t)\|_B \leq \epsilon.$$

Let us denote by $AP(\mathbb{R}; B)$ the set of all almost periodic functions in B on $\mathbb{R}$.

(ii) Let $T < \infty$ and $f \in BUC((-\infty, T); B)$. Then, we call f an almost periodic function in B on $(-\infty, T)$ if there exists a function $\tilde{f} \in AP(\mathbb{R}; B)$ such that $f = \tilde{f}$ on $(-\infty, T)$.

Let $AP((-\infty, T); B)$ denote the set of all almost periodic functions in B on $(-\infty, T)$.

(iii) Let $T \leq \infty$. A function $f \in BUC((-\infty, T); B)$ is called a backward asymptotically almost periodic function on $(-\infty, T)$ if there exist $f_1, f_2 \in BUC((-\infty, T); B)$ such that

$$f = f_1 + f_2, \quad f_1 \in AP((-\infty, T); B), \quad f_2 \in C_-((-\infty, T); B),$$

where

$$C_-((-\infty, T); B) := \{u \in BUC((-\infty, T); B)\ ;\ \lim_{t\to-\infty} \|u(t)\|_B = 0\}.$$

Let us denote by $AAP_-((-\infty, T); B)$ the set of all backward asymptotically almost periodic functions in B on $(-\infty, T)$.

Now our main results read as follows:

Theorem 1 ([9]). *Let Ω satisfy Assumption 1. Then, there exists an absolute constant $\delta > 0$ such that if u and v are almost periodic-in-time mild $L^{3,\infty}$-solutions to (N-S) on $(-\infty, \infty)$ for the same external force f, if*

$$(2.3)\qquad u, v \in L^2_{uloc}(\mathbb{R}; L^{6,2}(\Omega)),$$

and

$$\sup_t \|u\|_{3,\infty} < \delta, \tag{2.4}$$

then $u = v$.

Theorem 2 ([10]). *Let* Ω *satisfy Assumption 1. Then, there exists a constant* $\delta' = \delta'(\Omega) > 0$ *such that if* $T < \infty$, $u, v \in AAP_-((-\infty, T); L^{3,\infty})$ *are mild* $L^{3,\infty}$*-solutions to (N-S) on* $(-\infty, T)$ *for the same external force* f,

$$u, v \in L^2_{uloc}((-\infty, T); L^{6,2}(\Omega)), \tag{2.5}$$

and if

$$\limsup_{t \to -\infty} \|u(t)\|_{3,\infty} < \delta', \tag{2.6}$$

then $u = v$ *on* $(-\infty, T)$.

Remark. Our result is applicable to stationary solutions in L^3_w. It is known that if the external force $f = f(x)$ is steady and small in some functional space, then there exists a small steady solution $u(x)$ satisfying (2.5) and (2.6), see e.g. [21]. Theorem 2 shows that if f is a small steady force, the only possible backward asymptotically almost periodic $L^{3,\infty}$-solution with (2.5) is the steady state one.

Before coming to the main lemma of the proof, Lemma 2.3 below, let us recall several properties of almost periodic functions and of the Stokes semigroup. It is straightforward to see that Definition 1 on almost periodic functions is equivalent to the following one:

Proposition 2.1. $f \in C(\mathbb{R}; B)$ *is almost periodic in* B *if and only if for all* $\epsilon > 0$ *there exists* $l = l(\epsilon) > 0$ *with the following property: For all* $k \in \mathbb{Z}$, *there exists* $T_{\epsilon k} \in [-(k+1)l, -kl]$ *such that*

$$\sup_{t \in \mathbb{R}} \|f(t + T_{\epsilon k}) - f(t)\|_B \le \epsilon.$$

Proposition 2.2. *Assume that* u, v *are almost periodic in* $L^{3,\infty}$ *and* F *is almost periodic in* $L^{6/5}(\Omega)$.

(i) For all $\epsilon > 0$, there exists $l = l(\epsilon, u, v, F) > 0$ with the following property: For all $k \in \mathbb{Z}$, there exists $T_{\epsilon k} = T_{\epsilon k}(\epsilon, k, u, v, F) \in [-(k+1)l, -kl]$ such that

$$\begin{aligned} &\sup_{t\in\mathbb{R}} \|u(t+T_{\epsilon k}) - u(t)\|_{3,\infty} \le \epsilon, \\ &\sup_{t\in\mathbb{R}} \|v(t+T_{\epsilon k}) - v(t)\|_{3,\infty} \le \epsilon, \\ &\sup_{t\in\mathbb{R}} \|F(t+T_{\epsilon k}) - F(t)\|_{6/5} \le \epsilon. \end{aligned} \tag{2.7}$$

(ii) $w := u - v$ is almost periodic in $L^{3,\infty}$.

For the proof, see [5, Theorems 6.9 and 6.7].

Lemma 2.1 ([17],[41],[14],[15],[2],[3],[42],[25],[23]). *For all $t > 0$ and $a \in L^p_\sigma$, the following inequalities are satisfied:*

$$\|e^{-tA}a\|_{q,1} \le Ct^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\|a\|_{p,\infty} \quad \text{for } 1 < p < q < \infty, \tag{2.8}$$

$$\|\nabla e^{-tA}a\|_q \le Ct^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\|a\|_p \quad \text{for } 1 < p \le q \le 3, \tag{2.9}$$

where $C = C(p,q)$.

For all $\phi \in \dot{W}^{1,2}_{0,\sigma}$ it holds that

$$\|\nabla e^{-tA}\phi\|_2 \le \|\nabla\phi\|_2\,, \quad t > 0, \tag{2.10}$$

and for $\phi \in L^2_\sigma$

$$2\int_0^\infty \|\nabla e^{-\tau A}\phi\|_2^2\, d\tau = \|\phi\|_2^2. \tag{2.11}$$

For the proof of (2.10), (2.11) see e.g. [39, Proposition 2.1].

Lemma 2.2 ([22]). *Let $1 < p, q < \infty$ with $1/r := 1/p + 1/q < 1$. Then, for all $f \in L^{p,\infty}(\Omega)$ and $g \in L^{q,2}(\Omega)$, it holds that*

$$\|f \cdot g\|_{r,2} \le C\|f\|_{p,\infty}\|g\|_{q,2}, \tag{2.12}$$

where $C = C(p,q)$.

For $u \in \dot{W}^{1,2}_0(\Omega)$ it holds that

$$\|u\|_{6,2} \le C\|\nabla u\|_2, \tag{2.13}$$

where C is an absolute constant.

Finally, we come to the key lemma of the proof of uniqueness. If u and v are solutions to the Navier-Stokes equations, then $w := u - v$ satisfies

$$(U) \qquad \begin{cases} \partial_t w - \Delta w + w \cdot \nabla u + v \cdot \nabla w + \nabla p' = 0, & t \in \mathbb{R},\ x \in \Omega, \\ \text{div } w = 0, \quad t \in \mathbb{R},\ x \in \Omega, \\ w|_{\partial\Omega} = 0. \end{cases}$$

Hence, if Ω is a bounded domain and if u, v belong to the Leray-Hopf class, under the hypotheses of Theorem 1, the usual energy method and the Poincaré inequality yield $\|w(t)\|_2^2 \le e^{-(t-s)}\|w(s)\|_2^2$ for $t > s$. Consequently, in the case of *bounded* domains, Theorem 1 is obvious. In the case where Ω is an *unbounded* domain, u and v do not belong to the energy class in general and the Poincaré inequality does not hold in general. Hence, since we cannot use the energy method, we will use the argument of Lions-Masmoudi [26].

We recall the dual equations of the above system (U).

$$\text{(D)} \qquad \begin{cases} -\partial_t \psi - \Delta \psi - \sum_{i=1}^{3} u^i \nabla \psi^i - v \cdot \nabla \psi + \nabla \pi = F, & t < 0,\ x \in \Omega, \\ \nabla \cdot \psi = 0, & t < 0,\ x \in \Omega, \\ \psi|_{\partial\Omega} = 0. \end{cases}$$

In the following key lemma we construct a sequence of weak solutions of (D) having a property similar to that of almost periodic functions.

Lemma 2.3 ([9]). *Let u and v be almost periodic in $L^{3,\infty}$ and $L^{3,\infty}_\sigma$, respectively. Assume that F is almost periodic-in-time in $L^{6/5}(\Omega) \cap L^2(\Omega)$ and $\sup_t \|u\|_{3,\infty} < \delta$. Then, for all $\epsilon \in (0, \delta]$, there exists a constant $l = l(\epsilon) > 1$ with the following property: For all $k = 1, 2, \cdots$, there exist $T_{\epsilon k} \in [-(k+1)l, -kl]$ and generalized weak solutions $\psi_{\epsilon k} \in L^2(3T_{\epsilon k}, 0; \dot{W}^{1,2}_{0,\sigma})$ of (D) in the sense*

(2.14)

$$\int_{T_{\epsilon k}}^{0} \{ -(g(t+T_{\epsilon k}), \psi_{\epsilon k}(t+T_{\epsilon k})) + (g(t), \psi_{\epsilon k}(t)) \}\, dt$$
$$= \int_{T_{\epsilon k}}^{0} \int_{t+T_{\epsilon k}}^{t} \Big\{ (\frac{d}{dt} g, \psi_{\epsilon k}) + (\nabla g, \nabla \psi_{\epsilon k}) - (g, \sum_{i=1}^{3} u^i \nabla \psi^i_{\epsilon k}) - (g, v \cdot \nabla \psi_{\epsilon k}) - (g, F) \Big\}\, d\tau dt$$

for all $g \in L^2_{loc}(\mathbb{R}; D(A_2) \cap (\dot{W}^{1,2}_{0,\sigma})^*)$ with $\frac{d}{dt} g \in L^2_{loc}(\mathbb{R}; L^2_\sigma \cap (\dot{W}^{1,2}_{0,\sigma})^*)$. Moreover,

$$(2.15) \qquad \frac{1}{|T_{\epsilon k}|} \int_{3T_{\epsilon k}}^{0} \|\nabla \psi_{\epsilon k}\|_2^2\, d\tau \le C(1 + \sup_t \|F(t)\|_{6/5}^2),$$

$$(2.16)\qquad \frac{1}{|T_{\epsilon k}|}\int_{2T_{\epsilon k}}^{0}\|\nabla\psi_{\epsilon k}(t+T_{\epsilon k})-\nabla\psi_{\epsilon k}(t)\|_2^2\,d\tau \le C\epsilon^2(\sup_t\|F(t)\|_{6/5}^2+1),$$

where C is an absolute constant. Finally, (2.7) holds for those $T_{\epsilon k}$ and for u, v, F.

We note that this lemma does not require the divergence-free condition on u.

3 Outline of the proof of Theorem 1

The proof is based on the idea given by Lions-Masmoudi [26] whereby the uniqueness problem is reduced to the solvability of the dual equation. In order to prove Theorem 1, we establish the following two lemmata.

Lemma 3.1 ([9]). *Let w be an almost periodic function in $L^{3,\infty}(\Omega)$. Assume that for any almost periodic function F in $L^2(\Omega)\cap L^{6/5}(\Omega)$ and any number $\epsilon > 0$ there exists a sequence $\{T_{\epsilon k}\}_{k=1}^{\infty}$ such that*

$$(3.1)\qquad T_{\epsilon k}\to -\infty \quad as \quad k\to\infty,$$

$$(3.2)\qquad \limsup_{k\to\infty}\frac{1}{|T_{\epsilon k}|^2}\int_{T_{\epsilon k}}^{0}\int_{t+T_{\epsilon k}}^{t}(w(\tau),F(\tau))\,d\tau dt \le C\epsilon,$$

where C is independent of k and ϵ. Then $w\equiv 0$ in $\Omega\times\mathbb{R}$.

Lemma 3.2. *Let Ω, u, v satisfy the hypotheses of Theorem 1, F be an arbitrary almost periodic function in $L^2(\Omega)\cap L^{6/5}(\Omega)$ and let $T_{\epsilon k}=T_{\epsilon k}(u,v,F)\in[-(k+1)l,-kl]$, $k\in\mathbb{N}$, be the negative numbers given in Lemma 2.3. Then $w := u - v$ satisfies (3.2) for all $\epsilon\in(0,\delta]$.*

Outline of the proof of Lemma 3.2. Looking at the system (U) in Section 2, for $t > 3T_{\epsilon k}$ let

$$\begin{aligned}
(3.3)\qquad w(t) &= w_0(t)+w_1(t),\\
(3.4)\qquad w_0(t) &= e^{-(t-3T_{\epsilon k})A}w(3T_{\epsilon k})\\
(3.5)\quad (w_1(t),\phi) &= \int_{3T_{\epsilon k}}^{t}\{(w\cdot\nabla e^{-(t-\tau)A}\phi,u)+(v\cdot\nabla e^{-(t-\tau)A}\phi,w)\}\,d\tau \quad \text{for } \phi\in C_{0,\sigma}^{\infty}.
\end{aligned}$$

We note that (3.5) holds for all $\phi\in L^2_\sigma$, and from (2.10) we conclude that $|(w_1(t),\phi)|\le C(u,v)(t-3T_{\epsilon k})^{1/2}\|\nabla\phi\|_2$, since $w\otimes u, v\otimes w\in L^2(3T_{\epsilon k},0;L^2)$ by Lemma 2.2. Hence,

$w_1 \in L^\infty(3T_{\epsilon k}, 0; (\dot W^{1,2}_{0,\sigma})^*)$. Let $t > T_{\epsilon k}(> 3T_{\epsilon k})$ and let us write, using the notation $⨍$ for integral means,

$$
\begin{aligned}
&⨍_{T_{\epsilon k}}^{0} ⨍_{t+T_{\epsilon k}}^{t} (w(\tau), F(\tau))\, d\tau dt \\
=&⨍_{T_{\epsilon k}}^{0} ⨍_{t+T_{\epsilon k}}^{t} (w_0(\tau), F(\tau))\, d\tau dt + ⨍_{T_{\epsilon k}}^{0} ⨍_{t+T_{\epsilon k}}^{t} (w_1(\tau), F(\tau))\, d\tau dt \qquad (3.6)\\
=:&I_0 + I_1.
\end{aligned}
$$

By Lemma 2.1, we have

$$
\begin{aligned}
\left| \int_{t+T_{\epsilon k}}^{t} (w_0(\tau), F(\tau))\, d\tau \right| \quad &\le \int_{t+T_{\epsilon k}}^{t} \|e^{-(\tau-3T_{\epsilon k})A} w(3T_{\epsilon k})\|_6 \|F(\tau)\|_{6/5}\, d\tau \\
&\le C|T_{\epsilon k}|^{3/4} \sup_\tau \|w(\tau)\|_{3,\infty} \sup_\tau \|F(\tau)\|_{6/5}.
\end{aligned}
$$

Hence

$$
|I_0| = \left| ⨍_{T_{\epsilon k}}^{0} ⨍_{t+T_{\epsilon k}}^{t} (w_0(\tau), F(\tau))\, d\tau dt \right| \le C|T_{\epsilon k}|^{-1/4} \sup_\tau \|w(\tau)\|_{3,\infty} \sup_\tau \|F(\tau)\|_{6/5} \qquad (3.7)
$$

converges to 0 as $k \to \infty$ since $T_{\epsilon k} \to -\infty$.

In order to estimate I_1, we substitute w_1 into equation (2.14) for g. For details see [9]. Then, we obtain

(3.8)

$$
\begin{aligned}
&\int_{T_{\epsilon k}}^{0} \int_{t+T_{\epsilon k}}^{t} (F, w_1)\, d\tau dt \\
=&\int_{T_{\epsilon k}}^{0} \int_{t+T_{\epsilon k}}^{t} \Big\{ (w_0 \cdot \nabla \psi_{\epsilon k}, u) + (v \cdot \nabla \psi_{\epsilon k}, w_0) \Big\}\, d\tau dt \\
&\qquad + \int_{T_{\epsilon k}}^{0} (w_1(t+T_{\epsilon k}), \psi_{\epsilon k}(t+T_{\epsilon k})) - (w_1(t), \psi_{\epsilon k}(t))\, dt \\
\le&\int_{T_{\epsilon k}}^{0} \int_{t+T_{\epsilon k}}^{t} |(w_0 \cdot \nabla \psi_{\epsilon k}, u)|\, d\tau dt + \int_{T_{\epsilon k}}^{0} \int_{t+T_{\epsilon k}}^{t} |(v \cdot \nabla \psi_{\epsilon k}, w_0)|\, d\tau dt \\
&+ \left| \int_{T_{\epsilon k}}^{0} (w_1(t+T_{\epsilon k}), \psi_{\epsilon k}(t+T_{\epsilon k}) - \psi_{\epsilon k}(t))\, dt \right| + \left| \int_{T_{\epsilon k}}^{0} (w_1(t+T_{\epsilon k}) - w_1(t), \psi_{\epsilon k}(t))\, dt \right| \\
=&J_1 + J_2 + J_3 + J_4.
\end{aligned}
$$

By Lemma 2.1, we have

$$
\lim_{k\to\infty} \frac{1}{|T_{\epsilon k}|^2} J_1 = 0 \qquad (3.9)
$$

and

$$\lim_{k\to\infty} \frac{1}{|T_{\epsilon k}|^2} J_2 = 0. \tag{3.10}$$

By the definition (3.5) of w_1, (2.10), (2.7) (2.16), (2.15) we can show

$$\begin{aligned} \frac{1}{|T_{\epsilon k}|^2} J_3 \leq & C\big(\sup_t \|w\|_{3,\infty}\|u\|_{L^2_{uloc}(\mathbb{R};L^{6,2})} + \sup_t \|v\|_{3,\infty}\|w\|_{L^2_{uloc}(\mathbb{R};L^{6,2})}\big) \\ & \times \frac{1}{|T_{\epsilon k}|}\int_{T_{\epsilon k}}^0 \|\nabla(\psi_{\epsilon k}(t+T_{\epsilon k}) - \psi_{\epsilon k}(t))\|_2\, dt \\ \leq & C\epsilon. \end{aligned} \tag{3.11}$$

and

$$\limsup_{k\to\infty} \frac{1}{|T_{\epsilon k}|^2} J_4 \leq C\epsilon. \tag{3.12}$$

For details, see [9]. Finally, from (3.8), (3.9), (3.10), (3.11) and (3.12), we obtain

$$\limsup_{k\to\infty} |I_1| = \limsup_{k\to\infty} \left| ⨍_{T_{\epsilon k}}^0 ⨍_{t+T_{\epsilon k}}^t (w_1(\tau), F(\tau))\, d\tau dt \right| \leq C\epsilon.$$

This and (3.7) yield the assertion (3.2), which proves Lemma 3.2. □

Obviously, Lemmata 3.1 and 3.2 complete the proof of Theorem 1.

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