数理解析研究所講究録
第 1806 巻 2012 年 54-64

Results about dependence and convolution

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Abstract

A necessary and sufficient condition for two arithmetic functions to be linearly dependent over the set of prime-free functions is derived. A new kind of convolution is introduced and an application is given.

1 Introduction

The set $\mathcal{A}$ of arithmetic functions is a unique factorization domain under the usual addition and convolution (or Dirichlet product), [6], defined by

$$(f+g)(n):=f(n)+g(n), \quad (f*g)(n):=\sum_{ij=n} f(i)g(j) \quad (f,g\in\mathcal{A}, n\in\mathbb{N}).$$

The convolution identity I, is defined by $I(1)=1$ and $I(n)=0$ for all $n>1$.

For $r\in\mathbb{N}$, we say that $f_1, f_2, \ldots, f_r\in\mathcal{A}$ are algebraically dependent over $\mathbb{C}$, or $\mathbb{C}$-algebraically dependent, if there exists

$$P(X_1,\ldots,X_r):=\sum_{(i)} a_{(i)}X_1^{i_1}\cdots X_r^{i_r}\in\mathbb{C}[X_1,\ldots,X_r]\backslash\{0\}$$

such that

$$P(f_1,\ldots,f_r):=\sum_{(i)} a_{(i)}f_1^{i_1}*\cdots*f_r^{i_r}=0,$$

and are $\mathbb{C}$-algebraically independent otherwise. If the polynomial P is homogeneous of degree one in each variable, we say that $f_1, f_2, \ldots, f_r$ are $\mathbb{C}$-linearly dependent and $\mathbb{C}$-linearly independent otherwise.

A *derivation* d, over $\mathcal{A}$ is a map $d:\mathcal{A}\to\mathcal{A}$ satisfying

$$d(f*g)=df*g+f*dg, \quad d(c_1f+c_2g)=c_1df+c_2dg,$$

where $f,g\in\mathcal{A}$ and $c_1, c_2\in\mathbb{C}$. Derivations of higher orders are defined in the usual manner. Two typical examples of derivation are:

[1]Supported by a research grant for new scholar, the Thailand Research Fund MRG5380277.

- The *p-basic derivation*, p prime, defined by

$$(d_p f)(n) = f(np)\, \nu_p(np) \quad (n \in \mathbb{N}),$$

where $\nu_p(m)$ denotes the exponent of the highest power of p dividing m; for any primes p, q , we write $d_{pq}f$ instead of $d_p d_q f$.

- The *log-derivation* defined by

$$(d_L f)(n) = f(n) \log n \quad (n \in \mathbb{N}).$$

In 1986, Shapiro and Sparer [7] gave a systematic investigation of algebraic independence of Dirichlet series using the notion of Jacobian. Let $f_1, \ldots, f_r \in \mathcal{A}$ and $d_1, \ldots, d_r$ be derivations over $\mathcal{A}$, the *Jacobian* of f_i relative to d_i is the determinant

$$J(f_1, \ldots, f_r / d_1, \ldots, d_r) = \det(d_i(f_j)),$$

with multiplication being convolution. Clearly, a Jacobian is an element of $\mathcal{A}$. In the case where each d is a p-basic derivation corresponding to some prime p, we shall use the notation $J(f_1, \ldots, f_r / p_1, \ldots, p_r)$ for the corresponding Jacobian.

Shapiro-Sparer's criterion for $\mathbb{C}$-algebraic dependence of arithmetic functions states that:

Proposition 1. *Let $f_1, \ldots, f_r \in \mathcal{A}$ and $d_1, \ldots, d_r$ be distinct derivations over $\mathcal{A}$ which annihilate all elements of a subring $\mathcal{E} \subseteq \mathcal{A}$. If $J(f_1, \ldots, f_r / d_1, \ldots, d_r) \neq 0$, then $f_1, \ldots, f_r$ are algebraically independent over $\mathcal{E}$.*

In our earlier work, a necessary and sufficient criterion about $\mathbb{C}$-linear independence based, as guided by the real number case, on the notion of Wronskian was established.

Theorem 1. *Let $f_1, \ldots, f_r \in \mathcal{A}$ and let d be a derivation on $\mathcal{A}$. If $f_1, \ldots, f_r$ are $\mathbb{C}$-linearly dependent, then their Wronskian, relative to d,*

$$W_d(f_1, \ldots, f_r) := \begin{vmatrix} f_1 & f_2 & \ldots & f_r \\ df_1 & df_2 & \ldots & df_r \\ \vdots & & & \\ d^{r-1}f_1 & d^{r-1}f_2 & \ldots & d^{r-1}f_r \end{vmatrix}$$

vanishes, where, here an throughout, the multiplication involved in the determinant expansion is the Dirichlet product.

Theorem 2. *Let $f_1, \ldots, f_r \in \mathcal{A}\backslash\{0\}$. If their Wronskian $W = W_L(f_1, \ldots, f_r)$ relative to the log-derivation vanishes identically, then $f_1, \ldots, f_r$ are $\mathbb{C}$-linearly dependent.*

There are two investigations presented here. First, we consider Jacobians of two arithmetic functions for various p-basic derivations, but undergone an arbitrarily high order of derivations, and evaluate the resulting element at a single point 1. This enables us to obtain a necessary and sufficient condition for two arithmetic functions to be linearly dependent over the set of prime-free functions. Second, we consider a new kind of convolution, which was originated from the works of Haukkanen-Tóth, [8]. Our aim is to generalize this notion to the so-called Q_α-convolution and to connect it with a characterization problem.

2 Prime-free dependence

For $n \in \mathbb{N}$, let $\Omega(n)$ be the number of prime factors of n counting multiplicity. An arithmetic function f is said to be a *prime-free function* if $f(m) = f(n)$ for all $m, n \in \mathbb{N}$ having $\Omega(m) = \Omega(n)$. Examples of prime-free functions are abundant, for example, zero function, $\Omega(n)$, $2^{\Omega(n)}$, $\zeta(n) := 1$ $(n \in \mathbb{N})$ are prime-free functions.

It will be convenient to single out the set

$$\mathcal{A}^* := \{f \in \mathcal{A} : f(n) \neq 0 \text{ for all } n \in \mathbb{N}\}.$$

We say that two arithmetic functions $f, g \in \mathcal{A}^*$ are *prime-free dependent* if there exists a prime-free function H such that $f = Hg$. It is easy to check that prime-free dependence is an equivalence relation on A^*.

If f and g are $\mathbb{C}$-linearly dependent, then they are clearly prime-free dependent, but the converse is not true. For example, let $f(n) = 2^{\Omega(n)}n$ and $g(n) = n$, then f and g are prime-free dependent. But

$$\begin{aligned} W(f,g)(2) &= \begin{vmatrix} f & g \\ d_L f & d_L g \end{vmatrix}(2) = (f * d_L g - g * d_L f)(2) \\ &= f(1)g(2) - f(2)g(1) = -2 \neq 0, \end{aligned}$$

that is, f and g are $\mathbb{C}$-linearly independent.

Let $f, g \in \mathcal{A}$ and $k, \ell \in \mathbb{N}$. An (k, ℓ)-Jacobian of f, g with respect to distinct primes $p_1, \dots, p_r$ and distinct prime $q_1, \dots, q_s$ is denoted by

$$J(p_1^{\alpha_1} \cdots p_r^{\alpha_r}, q_1^{\beta_1} \cdots q_s^{\beta_s}) = \begin{vmatrix} d_{p_1^{\alpha_1} \dots p_r^{\alpha_r}} f & d_{p_1^{\alpha_1} \dots p_r^{\alpha_r}} g \\ d_{q_1^{\beta_1} \dots q_s^{\beta_s}} f & d_{q_1^{\beta_1} \dots q_s^{\beta_s}} g \end{vmatrix},$$

where $0 \le \alpha_i \le k, 0 \le \beta_j \le \ell$, $\sum_{i=1}^r \alpha_i = k$, $\sum_{j=1}^s \beta_j = \ell$. In the same manner, let $f_1, \dots, f_s \in \mathcal{A}$ and $k_1, \dots, k_s \in \mathbb{N}$. An $(k_1, \dots, k_s)$-Jacobian of $f_1, \dots, f_s$ with respect to distinct primes $p_{11}, \dots, p_{1r}, \dots, p_{s1}, \dots, p_{sr}$ is denoted by

$$J(p_{11}^{\alpha_{11}} \cdots p_{1r}^{\alpha_{1r}}, \dots, p_{s1}^{\alpha_{s1}} \cdots p_{sr}^{\alpha_{sr}}) = \begin{vmatrix} d_{p_{11}^{\alpha_{11}} \dots p_{1r}^{\alpha_{1r}}} f_1 & \cdots & d_{p_{11}^{\alpha_{11}} \dots p_{1r}^{\alpha_{1r}}} f_s \\ \vdots & & \\ d_{p_{s1}^{\alpha_{s1}} \dots p_{sr}^{\alpha_{sr}}} f_1 & \cdots & d_{p_{s1}^{\alpha_{s1}} \dots p_{sr}^{\alpha_{sr}}} f_s \end{vmatrix}$$

where $0 \le \alpha_{ij} \le k_i$, $\sum_{j=1}^r \alpha_{ij} = k_i \quad (i = 1, \dots, s;\ j = 1, \dots, r)$.

Our first main result is:

Theorem 3. *Let $f, g \in \mathcal{A}^*$.*

(1) If f and g are prime-free dependent, then with $k \in \mathbb{N}$, the (k,k)-Jacobian, $J(p^k, p_1^{\beta_1} \cdots p_r^{\beta_r})$, vanishes at 1 for all $r \in \mathbb{N}$ and primes $p, p_1, \dots, p_r$.

(2) If there exists a prime p such that for all $k \in \mathbb{N}$, the (k,k)-Jacobian, $J(p^k, p_1^{\beta_1} \cdots p_r^{\beta_r})$, vanishes at 1 for all $r \in \mathbb{N}$ and primes $p_1, \dots, p_r$, then f and g are prime-free dependent.

Proof. (1) If f and g are prime-free dependent, then there exists a prime-free function H such that $f = Hg$. Let p be a prime. Then with $k, r \in \mathbb{N}$, for all primes $p_1, \dots p_r$ and $\beta_1, \dots, \beta_r \in \mathbb{N}$ such that $0 \le \beta_1, \dots, \beta_r \le k$, $\sum_{i=1}^r \beta_i = k$, we have

$$\begin{aligned} &J(p^k, p_1^{\beta_1} \cdots p_r^{\beta_r})(1) \\ &= d_{p^k} f(1) d_{p_1^{\beta_1} \dots p_r^{\beta_r}} g(1) - d_{p^k} g(1) d_{p_1^{\beta_1} \dots p_r^{\beta_r}} g(1) \\ &= k! \beta_1! \cdots \beta_r! \Big(f(p^k) g(p_1^{\beta_1} \cdots p_r^{\beta_r}) - g(p^k) f(p_1^{\beta_1} \cdots p_r^{\beta_r}) \Big) \\ &= k! \beta_1! \cdots \beta_r! \Big(H(p^k) g(p^k) g(p_1^{\beta_1} \cdots p_r^{\beta_r}) - g(p^k) H(p_1^{\beta_1} \cdots p_r^{\beta_r}) g(p_1^{\beta_1} \cdots p_r^{\beta_r}) \Big) \\ &= 0. \end{aligned}$$

(2) Assume that there exists a prime p such that for all $k \in \mathbb{N}$, the (k,k)-Jacobian, $J(p^k, p_1^{\beta_1} \cdots p_r^{\beta_r})$ vanishes at 1 for all $r \in \mathbb{N}$ and primes $p_1, \dots, p_r$, that

is,

$$0 = J(p^k, p_1^{\beta_1} \cdots p_r^{\beta_r})(1) = d_{p^k} f(1) d_{p_1^{\beta_1} \cdots p_r^{\beta_r}} g(1) - d_{p^k} g(1) d_{p_1^{\beta_1} \cdots p_r^{\beta_r}} f(1)$$
$$= k! \beta_1! \cdots \beta_r! \left(f(p^k) g(p_1^{\beta_1} \cdots p_r^{\beta_r}) - g(p^k) f(p_1^{\beta_1} \cdots p_r^{\beta_r}) \right)$$

Thus,

$$f(p_1^{\beta_1} \cdots p_r^{\beta_r}) = \frac{f(p^k)}{g(p^k)} g(p_1^{\beta_1} \cdots p_r^{\beta_r}),$$

i.e.,

$$f(n) = \frac{f}{g}(p^k) g(n) \quad \text{for all } n \in \mathbb{N} \quad \text{with} \quad \Omega(n) = k.$$

Taking

$$H(n) = \frac{f}{g}(p^k) \quad \text{for all } n \in \mathbb{N} \text{ with } \Omega(n) = k,$$

the desired result follows. □

The method of proof in Theorem 3 extends easily to the following more general case.

Theorem 4. *Let $f, g \in \mathcal{A}^*$.*

1. *If f and g are $\mathbb{C}$-linearly dependent, then with $k, j \in \mathbb{N}$, the (j,k)-Jacobian, $J(p^j, p_1^{\beta_1} \cdots p_r^{\beta_r})$, vanishes at 1 for all $r \in \mathbb{N}$ and all primes $p, p_1, \ldots, p_r$.*
2. *If there exist a prime p and $j \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, the (j,k)-Jacobian, $J(p^j, p_1^{\beta_1} \cdots p_r^{\beta_r})$, vanishes at 1 for all $r \in \mathbb{N}$ and all primes $p_1, \ldots, p_r$, then f and g are $\mathbb{C}$-linearly dependent.*

Proof. (1) Assume that f and g are $\mathbb{C}$-linearly dependent. Then $f = cg$ for some constant $c \in \mathbb{C}$. Let $k, j \in \mathbb{N}$. Then for all $r \in \mathbb{N}$, for all primes $p, p_1, \ldots p_r$ and $\beta_1, \ldots, \beta_r \in \mathbb{N}$ such that $0 \le \beta_1, \ldots, \beta_r \le k$, $\sum_{i=1}^r \beta_i = k$, we have

$$J(p^j, p_1^{\beta_1} \cdots p_r^{\beta_r})(1) = d_{p^j} f(1) d_{p_1^{\beta_1} \cdots p_r^{\beta_r}} g(1) - d_{p^j} g(1) d_{p_1^{\beta_1} \cdots p_r^{\beta_r}} g(1)$$
$$= j! \beta_1! \cdots \beta_r! \left(f(p^j) g(p_1^{\beta_1} \cdots p_r^{\beta_r}) - g(p^j) f(p_1^{\beta_1} \cdots p_r^{\beta_r}) \right)$$
$$= j! \beta_1! \cdots \beta_r! \left(cg(p^j) g(p_1^{\beta_1} \cdots p_r^{\beta_r}) - g(p^j) cg(p_1^{\beta_1} \cdots p_r^{\beta_r}) \right) = 0.$$

(2) Assume that there exist a prime p and $j \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, the (j,k)-Jacobian, $J(p^j, p_1^{\beta_1} \cdots p_r^{\beta_r})$, vanishes at 1 for all $r \in \mathbb{N}$ and all primes $p_1, \ldots, p_r$. Then

$$0 = J(p^j, p_1^{\beta_1} \cdots p_r^{\beta_r})(1) = d_{p^j} f(1) d_{p_1^{\beta_1} \cdots p_r^{\beta_r}} g(1) - d_{p^j} g(1) d_{p_1^{\beta_1} \cdots p_r^{\beta_r}} f(1)$$
$$= j! \beta_1! \cdots \beta_r! \left(f(p^j) g(p_1^{\beta_1} \cdots p_r^{\beta_r}) - g(p^j) f(p_1^{\beta_1} \cdots p_r^{\beta_r}) \right),$$

i.e.,

$$f(p_1^{\beta_1}\cdots p_r^{\beta_r}) = \frac{f(p^j)}{g(p^j)} g(p_1^{\beta_1}\cdots p_r^{\beta_r}).$$

Thus,

$$f(n) = cg(n), \quad c = \frac{f(p^j)}{g(p^j)} \in \mathbb{C} \quad (n \in \mathbb{N}),$$

i.e., f and g are $\mathbb{C}$-linearly dependent. □

Pushing our investigation in another direction, we have:

Theorem 5. *Let $f_1, \ldots, f_s \in \mathcal{A}\backslash\{0\}$.*

(1) If $f_1, \ldots, f_s$ are $\mathbb{C}$-linearly dependent, then with $k \in \mathbb{N}$, the $(1, \ldots, 1, k)$-Jacobian, $J(q_1, \ldots, q_{s-1}, p_1^{\beta_1}\cdots p_r^{\beta_r})$, vanishes at 1 for all $r \in \mathbb{N}$ and all primes $p_1, \ldots, p_r$, $q_1 \ldots, q_{s-1}$.

(2) Assume that there is a set of $s-1$ primes $\{q_1 \leq \cdots \leq q_{s-1}\}$ such that one of the sets of $s-1$ vectors

$$\{(f_{i_1}(q_1), \ldots, f_{i_{s-1}}(q_{s-1}))^t : 1 \leq i_1 < i_2 < \cdots < i_{s-1} \leq s\}$$

is linearly independent over $\mathbb{C}$. If, for all $k \in \mathbb{N}$, the $(1, \ldots, 1, k)$-Jacobian, $J(q_1 \ldots, q_{s-1}, p_1^{\beta_1}\cdots p_r^{\beta_r})$, vanishes at 1 for all $r \in \mathbb{N}$ and all primes $p_1, \ldots, p_r$, then $f_1, \ldots, f_s$ are $\mathbb{C}$-linearly dependent.

Proof. (1) If $f_1, \ldots, f_s$ are $\mathbb{C}$-linearly dependent, then there are complex numbers $c_1, \ldots, c_s$, not all zero, such that

$$c_1 f_1 + \ldots + c_s f_s = 0.$$

Let $q_1, \ldots, q_{s-1}$ be primes and $k \in \mathbb{N}$. Thus, for all $r \in \mathbb{N}$, $0 \leq \beta_1, \ldots, \beta_r \leq k$ with $\sum_{i=1}^r \beta_i = k$ and all primes $p_1, \ldots, p_r$, we have

$$c_1 \begin{bmatrix} f_1(q_1) \\ \vdots \\ f_1(q_{s-1}) \\ f_1(p_1^{\beta_1}\cdots p_r^{\beta_r}) \end{bmatrix} + \cdots + c_s \begin{bmatrix} f_s(q_1) \\ \vdots \\ f_s(q_{s-1}) \\ f_s(p_1^{\beta_1}\cdots p_r^{\beta_r}) \end{bmatrix} = 0,$$

i.e., the s column vectors are linearly dependent implying that

$$\begin{vmatrix} f_1(q_1) & \cdots & f_s(q_1) \\ \vdots & & \\ f_1(q_{s-1}) & \cdots & f_s(q_{s-1}) \\ f_1(p_1^{\beta_1}\cdots p_r^{\beta_r}) & \cdots & f_s(p_1^{\beta_1}\cdots p_r^{\beta_r}) \end{vmatrix} = 0,$$

and consequently,

$$J(q_1,\dots,q_{s-1},p_1^{\beta_1}\cdots p_r^{\beta_r})(1)=\begin{vmatrix} d_{q_1}f_1 & \cdots & d_{q_1}f_s \\ \vdots & & \\ d_{q_{s-1}}f_1 & \cdots & d_{q_{s-1}}f_s \\ d_{p_1^{\beta_1}\cdots p_r^{\beta_r}}f_1 & \cdots & d_{p_1^{\beta_1}\cdots p_r^{\beta_r}}f_s \end{vmatrix}$$

$$=\beta_1!\cdots\beta_r!\begin{vmatrix} f_1(q_1) & \cdots & f_s(q_1) \\ \vdots & & \\ f_1(q_{s-1}) & \cdots & f_s(q_{s-1}) \\ f_1(p_1^{\beta_1}\cdots p_r^{\beta_r}) & \cdots & f_s(p_1^{\beta_1}\cdots p_r^{\beta_r}) \end{vmatrix}=0$$

(2) Since, for all $k\in\mathbb{N}$, the $(1,\dots,1,k)$-Jacobian, $J(q_1\dots,q_{s-1},p_1^{\beta_1}\cdots p_r^{\beta_r})$, vanishes at 1 for all $r\in\mathbb{N}$ and all primes $p_1,\dots,p_r$, we have

$$0=J(q_1\dots,q_{s-1},p_1^{\beta_1}\cdots p_r^{\beta_r})(1)=\beta_1!\cdots\beta_r!\begin{vmatrix} f_1(q_1) & \cdots & f_s(q_1) \\ \vdots & & \\ f_1(q_{s-1}) & \cdots & f_s(q_{s-1}) \\ f_1(p_1^{\beta_1}\cdots p_r^{\beta_r}) & \cdots & f_s(p_1^{\beta_1}\cdots p_r^{\beta_r}) \end{vmatrix}.$$

Expanding via the last row, we get

$$0=f_1(p_1^{\beta_1}\cdots p_r^{\beta_r})\begin{vmatrix} f_2(q_1) & \cdots & f_s(q_1) \\ \vdots & & \\ f_2(q_{s-1}) & \cdots & f_s(q_{s-1}) \end{vmatrix}+\cdots+f_s(p_1^{\beta_1}\cdots p_r^{\beta_r})\begin{vmatrix} f_1(q_1) & \cdots & f_{s-1}(q_1) \\ \vdots & & \\ f_1(q_{s-1}) & \cdots & f_{s-1}(q_{s-1}) \end{vmatrix},$$

i.e., for all $n\in\mathbb{N}$, we have

$$0=f_1(n)\begin{vmatrix} f_2(q_1) & \cdots & f_s(q_1) \\ \vdots & & \\ f_2(q_{s-1}) & \cdots & f_s(q_{s-1}) \end{vmatrix}+\cdots+f_s(n)\begin{vmatrix} f_1(q_1) & \cdots & f_{s-1}(q_1) \\ \vdots & & \\ f_1(q_{s-1}) & \cdots & f_{s-1}(q_{s-1}) \end{vmatrix}.$$

Since one of the sets of $s-1$ vectors

$$\{(f_{i_1}(q_1),\dots,f_{i_{s-1}}(q_{s-1}))^t : 1\le i_1<i_2<\cdots<i_{s-1}\le s\}$$

is linearly independent over $\mathbb{C}$, then one of the determinant-coefficients on the right-hnad side is nonzero, i.e., $f_1,\dots,f_s$ are $\mathbb{C}$-linearly dependent. □

3 Q_α-convolution

Let $n = \prod_p p^{\nu_p(n)}$ denote the prime factorization of $n \in \mathbb{N}$. Haukkanen-Tóth, [8], introduced the binomial convolution of arithmetic function f and g as

$$(f \circ g)(n) = \sum_{d|n} \left(\prod_p \binom{\nu_p(n)}{\nu_p(d)} \right) f(d)g(n/d)$$

where $\binom{a}{b}$ denotes the usual binomial coefficient. Observe that $f \circ g$ can also be put under the form

$$(f \circ g)(n) = \sum_{xy=n} \frac{\xi(n)}{\xi(x)\xi(y)} f(x)g(y)$$

where $\xi(n) = \prod_p (\nu_p(n)!)$. This convolution first appeared in 1996 in [1] and later in [8], where more properties are derived under this convolution, such as, $(\mathcal{A}, +, \circ, \mathbb{C})$ is a $\mathbb{C}$-algebra under addition and binomial convolution.

We can generalize the binomial convolution even further to a new kind of convolution by replacing the function ξ with an arbitrary function. Let $\alpha \in \mathcal{A}^*$. The Q_α-*convolution* of two arithmetic function f and g is defined as

$$(f \diamond g)(n) = \sum_{xy=n} \frac{\alpha(n)}{\alpha(x)\alpha(y)} f(x)g(y).$$

The Q_α-convolution identity is the function αI. Two remarks which justifies its introduction are:

1. if α is a completely multiplicative function, then $f \diamond g = f * g$, the classical Dirichlet convolution;

2. if $\alpha = \xi$, then $f \diamond g = f \circ g$, the Haukkanen-Tóth convolution.

The most important result for this concept, which somewhat renders this convolution not too exciting is:

Proposition 2. *The algebra* $(\mathcal{A}, +, \diamond, \mathbb{C})$ *and* $(\mathcal{A}, +, *, \mathbb{C})$ *are isomorphic under the mapping* $f \longmapsto f/\alpha$.

With this isomorphism, we can express the Q_α-convolution in terms of Dirichlet convolution as

$$f \diamond g = \alpha \left(\frac{f}{\alpha} * \frac{g}{\alpha} \right)$$

or equivalently,

$$f * g = \frac{\alpha f \diamond \alpha g}{\alpha}.$$

If f^{-1*} and f^{-1} denote the inverses of f under the Dirichlet convolution and the Q_α-convolution, respectively, both of which exist if and only if $f(1) \neq 0$, then we have:

Theorem 6. *If $f \in \mathcal{A}$ be such that $f(1) \neq 0$, then*

$$f^{-1*} = \frac{(\alpha f)^{-1}}{\alpha}, \quad f^{-1} = \alpha \left(\frac{f}{\alpha}\right)^{-1*}.$$

Proof. From

$$I = f * f^{-1*} = \frac{\alpha f \diamond \alpha f^{-1*}}{\alpha},$$

we get $\alpha I = \alpha f \diamond \alpha f^{-1*}$, i.e., $\alpha f^{-1*} = (\alpha f)^{-1}$. From

$$\alpha I = f \diamond f^{-1} = \alpha \left(\frac{f}{\alpha} * \frac{f^{-1}}{\alpha}\right),$$

we get $I = \left(\frac{f}{\alpha} * \frac{f^{-1}}{\alpha}\right)$, i.e., $\frac{f^{-1}}{\alpha} = \left(\frac{f}{\alpha}\right)^{-1*}$. □

The following characterization of completely multiplicative functions has been proved by many authors, see e.g. [2], [4], [5].

Proposition 3. *Let $f \in \mathcal{A}$ be a multiplicative function. Then f is completely multiplicative if and only if*

$$f(g * h) = fg * fh \text{ for all } g, h \in \mathcal{A}.$$

We end our presentation with some characterizations of completely multiplicative functions using a distributive property through Q_α-convolution.

Theorem 7. *Let $f \in \mathcal{A}$ be a multiplicative function. Then f is completely multiplicative if and only if*

$$f(g \diamond h) = fg \diamond fh \text{ for all } g, h \in \mathcal{A}.$$

Proof. Assume that f is completely multiplicative. Let $g, h \in \mathcal{A}$. Then

$$f(g \diamond h) = f\alpha \left(\frac{g}{\alpha} * \frac{h}{\alpha}\right) = \alpha \left(\frac{fg}{\alpha} * \frac{fh}{\alpha}\right) = fg \diamond fh$$

Assume that $f(g \diamond h) = fg \diamond fh$ for all $g, h \in \mathcal{A}$. Then

$$\alpha f(g * h) = f(\alpha g \diamond \alpha i) = \alpha fg \diamond \alpha fh = \alpha \left(\frac{\alpha fg}{\alpha} * \frac{\alpha fh}{\alpha}\right) = \alpha(fg * fh)$$

so $f(g*h) = fg*fh$, and so by Proposition 3, f is a completely multiplicative. □

In 1973, E. Langford [3] gave a characterization of completely multiplicative functions using a distributive property over a Dirichlet product. We do the same here through Q_α-convolution. Let $g, h \in \mathcal{A}$ and $k = g \diamond h$. We notice that

$$\alpha(1)k(p) = g(1)h(p) + g(p)h(1)$$

for prime p. If the relation

$$\alpha(1)k(n) = g(1)h(n) + g(n)h(1)$$

holds only when n is a prime, we say that the product $k = g \diamond h$ is Q_α-discriminative.

Theorem 8. *Let $f \in \mathcal{A}$ be such that $f(1) \neq 0$. Then f is completely multiplicative if and only if it distributes over a Q_α-discriminative product.*

Proof. The necessity part follows from Theorem 7. To prove the sufficiency part, assume that f distributes over a Q_α-discriminative product $k = g \diamond h$. First we show that $f(1) = 1$. If $k(1) = 0$, then

$$0 = \alpha(1)k(1) = \alpha(1)(g \diamond h)(1) = g(1)h(1),$$

and so

$$g(1)h(1) + g(1)h(1) = 0 = \alpha(1)k(1)$$

which contradicts the property of k. Hence, $k(1) \neq 0$. From

$$f(1)k(1) = fk(1) = f(g \diamond h)(1) = (fg \diamond fh)(1) = f(1)^2\alpha(1)\frac{g(1)}{\alpha(1)}\frac{h(1)}{\alpha(1)} = f(1)^2k(1),$$

we get $f(1) = 1$. To finish the proof it suffices to show that

$$f(p_1 \cdots p_r) = f(p_1) \cdots f(p_r) \tag{1}$$

for all primes $p_1, \ldots, p_r$, $r \in \mathbb{N}$ (not necessary distinct). We do this by induction on r. Clearly, (1) holds when $r = 1$. Now, let $r > 1$ and assume that (1) holds for all $1 \leq s < r$. Let $p_1, \ldots, p_r$ be primes and $n = p_1 \cdots p_r$. By induction hypothesis and $f(g \diamond h) = fg \diamond fh$, we obtain

$$0 = f(g \diamond h)(n) - (fg \diamond fh)(n) = (f(p_1 \cdots p_r) - f(p_1) \cdots f(p_r)) \sum_{\substack{xy=n \\ x,y<n}} \alpha(n)\frac{g(x)h(y)}{\alpha(x)\alpha(y)}.$$

If

$$\sum_{\substack{xy=n \\ x,y<n}} \alpha(n)\frac{g(x)h(y)}{\alpha(x)\alpha(y)} = 0,$$

then

$$k(n) = (g \diamond h)(n) = \alpha(n)\left(\frac{g(1)h(n) + g(n)h(1)}{\alpha(1)\alpha(n)}\right),$$

yielding $\alpha(1)k(1) = g(1)h(n) + g(n)h(1)$, which is impossible for non-prime n. Thus,

$$\sum_{\substack{xy=n \\ x,y<n}} \alpha(n)\frac{g(x)h(y)}{\alpha(x)\alpha(y)} \neq 0,$$

and consequently, $f(p_1 \cdots p_r) = f(p_1) \cdots f(p_r)$, as to be proved. □

References

[1] P. Haukkanen, *On a binomial convolution of arithmetical functions*, Nieuw Arch. Wisk. (IV) 14(1996), 209-216.

[2] P. Haukkanen, *On characterizations of completely multiplicative arithmetical functions*, in: Number Theory (Turku, 1999), de Gruyter, Berlin (2001), 115-123.

[3] E. Langford, *Distributivity over the Dirichlet product and completely multiplicative arithmetical functions*, Amer. Math. Monthly 80 (1973), 411-414.

[4] P.J. McCarthy, Introduction to Arithmetical Functions, Springer, 1986.

[5] N. Pabhapote, V. Laohakosol, *Distributive property of completely multiplicative functions*, Lithuanian Math. J, 50 (2010), 312-322.

[6] H. N. Shapiro, Introduction to the Theory of Numbers, John Wiley and Sons, New York, 1983.

[7] H. N. Shapiro and G. H. Sparer, *On algebraic independence of Dirichlet series*, Comm. Pure Appl. Math. 39(1986), 695-745.

[8] L. Tóth and P. Haukkanen, *On the binomial convolution of arithmetical functions*, J. Combinatorics and Number Theory 1(2009), 31-48.

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