数理解析研究所講究録
第 1806 巻 2012 年 156-163

Higher dimensional Dedekind sums in positive characteristic

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1 Introduction

For relatively prime integers $c>0$, a, the *inhomogeneous Dedekind sum* is defined by

$$s(a,c)=\frac{1}{4c}\sum_{k=1}^{c-1}\cot\left(\frac{\pi ka}{c}\right)\cot\left(\frac{\pi k}{c}\right).$$

This satisfies the reciprocity law

$$s(a,c)+s(c,a)=\frac{a^{2}+c^{2}+1-3ac}{12ac}$$

if $a, c>0$ are coprime. For basic facts, see the book [10]. For $a, b\in \mathbb{Z}$ relatively prime to an integer $c>0$, H. Rademacher defined the *homogeneous Dedekind sum* by

$$s(c;a,b)=\frac{1}{4c}\sum_{k=1}^{c-1}\cot\left(\frac{\pi ka}{c}\right)\cot\left(\frac{\pi kb}{c}\right).$$

which satisfies the reciprocity law

$$s(c;a,b)+s(b;a,c)+s(a;a,b)=\frac{a^{2}+b^{2}+c^{2}-3abc}{12abc}$$

if a, b, c are pairwise coprime. For $a_{1}, \ldots, a_{d}\in \mathbb{Z}$ relatively prime to an integer $a_{0}>0$, D. Zagier [11] defined the *higher dimensional Dedekind sum* by

$$d(a_{0};a_{1},\ldots,a_{d})=(-1)^{d/2}\frac{1}{a_{0}}\sum_{k=1}^{a_{0}-1}\cot\left(\frac{\pi ka_{1}}{a_{0}}\right)\cdots\cot\left(\frac{\pi ka_{d}}{a_{0}}\right).$$

If $a_{0}, a_{1}, \ldots, a_{d}$ are pairwise coprime, it satisfies the reciprocity law

$$\sum_{j=0}^{d}d(a_{j};a_{1},\ldots,a_{j-1},a_{j+1},\ldots,a_{n})=1-\frac{l_{d}(a_{0},\ldots,a_{d})}{a_{0}\cdots a_{d}},$$

where $l_{d}(a_{0},\ldots,a_{d})$ are polynomials in $a_{0}, \ldots, a_{d}$. We denote by $\cot^{(m)}(z)$ the mth derivative of $\cot(z)$. Let $a_{1}\ldots, a_{d}\in \mathbb{Z}$ be relatively prime to $a_{0}\in \mathbb{N}$, and let

$m_0, \ldots, m_d \geq 0$. A. Bayad and A. Raouj [3] investigated the *multiple Dedekind-Rademacher sum* given by

$$C\left(\begin{array}{c|ccc} a_0 & a_1, & \ldots, & a_d \\ m_0 & m_1, & \ldots, & m_d \end{array}\right) = \frac{1}{a_0^{m_0+1}} \sum_{k=1}^{a_0-1} \cot^{(m_1)}\left(\frac{\pi k a_1}{a_0}\right) \cdots \cot^{(m_d)}\left(\frac{\pi k a_d}{a_0}\right),$$

which satisfies the reciprocity law. See also M. Beck [4] for related topics. These Dedekind sums are rational numbers and satisfy the reciprocity law. They can be applied to number theory and combinatorial theory.

The aim of our paper is to introduce two kinds of multiple Dedekind-Rademacher sums in function fields. These are related to two kinds of "lattices", i.e.,

– A-lattices, which are associated with Drinfeld modules,

– $\mathbb{F}_q$-vector spaces of finite dimension, which are not associated with Drinfeld modules.

It should be remarked that the first kind of multiple Dedekind-Rademacher sums is an extension of Dedekind sums in [1, 8, 9]. We are going to discuss the reciprocity law and the rationality. Some results are influenced by the joint works with A. Bayad (cf. [1, 2]).

The author would like to thank Professor T. Noda for giving the opportunity to write this report.

2 A-lattices and Drinfeld modules

In this section, we recall some basic facts on A-lattices and Drinfeld modules investigated in the theory of function fields. We refer to D. Goss [7] for details. Let $\mathcal{C}$ be a smooth, projective, geometrically connected curve over $\mathbb{F}_q$. Denote by K the function field of $\mathcal{C}$ over $\mathbb{F}_q$. We take and fix a closed point $\infty \in \mathcal{C}$ of degree d_∞ over $\mathbb{F}_q$. Let v_∞ be the valuation associated to ∞, and let $|\ |_\infty$ be the normalized absolute value corresponding to v_∞. Here "normalized" means that $|x|_\infty = q^{\deg(x)} = q^{-d_\infty v_\infty(x)}$ for any $x \in K$. Let K_∞ be the completion of K with respect to $|\ |_\infty$. We denote by C_∞ the completion of an algebraic closure of K_∞. Put $A = H^0(\mathcal{C} - \infty, \mathcal{O}_\mathcal{C})$.

2.1. A-lattices. The subset Λ of C_∞ is an *A-lattice* of rank r if it is a finitely generated A-submodule of rank r in C_∞ that is discrete in the topology of C_∞. Define

$$e_\Lambda(z) = z \prod_{0 \neq \lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right),$$

which has the following properties:

(1) e_Λ is entire in the rigid analytic sense, and the map $e_\Lambda : C_\infty \to C_\infty$, $z \mapsto e_\Lambda(z)$ is surjective;

(2) e_Λ is $\mathbb{F}_q$-linear and Λ-periodic;

(3) e_Λ has simple zeros at the points of Λ, and no other zeros;

(4) $de_\Lambda(z)/dz = e'_\Lambda(z) = 1$.

2.2. Drinfeld modules. For a field L, let $L\{\tau\}$ be the non-commutative ring of polynomials in τ over L such that $\tau a = a^q \tau$ $(a \in L)$. An $\mathbb{F}_q$-algebra homomorphism $\phi : A \to L\{\tau\}$, $a \mapsto \phi_a$ is said to be a *Drinfeld module* of rank r over L if ϕ satisfies

(i) $D \circ \phi = \iota$, where D is the derivation $D(f) = a_0$ for $f(\tau) = \sum_{i=0}^{l} a_i \tau^i \in L\{\tau\}$, and ι is the inclusion map $\iota : A \hookrightarrow C_\infty$;

(ii) For some $a \in A$, $\phi_a \neq \iota(a)\tau^0$;

(iii) For all $a \in A$, $\deg \phi_a = r \deg(a) = -r d_\infty v_\infty(a)$.

2.3. For any rank r A-lattice Λ, there exists a unique rank r Drinfeld module ϕ^Λ such that $e_\Lambda(az) = \phi_a^\Lambda(e_\Lambda(z))$ $(a \in A)$. The association $\Lambda \mapsto \phi^\Lambda$ yields a bijection between the set of A-lattices of rank r in C_∞ and the set of Drinfeld modules of rank r over C_∞.

3 Multiple Dedekind-Rademacher sums

In this section, we introduce a generalization of the higher dimensional Dedekind sum defined in [1].

3.1. Multiple Dedekind-Rademacher sums. Let Λ be an A-lattice. Select $a_0, a_1, \ldots, a_d \in A \setminus \{0\}$ such that $a_1, a_2, \ldots, a_d$ is relatively prime to a_0. Let $m_0, \ldots, m_d$ be non-negative integers. Then

Definition 3.1 *We call*

$$s_\Lambda\left(\begin{array}{c|ccc} a_0 & a_1, & \ldots, & a_d \\ m_0 & m_1, & \ldots, & m_d \end{array}\right) = \frac{1}{a_0^{m_0+1}} \sum_{0 \neq \lambda \in \Lambda/a_0\Lambda} e_\Lambda\left(\frac{\lambda a_1}{a_0}\right)^{-m_1-1} \cdots e_\Lambda\left(\frac{\lambda a_d}{a_0}\right)^{-m_d-1}$$

the multiple Dedekind-Rademacher sum. When $\Lambda/a_0\Lambda = 0$, the sum is defined to be zero.

This Dedekind sum gives some special Dedekind sums analogous to the classical ones. For example,

$$\begin{aligned} s_\Lambda(a_0; a_1, \ldots, a_d) &= (-1)^d s_\Lambda \left(\begin{array}{c|ccc} a_0 & a_1, & \ldots, & a_d \\ 0 & 0, & \ldots, & 0 \end{array} \right) \\ &= \frac{(-1)^d}{a_0} \sum_{0 \neq \lambda \in \Lambda / a_0 \Lambda} e_\Lambda \left(\frac{\lambda a_1}{a_0} \right)^{-1} \cdots e_\Lambda \left(\frac{\lambda a_d}{a_0} \right)^{-1} \end{aligned}$$

is an analog of Zagier's higher dimensional Dedekind sum.

$$s_\Lambda(c; a, b) = \frac{1}{c} \sum_{0 \neq \lambda \in \Lambda / c\Lambda} e_\Lambda \left(\frac{\lambda a}{c} \right)^{-1} e_\Lambda \left(\frac{\lambda b}{c} \right)^{-1}$$

is an analog of the homogeneous Dedekind sum, and

$$s_\Lambda(a, c) = \frac{1}{c} \sum_{0 \neq \lambda \in \Lambda / c\Lambda} e_\Lambda \left(\frac{\lambda}{c} \right)^{-1} e_\Lambda \left(\frac{\lambda a}{c} \right)^{-1}$$

is an analog of the inhomogeneous Dedekind sum.

3.2. The reciprocity law. We now state the reciprocity law for our Dedekind sums.

Theorem 3.2 *If $a_0, \ldots, a_d \in A \setminus \{0\}$ are pairwise coprime,*

$$\begin{aligned} &\sum_{i=0}^{d} \sum_{\substack{l_0, \ldots, \widehat{l_i}, \ldots, l_d \geq 0 \\ l_0 + \cdots + \widehat{l_i} + \cdots + l_d = m_i}} \left(\prod_{j \neq i} \binom{m_j + l_j}{m_j} (-a_j)^{l_j} \right) \\ &\qquad \times s_\Lambda \left(\begin{array}{c|ccccc} a_i & a_0, & \ldots, & \widehat{a_i}, & \ldots, & a_d \\ m_i & m_0 + l_0, & \ldots, & \widehat{m_i + l_i}, & \ldots, & m_d + l_d \end{array} \right) \\ &= \frac{(-1)^{m_0 + \cdots + m_d + d}}{a_0^{m_0+1} \cdots a_d^{m_d+1}} \sum_{\substack{j_0, \ldots, j_d \geq 0 \\ j_0 + \cdots + j_d \\ = m_0 + \cdots + m_d + d}} A_{0,j_0} A_{1,j_1} \cdots A_{d,j_d}. \end{aligned}$$

Here $\widehat{\bullet}$ is omitting of $\bullet$ and

$$A_{i,j_i} = \begin{cases} (-1)^{m_i+1} & (\text{if } j_i = 0) \\ \binom{j_i - 1}{m_i} E_{j_i}(\phi[a_i]) & (\text{if } j_i \geq m_i) \\ 0 & (\text{otherwise}) \end{cases},$$

where $\phi[a_i] = \{x \in C_\infty \mid \phi_{a_i}(x) = 0\}$ and $E_{j_i}(\phi[a_i]) = \sum_{0 \neq x \in \phi[a_i]} \frac{1}{x^{j_i}}$.

3.3. Outline of Proof of Theorem 3.2. Consider

$$F(z) = \frac{1}{\phi_{a_0}(z)^{m_0+1} \cdots \phi_{a_d}(z)^{m_d+1}}.$$

Its poles are $R = \bigcup_{i=0}^{d} \phi[a_i]$. We calculate the residue $\mathrm{Res}(F(z)dz, z = x)$ for each $x \in R$. Using the residue theorem

$$\sum_{x \in R} \mathrm{Res}(F(z)dz, z = x) = 0,$$

we obtain the theorem stated above.

4 The rationality

4.1. Let ϕ be the rank r Drinfeld module associated to Λ. Dedekind sums in function feilds are not always rational. When is $s_\Lambda \left(\begin{array}{c|ccc} a_0 & a_1, & \ldots, & a_{q^i-1} \\ m_0 & m_1, & \ldots, & m_{q^i-1} \end{array} \right)$ rational, i.e, $s_\Lambda \left(\begin{array}{c|ccc} a_0 & a_1, & \ldots, & a_{q^i-1} \\ m_0 & m_1, & \ldots, & m_{q^i-1} \end{array} \right) \in K$? We find that the rationality of the Dedekind sum is related to the field of definition of ϕ.

Proposition 4.1 *If ϕ is defined over K, then $s_\Lambda \left(\begin{array}{c|ccc} a_0 & a_1, & \ldots, & a_{q^i-1} \\ m_0 & m_1, & \ldots, & m_{q^i-1} \end{array} \right)$ is rational.*

Now let $\mathcal{C} = \mathbb{P}^1$, and we consider a rational function field $K = \mathbb{F}_q(T)$. Let $A = \mathbb{F}_q[T]$. If we restrict ourselves to the higher dimensional Dedekind sum, then we obtain the following good result.

Theorem 4.2 *The following conditions are equivalent:*
(i) For all d, $s_\Lambda \left(\begin{array}{c|ccc} a_0 & a_1, & \ldots, & a_d \\ 0 & 0, & \ldots, & 0 \end{array} \right)$ are rational.
(ii) ϕ is defined over K.

4.2. Outline of proofs of Proposition 4.1 and Theorem 4.2. For Proposition 4.1 and (i)$\Rightarrow$(ii) of Theorem 4.2, we apply Galois theory to

$$s_\Lambda \left(\begin{array}{c|ccc} a_0 & a_1, & \ldots, & a_{q^i-1} \\ m_0 & m_1, & \ldots, & m_{q^i-1} \end{array} \right) = \frac{1}{a_0^{m_0+1}} \sum_{x \in \phi[a_0] \setminus \{0\}} \frac{1}{\phi_{a_0}(x)^{m_0+1} \cdots \phi_{a_d}(x)^{m_d+1}}.$$

It is invariant under the action of all elements of $\mathrm{Gal}(K(\phi[a_0])/K)$. The claim follows from it.

For (ii)⇒(i) of Theorem 4.2, from the assumption, for all $a \in A \setminus \{0\}$ and $j < i$,

$$E_{q^i-q^j}(\phi[a]) = \sum_{x\in\phi[a]\setminus\{0\}} \frac{1}{x^{q^i-q^j}} \in K.$$

We recall the Newton formula for the power sums of the zeros of a given polynomial.

Lemma 4.3 (Newton formula cf. [5, 6]) *Let*

$$f(X) = X^n + c_1X^{n-1} + \cdots + c_{n-1}X + c_n$$

be a polynomial over a field F, and $\alpha_1, \ldots, \alpha_n$ be the roots of $f(X)$. For each non-negative integer k, put $T_k = \alpha_1^k + \cdots + \alpha_n^k$. Then it holds that

$$T_k + c_1T_{k-1} + \cdots + c_{k-1}T_1 + kc_k = 0 \quad (k \le n).$$

We now return to the proof of the theorem. If $\phi_T(z) = l_0(T)z + l_1(T)z^q + \cdots + l_r(T)z^{q^r}$ ($l_0(T) = T$), then we have

$$\frac{1}{T}\phi_T(z^{-1})z^{q^r} = z^{q^r-1} + \frac{l_1(T)}{T}z^{q^r-q} + \cdots + \frac{l_{r-1}(T)}{T}z^{q^r-q^{r-1}} + \frac{l_r(T)}{T}.$$

Since $\{1/x \mid x \in \phi[T] \setminus \{0\}\}$ is the set of roots of this polynomial, using Newton formula, it follows that

$$E_{q^i-1}(\phi[T]) + \frac{l_1(T)}{T}E_{q^i-q}(\phi[T]) + \cdots + \frac{l_{i-1}(T)}{T}E_{q^i-q^{i-1}}(\phi[T]) = \frac{l_i(T)}{T}$$

for $i = 1, \ldots, r$. Using this identity repeatedly, we deduce $l_1(T), \ldots, l_r(T) \in K$. Since ϕ is determined by ϕ_T, we conclude that ϕ is defined over K.

5 Another type of Dedekind sums

In Section 3, we introduced multiple Dedekind-Rademacher sums for a given A-lattice. In this section, we introduce multiple Dedekind-Rademacher sums for an $\mathbb{F}_q$-vector space of finite dimension.

5.1. Let V be an $\mathbb{F}_q$-vector space of finite dimension, and $e_V(z) = z\prod_{0\neq v\in V}\left(1 - \frac{z}{v}\right)$ be its exponential function. The field $\mathbb{F}_q(V)$ stands for the field generated by V over $\mathbb{F}_q$. We take $a_0, \ldots, a_d \in C_\infty$ such that $a_i/a_0 \notin \mathbb{F}_q(V)$ if $i \neq 0$. Let $m_0, \ldots, m_d$ be non-negative integers. Then the *multiple Dedekind-Rademacher sum* for V is defined as follows.

Definition 5.1 *We define*

$$s_V\left(\begin{array}{c|ccc} a_0 & a_1, & \dots, & a_d \\ m_0 & m_1, & \dots, & m_d \end{array}\right) = \frac{1}{a_0^{m_0+1}} \sum_{0\neq v\in V} e_V\left(\frac{va_1}{a_0}\right)^{-m_1-1} \cdots e_V\left(\frac{va_d}{a_0}\right)^{-m_d-1}.$$

This kind of Dedekind sums also have the reciprocity law.

Theorem 5.2 *If $a_i/a_j \notin \mathbb{F}_q(V)$ for $i \neq j$, then*

$$\sum_{i=0}^{d} \sum_{\substack{l_0,\dots,\widehat{l_i},\dots,l_d\geq 0 \\ l_0+\cdots+\widehat{l_i}+\cdots+l_d=m_i}} \left(\prod_{j\neq i}\binom{m_j+l_j}{m_j}(-a_j)^{l_j}\right)$$
$$\times s_V\left(\begin{array}{c|ccccc} a_i & a_0, & \dots, & \widehat{a_i}, & \dots, & a_d \\ m_i & m_0+l_0, & \dots, & \widehat{m_i+l_i}, & \dots, & m_d+l_d \end{array}\right)$$
$$= \frac{(-1)^{m_0+\cdots+m_d+d}}{a_0^{m_0+1}\cdots a_d^{m_d+1}} \sum_{\substack{j_0,\dots,j_d\geq 0 \\ j_0+\cdots+j_d \\ =m_0+\cdots+m_d+d}} A_{0,j_0}A_{1,j_1}\cdots A_{d,j_d},$$

where

$$A_{i,j_i} = \begin{cases} (-1)^{m_i+1} & (\text{if } j_i = 0) \\ \binom{j_i-1}{m_i} a_i^{j_i} E_{j_i}(V) & (\text{if } j_i \geq m_i+1) \\ 0 & (\text{otherwise}) \end{cases}, \quad E_{j_i}(V) = \sum_{0\neq v\in V} \frac{1}{v^{j_i}}.$$

We can prove it in the same way as the proof of Theorem 3.2.

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