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On Automorphism Groups of Petri Nets Based on Place Connectivity

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1 Introduction

A Petri net is a mathematical model which is applied to descriptions of parallel processing systems. So far, some types of morphisms related to Petri nets (or condition/event net) have been studied in terms of the category theory, in order to simplify the behavior of more complicated Petri nets and understand the concurrency in other computation models [4][10].

Studying how the structure of Petri nets have an effect on Petri net languages and codes, we often realize that the ratio between the number of tokens in a place and the weights of edges connected to the place is important. So we give our definition of morphims between Petri nets focusing on the connection state/level of edges which come in or go out a place. This is an extension of an automorphism which we used to introduce to a net in [5][6].

In the second section we introduce morphims between two Petri nets. The set of all morphisms of a Petri net forms a monoid expressed by a semi-direct product. Especially, the set of all automorphisms of a Petri net forms a group. We investigate the inclusion relations among such monoids and groups. The third section deals with a pre-order induced by a surjective morphism. Two diamond properties are proved. It is a common case that one gives some redundancy or multiple provisions to systems to improve their reliability and safety. Surjective morphism will be effective to analyze such redundant systems. In the last section we show the properties of languages generated by two Petri nets ordered by a surjective morphism. The languages generated by them and their reachability sets have close correspondence.

2 Preliminaries

Here we give our definition of morphisms of a Petri net and state the properties of some monoids composed of these morphisms.

2.1 Petri Nets and Morphisms

In this section, we give definitions and fundamental properties related to Petri nets. We denote the set of all nonnegative integers by $\boldsymbol{N}_0$, that is, $\boldsymbol{N}_0 = \{0, 1, 2, \dots\}$.

First of all, a Petri net is viewed as a particular kind of directed graph, together with an initial state μ_0, called the *initial marking*. The underlying graph N of a Petri net is a directed, weighted, bipartite graph consisting of two kinds of nodes, called *places* and *transitions*, where arcs are either from a place to a transition or from a transition to a place.

DEFINITION 2.1 (Petri net) A *Petri net* is a 4-tuple (P, T, W, μ_0) where
(1) $P = \{p_1, p_2, \dots, p_m\}$ is a finite set of places,
(2) $T = \{t_1, t_2, \dots, t_n\}$ is a finite set of transitions,
(3) $W : \boldsymbol{E}(P,T) \to \{0, 1, 2, 3, \dots\}$, i.e.,$W \in \boldsymbol{N}_0{}^{\boldsymbol{E}(P,T)}$, is a *weight function*, where $\boldsymbol{E}(P,T) = (P \times T) \cup (T \times P)$,
(4) $\mu_0 : P \to \{0, 1, 2, 3, \dots\}$, i.e., $\mu_0 \in \boldsymbol{N}_0{}^P$, is the initial marking,
(5) $P \cap T = \emptyset$ and $P \cup T \neq \emptyset$.

A Petri net structure (net, for short) $N = (P, T, W)$ without any specific initial marking is denoted by N, a Petri net with a given initial marking μ_0 is denoted by (N, μ_0). □

In the graphical representation, the places are drawn as circles and the transitions are drawn as bars or boxes. Arcs are labeled with their weights(positive integers), where a k-weighted arc can be interpreted as the set of k parallel arcs. Labels for unity weights are usually omitted. A marking (state) assigns a nonnegative integer k to each place. If a marking assigns a nonnegative integer k to a place p, we say that p is *marked with* k tokens. Pictorially, we put k black dots (tokens) in place p. A marking is denoted by μ, an n-dimensional row vector, where n is the total number of places. The i-th component of μ, denoted by $\mu(p_i)$, is the number of tokens in the i-th place p_i.

EXAMPLE 2.1 Fig. 1 shows a graphical representation of a Petri net $\mathcal{P} = (P, T, W, \mu_0)$. $P = \{\mathbf{a}, \mathbf{b}\}$ and $T = \{\mathbf{t}\}$. $(\mathbf{a}, \mathbf{t})$ and $(\mathbf{t}, \mathbf{b})$ are arcs of weights 2 and 1 respectively. $(\mathbf{t}, \mathbf{a})$ and $(\mathbf{b}, \mathbf{t})$ are arcs of weight 0, which are not usually drawn in the picture. Note that the weight of $(\mathbf{t}, \mathbf{b})$ is omitted since it is unity. That is, $W(\mathbf{a}, \mathbf{t}) = 2, W(\mathbf{b}, \mathbf{t}) = 1, W(\mathbf{t}, \mathbf{a}) = W(\mathbf{b}, \mathbf{t}) = 0$. The initial marking μ_0 with $\mu_0(\mathbf{a}) = 3, \mu_0(\mathbf{b}) = 0$ is often written like a row vector $\mu_0 = (3, 0)$. □

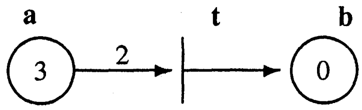

Figure 1. Graphical representation of a Petri net

Now we introduce a Petri net morphism based on place connectivity. We denote the set of all positive rational numbers by $\boldsymbol{Q}_+$.

DEFINITION 2.2 Let $\mathcal{P}_1 = (P_1, T_1, W_1, \mu_1)$ and $\mathcal{P}_2 = (P_2, T_2, W_2, \mu_2)$ be Petri nets. Then a triple $(f, (\alpha, \beta))$ of maps is called a *morphism* from $\mathcal{P}_1$ to $\mathcal{P}_2$ if the maps $f : P_1 \longrightarrow \boldsymbol{Q}_+, \alpha : P_1 \longrightarrow P_2$ and $\beta : T_1 \longrightarrow T_2$ satisfy the condition that for any $p \in P_1$ and $t \in T_1$,

$$\begin{aligned} &W_2(\alpha(p), \beta(t)) = f(p)W_1(p, t), \\ &W_2(\beta(t), \alpha(p)) = f(p)W_1(t, p), \\ &\mu_2(\alpha(p)) = f(p)\mu_1(p). \end{aligned} \tag{2.1}$$

In this case we write $(f, (\alpha, \beta)) : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$. □

The morphism $(f, (\alpha, \beta)) : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ is called *injective* (resp. *surjective*) if both α and β are injective (resp. surjective). In particular, it is called an *isomorphism* from $\mathcal{P}_1$ to $\mathcal{P}_2$ if it is injective and surjective. Then $\mathcal{P}_1$ is said to be *isomorphic* to $\mathcal{P}_2$ and we write $\mathcal{P}_1 \simeq \mathcal{P}_2$. Moreover, in case of $\mathcal{P}_1 = \mathcal{P}_2$, an isomorphism is called an *automorphism* of $\mathcal{P}_1$. By $\mathbf{Aut}(\mathcal{P})$ we denote the set of all the automorphisms of $\mathcal{P}$.

For Petri nets $\mathcal{P}_1$ and $\mathcal{P}_2$, we write $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2$ if there exists a surjective morphism from $\mathcal{P}_1$ to $\mathcal{P}_2$. The relation $\sqsupseteq$ forms a pre-order (a relation satisfying the reflexive law and the transitive law) as shown below. Of course, the pre-order is regarded as an order by identifying isomorphisms.

PROPOSITION 2.1 *Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ be Petri nets. Then,*
(1) $\mathcal{P}_1 \sqsupseteq \mathcal{P}_1$.
(2) $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2$ *and* $\mathcal{P}_2 \sqsupseteq \mathcal{P}_1 \iff \mathcal{P}_1 \simeq \mathcal{P}_2$.
(3) $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2$ *and* $\mathcal{P}_2 \sqsupseteq \mathcal{P}_3$ *imply* $\mathcal{P}_1 \sqsupseteq \mathcal{P}_3$.

Proof) Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)\ (i = 1, 2, 3)$ through the proof. The proof complete in the order (1), (3), (2).
(1) Trivial.
(3) There exist surjective morphisms $(f_i, (\alpha_i, \beta_i)) : \mathcal{P}_i \longrightarrow \mathcal{P}_{i+1}\ (i = 1, 2)$. We define a map $f : P_1 \longrightarrow \boldsymbol{Q}_+$ by $f(p) = f_1(p) \cdot f_2(\alpha(p))$. Then $(f, (\alpha_1\alpha_2, \beta_1\beta_2))$ is a surjective morphism from $\mathcal{P}_1$ to $\mathcal{P}_2$.

(2) ($\Rightarrow$) There exist surjective morphisms $(f,\ (\alpha,\beta)) : \mathcal{P}_1 \to \mathcal{P}_2$ and $(g,\ (\alpha',\beta')) : \mathcal{P}_2 \to \mathcal{P}_1$. Since $\alpha\alpha'$ is surjective by (3) above and P_1 is finite, both α and α' are bijections. β and β' are also. Therefore $\mathcal{P}_1 \simeq \mathcal{P}_3$.

($\Leftarrow$) If $(f,\ (\alpha,\beta))$ be a isomorphism from $\mathcal{P}_1$ to $\mathcal{P}_2$, then it is easily shown that $(\alpha^{-1}f^{-1},\ (\alpha^{-1},\beta^{-1}))$ is a isomorphism from $\mathcal{P}_2$ to $\mathcal{P}_1$, where $f^{-1} : P_2 \to \boldsymbol{Q}_+, p \mapsto 1/f(p)$. □

DEFINITION 2.3 (Similar) *Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net. Two places $p, q \in P$ are said to be similar if there exists some positive rational number r such that $\mu(p) = r\mu(q)$, $W(q,t) = rW(p,t)$ and $W(t,q) = rW(t,p)$ for all $t \in T$. Two transitions $s, t \in T$ are said to be similar if $W(p,s) = W(p,t)$ and $W(s,p) = W(t,p)$ for all $p \in P$.* □

The similarity defined above is obviously an equivalence relation on $P \cup T$. We denote this relation by $\sim_{\mathcal{P}}$ or simply $\sim$ and the $\sim_{\mathcal{P}}$-class of a place or a transition u by $C(u)$. A place (resp. a transition) is said to be *isolated* if it has no connection to any transitions (resp. any places). Especially, a place p is 0-isolated if it is isolated and $\mu(p) = 0$. Note that two 0-isolated places p and q are similar because for any positive rational number r $\mu(p) = 0 = r\mu(q)$, $W(q,t) = 0 = rW(p,t)$ and $W(t,q) = 0 = rW(t,p)$ for all $t \in T$.

2.2 Monoids $\mathcal{S}^1$ of Surjective Morphisms of Petri Nets

We introduce a composition of morphisms; all the morphisms between Petri nets form a monoid under this composition.

Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(i = 1,2,3)$ be Petri nets, $(f,\ (\alpha,\beta)) : \mathcal{P}_1 \to \mathcal{P}_2$ and $(g,\ (\gamma,\delta)) : \mathcal{P}_2 \to \mathcal{P}_3$ be morphisms. Then,

$$\begin{aligned}
W_3(\gamma(\alpha(p)), \delta(\beta(t))) &= g(\alpha(p))W_2(\alpha(p), \beta(t)) \\
&= g(\alpha(p))f(p)W_1(p,t), \\
W_3(\delta(\beta(t)), \gamma(\alpha(p))) &= g(\alpha(p))W_2(\beta(t), \alpha(p)) \\
&= g(\alpha(p))f(p)W_1(t,p), \\
\mu_3(\gamma(\alpha(p))) &= g(\alpha(p))\mu_2(\alpha(p)) = g(\alpha(p))f(p)\mu_1(p)
\end{aligned}$$

hold.

In this manuscript, by writing compositions of maps like $g\circ\alpha$, $\gamma\circ\alpha$ and $\delta\circ\beta$ in the form of multiplications like αg, $\alpha\gamma$ and $\beta\delta$ respectively, the *composition* of morphisms is written as $(f \otimes_{P_1} (\alpha g), (\alpha\gamma, \beta\delta))$, where $\otimes_{P_1}$ is the operation in the following fundamental commutative group $({\boldsymbol{Q}_+}^P, \otimes_P)$.

The set $({\boldsymbol{Q}_+}^P, \otimes_P)$ of all maps from a set P to $\boldsymbol{Q}_+$ forms a commutative group under the operation $\otimes_P$ defined by $f \otimes_P g : p \mapsto f(p)g(p)$. $\mathbf{1}_{\otimes_P} : P \to \boldsymbol{Q}_+ : p \mapsto 1$ is the identity and $f^{-1} : P \to \boldsymbol{Q}_+ : p \mapsto 1/f(p)$ is the inverse of a $f \in {\boldsymbol{Q}_+}^P$. Whenever it does not cause confusion, we write $\otimes$ instead of $\otimes_P$. Immediately we obtain the following lemma.

LEMMA 2.1 Let α and β be arbitrary maps on P and $f, g : P \to \boldsymbol{Q}_+$. Then the following equations are true.

(1) $(\alpha\beta)f = \alpha(\beta f)$.
(2) $\alpha(f \otimes g) = (\alpha f) \otimes (\alpha g)$.
(3) $\alpha\mathbf{1}_\otimes = \mathbf{1}_\otimes$.
(4) $(\alpha f) \otimes (\alpha f^{-1}) = \mathbf{1}_\otimes$.
(5) $(\alpha f)^{-1} = \alpha f^{-1}$.

Proof) For each $p \in P$, the following equations hold.

(1) $((\alpha\beta)f)(p) = f(\beta(\alpha(p))) = (\beta f)(\alpha(p)) = (\alpha(\beta f))(p)$.
(2) $(\alpha(f \otimes g))(p) = f(\alpha(p)) \cdot g(\alpha(p)) = (\alpha f)(p) \cdot (\alpha g)(p) = ((\alpha f) \otimes (\alpha g))(p)$.
(3) $(\alpha\mathbf{1}_\otimes)(p) = \mathbf{1}_\otimes(\alpha(p)) = \mathbf{1}_\otimes(p)$.
(4) By (2) and (3) above, $(\alpha f) \otimes (\alpha f^{-1}) = \alpha(f \otimes f^{-1}) = \alpha\mathbf{1}_\otimes = \mathbf{1}_\otimes$.
(5) $(\alpha f)^{-1}(p) = 1/f(\alpha(p)) = f^{-1}(\alpha(p)) = (\alpha f^{-1})(p)$. □

For a surjective morphim $x : \mathcal{P}_1 \to \mathcal{P}_2$, $\mathcal{P}_1$ is called the domain of x, denoted by $Dom(x)$, and $\mathcal{P}_2$ is called the image(or range) of x, denoted by $Im(x)$. Especially $Dom(0) = Im(0) = \emptyset$.

We denote the set of all surjective morphisms between two Petri nets and a zero element 0, by $\mathcal{S}$. $\mathcal{S}$ forms a semigroup, equipped with the multiplication of $x = (f, (\alpha, \beta))$ and $y = (g, (\gamma, \delta))$:

$$x \cdot y \stackrel{\text{def}}{=} \begin{cases} (f \otimes_{\mathcal{P}} \alpha g, (\alpha\gamma, \beta\delta)) & \text{if } \; Im(x) = Dom(y). \\ 0 & \text{otherwise.} \end{cases}$$

$\mathcal{S}^1 = \mathcal{S} \cup \{1\}$ is the monoid obtained from $\mathcal{S}$ by adjoining an (extra) identity 1, that is, $1 \cdot s = s \cdot 1 = s$ for all $s \in \mathcal{S}$ and $1 \cdot 1 = 1$.

EXAMPLE 2.2 Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(1 \le i \le 3)$ be Petri nets shown in Figure 2. The four morphisms $x_i = (f_i, (\alpha_i, \beta_i))$ $(0 \le i \le 3)$ are from $\mathcal{P}_1$ to $\mathcal{P}_2$, where

$$f_0 = \begin{pmatrix} p_1 & p_2 \\ 1/2 & 1 \end{pmatrix}, \quad \alpha_0 = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix},$$
$$f_1 = \begin{pmatrix} p_1 & p_2 \\ 3/2 & 1/3 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} p_1 & p_2 \\ q_2 & q_1 \end{pmatrix},$$
$$f_2 = \begin{pmatrix} p_1 & p_2 \\ 1/2 & 1/3 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_1 \end{pmatrix},$$
$$f_3 = \begin{pmatrix} p_1 & p_2 \\ 3/2 & 1 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} p_1 & p_2 \\ q_2 & q_2 \end{pmatrix},$$

and $\beta_0 = \beta_1 = \beta_2 = \beta_3 : T_1 \longrightarrow T_2, t_1 \mapsto s,\ t_2 \mapsto s$. Especially only x_0 and x_1 are surjective morphisms.
Only one morphism $y = (g, (\gamma, \delta))$ exists from $\mathcal{P}_2$ to $\mathcal{P}_3$, where

$$\begin{aligned} &g : P_2 \longrightarrow \boldsymbol{Q}_+, q_1 \mapsto 1,\ q_2 \mapsto 1/3, \\ &\gamma : P_2 \longrightarrow P_3, q_1 \mapsto r, q_2 \mapsto r, \\ &\delta : T_2 \longrightarrow T_3, s \mapsto u. \end{aligned}$$

This is a surjective morphism. The compositions of morphisms x_i $(0 \le i \le 3)$ and y are the same surjective morphism $(h, (\sigma, \tau))$ from $\mathcal{P}_1$ to $\mathcal{P}_3$, where

$$\begin{aligned} &h : P_1 \longrightarrow \boldsymbol{Q}_+, p_1 \mapsto 1/2,\ p_2 \mapsto 1/3, \\ &\sigma = \alpha_i\gamma : P_1 \longrightarrow P_3, p_1 \mapsto r, p_2 \mapsto r, \\ &\tau = \beta_i\delta : T_1 \longrightarrow T_3, t_1 \mapsto u, t_2 \mapsto u. \end{aligned}$$

for any $i = 1, 2, 3, 4$. Note that h is expressed as $h = f_i \otimes (\alpha_i g)$. □

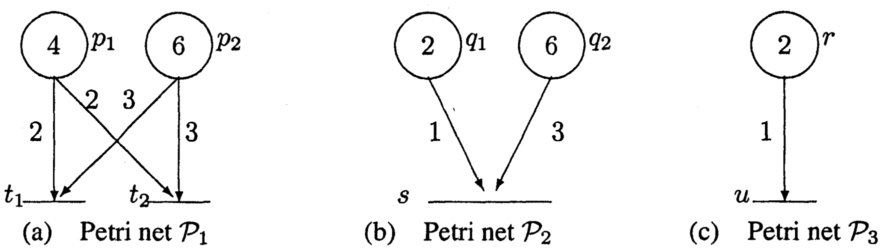

Figure 2. Petri nets $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ with $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2 \sqsupseteq \mathcal{P}_3$.

3 Ideals in the monoid $\mathcal{S}^1$

In this section we consider ideals and Green's relations on the monoid $\mathcal{S}^1$.
At first, we consider some properties of the structure of the automorphism group of a Petri net $\mathcal{P}$.

3.1 Green's equivalences on the monoid $\mathcal{S}^1$

In general, Green's equivalences $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}, \mathcal{D}$ on a monoid M, which are well-known and important equivalence relations in the development of semigroup theory, are defined as follows:

$$\begin{aligned}
&x\mathcal{L}y \iff Mx = My,\\
&x\mathcal{R}y \iff xM = yM,\\
&x\mathcal{J}y \iff MxM = MyM,\\
&\mathcal{H} = \mathcal{L} \cap \mathcal{R},\\
&\mathcal{D} = (\mathcal{L} \cup \mathcal{R})^*,
\end{aligned}$$

where $(\mathcal{L} \cup \mathcal{R})^*$ means the reflexive and transitive closure of $\mathcal{L} \cup \mathcal{R}$. Mx (resp. xM) is called the *principal left* (resp. *right*) *ideal generated by* x and MxM the it principal (two-sided) ideal generated by x. Then, the following facts are generally true[2, 1].

FACT 1 *The following relations are true.*

$$\begin{aligned}
&(1)\, \mathcal{D} = \mathcal{L}\mathcal{R} = \mathcal{R}\mathcal{L}\\
&(2)\, \mathcal{H} \subset \mathcal{L}\ (\mathit{resp.}\mathcal{R}) \subset \mathcal{D} \subset \mathcal{J}
\end{aligned}$$

FACT 2 *An $\mathcal{H}$-class of a monoid M is a group if and only if it contains an idempotent.*

Now we consider the case of $M = \mathcal{S}^1$ in the rest of the maniscript. The following lemma is obviously true.

LEMMA 3.1 *Let $x : \mathcal{P}_1 \to \mathcal{P}_2$, $y : \mathcal{P}_3 \to \mathcal{P}_4 \in \mathcal{S}^1$. Then,*
(1) $x\mathcal{S}^1 \subset y\mathcal{S}^1 \Longrightarrow \mathcal{P}_1 = \mathcal{P}_3$ *and* $\mathcal{P}_2 \sqsubseteq \mathcal{P}_4$.
(2) $\mathcal{S}^1 x \subset \mathcal{S}^1 y \Longrightarrow \mathcal{P}_1 \sqsubseteq \mathcal{P}_3$ *and* $\mathcal{P}_2 = \mathcal{P}_4$.
(3) $x\mathcal{S}^1 = y\mathcal{S}^1 \Longrightarrow \mathcal{P}_1 = \mathcal{P}_3$ *and* $\mathcal{P}_2 \simeq \mathcal{P}_4$.
(4) $\mathcal{S}^1 x = \mathcal{S}^1 y \Longrightarrow \mathcal{P}_1 \simeq \mathcal{P}_3$ *and* $\mathcal{P}_2 = \mathcal{P}_4$. □

Note that any reverses of the implications above are not necessarily true.

PROPOSITION 3.1 The following conditions are equivalent.
(1) H is an $\mathcal{H}$-class and a group.
(2) $H = \mathbf{Aut}(\mathcal{P})$ for some Petri net $\mathcal{P}$.

Proof) (1)$\Longrightarrow$(2) By FACT2, H contains an idempotent e, that is $e^2 = e$. This implies $Dom(e) = Im(e) = \mathcal{P}$ for some Petri net $\mathcal{P}$. By (3) and (4) of LEMMA 3.1, $Dom(x) = Dom(e) = \mathcal{P}$ and $Im(x) = Im(e) = \mathcal{P}$ for any $x \in H$ because $x\mathcal{S}^1 = e\mathcal{S}^1$ and $\mathcal{S}^1 x = \mathcal{S}^1 e$ hold. Therefore each element of H is an automorphism of $\mathcal{P}$. Conversely, for an automorphism x of $\mathcal{P}$, $x \in H$ because x is a surjective morphism with $Dom(x) = Im(x) = \mathcal{P}$. Hence we have $H = \mathbf{Aut}(\mathcal{P})$.
(2)$\Longrightarrow$(1) For $x, y \in H = \mathbf{Aut}(\mathcal{P})$, there exists $z, w \in H$ such that $x = zy$ and $wx = y$. This implies $\mathcal{S}^1 x = \mathcal{S}^1 y$. Similarily we have $x\mathcal{S}^1 = y\mathcal{S}^1$. Therefore $x\mathcal{H}y$. Conversely, $x\mathcal{H}y$ and $x \in H$ implies $y \in H$ because y is a surjective morphism with $Dom(y) = Im(y) = \mathcal{P}$. Hence H is an $\mathcal{H}$-class and a group. □

PROPOSITION 3.2 *On the monoid $\mathcal{S}^1$, $\mathcal{J} = \mathcal{D}$.*

Proof) Since $\mathcal{D} \subset \mathcal{J}$ holds, it is enough to show the reverse inclusion.

$$\begin{aligned}
x\mathcal{J}y &\iff \mathcal{S}^1 x \mathcal{S}^1 = \mathcal{S}^1 y \mathcal{S}^1\\
&\iff \exists u, v, z, w \in \mathcal{S}^1\ (x = uyv, y = zxw)
\end{aligned}$$

implies that $x = uzxwv, y = zuyvw$. Setting
$P = Dom(x)$,$Q = Dom(y)$,$R = Im(x)$ and $S = Im(y)$, $uz : P \to P$, $zu : Q \to Q$, $wv : R \to R$, $vw : S \to S$ are automorphisms. This implies that u, v, z, w are isomorphisms and $u^{-1} = z, v^{-1} = w$. Let $t = xw$. Then,

$$\begin{aligned}
&x = x(ww^{-1}) = (xw)w^{-1} = tw^{-1}\\
&y = z(xw) = zt\\
&t = (z^{-1}z)t = z^{-1}(zt) = z^{-1}y
\end{aligned}$$

Therefore $x\mathcal{S}^1 = t\mathcal{S}^1$ and $\mathcal{S}^1 t = \mathcal{S}^1 y$, that is, $x\mathcal{R}t\mathcal{L}y$. Thus $\mathcal{D} \subset \mathcal{J}$. □

3.2 Intersection of principal ideals

The aim here is that for given $x, y \in \mathcal{S}^1$ we find a elements z such that $\mathcal{S}^1x \cap \mathcal{S}^1y = \mathcal{S}^1z$ (resp. $x\mathcal{S}^1 \cap y\mathcal{S}^1 = z\mathcal{S}^1$). $x\mathcal{S}^1 \cap y\mathcal{S}^1 = \{0\}$ (resp. $\mathcal{S}^1x \cap \mathcal{S}^1y = \{0\}$) is a trivial case($z = 0$). We should only consider the non-trivial case.

LEMMA 3.2 *Let $\mathcal{P}_i = (P_i, T_i.W_i, \mu_i)(i = 1,2,3)$ be Petri nets, $x = (f, (\alpha, \beta)) : \mathcal{P}_1 \to \mathcal{P}_3$, $y = (g, (\gamma, \delta)) : \mathcal{P}_2 \to \mathcal{P}_3$ be elements of $\mathcal{S}^1$. If $|\alpha^{-1}(p)| \le |\gamma^{-1}(p)|$ and $|\beta^{-1}(t)| \le |\delta^{-1}(t)|$ for any $p \in P_3$ and $t \in T_3$, then $\mathcal{S}^1y \subset \mathcal{S}^1x$.*

Proof) By the assumption, we can choose arbitrary surjective morphisms $\xi : P_2 \to P_1$ and $\eta : T_2 \to T_1$ such that $\xi(\gamma^{-1}(p)) = \alpha^{-1}(p)$ for any $p \in P_3$ and $\eta(\delta^{-1}(t)) = \beta^{-1}(t)$ for any $t \in T_3$. $h : P_2 \to \mathbf{Q}_+$ is defined by $h(q) = g^{-1}(q)f(\xi(q))$ for each $q \in P_2$. Then, we can verify that $z = (h, (\xi, \eta))$ is a surjective morphism from $\mathcal{P}_2$ to $\mathcal{P}_1$ and thus $z \in \mathcal{S}^1$, $y = zx$. Therefore $\mathcal{S}^1y \subset \mathcal{S}^1x$. □

LEMMA 3.3 *Let $\mathcal{P}_i = (P_i, T_i.W_i, \mu_i)(i = 0,1,2)$ be Petri nets, $x = (f, (\alpha, \beta)) : \mathcal{P}_0 \to \mathcal{P}_1$, $y = (g, (\gamma, \delta)) : \mathcal{P}_0 \to \mathcal{P}_2$ be elements of $\mathcal{S}^1$. If for any $p \in P_1$ and $t \in T_1$, there exist $q \in P_2$ and $s \in T_2$ such that $\alpha^{-1}(p) \subset \gamma^{-1}(q)$ and $\beta^{-1}(t) \subset \delta^{-1}(s)$, then $y\mathcal{S}^1 \subset x\mathcal{S}^1$.*

Proof) Let p and t be arbitrary elements of P_1 and T_1, respectively. By the assumption, $q \in P_2$ and $s \in T_2$ is uniquely defined and

$$\begin{aligned}\alpha^{-1}(p) &= \{p_1, p_2, \ldots, p_n\} \subset \gamma^{-1}(q),\\ \beta^{-1}(p) &= \{t_1, t_2, \ldots, t_m\} \subset \delta^{-1}(s).\end{aligned}$$

Then we can easily chech that $\mu_2(q) = g(p_i)f^{-1}(p_i)\mu_1(p)$, $W_2(q,s) = g(p_i)f^{-1}(p_i)W_1(p,t_j)$ and $W_2(s,q) = g(p_i)f^{-1}(p_i)\ W_1(t_j,p)$ for any $i\,(1 \le i \le n)$ and any $j\,(1 \le j \le m)$. Since the values of $g(p_i)f^{-1}(p_i)$ are the same rational number determined only depending on $p \in P_1$, the maps

$$\begin{aligned}&\xi : P_1 \to P_2,\ p \mapsto q, \text{ where } \alpha^{-1}(p) \subset \gamma^{-1}(q),\\ &\eta : T_1 \to T_2,\ t \mapsto s, \text{ where } \beta^{-1}(t) \subset \delta^{-1}(s) \text{ and}\\ &h : P_1 \to \mathbf{Q}_+,\ p \mapsto g(p_i)f^{-1}(p_i), \text{ where } \alpha(p_i) = p\end{aligned}$$

are well-defined. Therefore we have $z = (h, (\xi, \eta)) \in \mathcal{S}^1$ and thus $y = xz$, that is, $y\mathcal{S}^1 \subset x\mathcal{S}^1$. □

PROPOSITION 3.3 (Intersection of Principal Left Ideals) *Let $\mathcal{P}_i = (P_i, T_i.W_i, \mu_i)(i = 1,2,3)$ be Petri nets, $x = (f_1, (\alpha_1, \beta_1)) : \mathcal{P}_1 \to \mathcal{P}_3$, $y = (f_2, (\alpha_2, \beta_2)) : \mathcal{P}_2 \to \mathcal{P}_3$ be elements of $\mathcal{S}^1$, $P_3 = \{c_1, c_2, \ldots, c_N\}$ and $T_3 = \{d_1, d_2, \ldots, d_M\}$.*

$$\begin{aligned}n_i &= \max\{|\alpha_1^{-1}(c_i)|, |\alpha_2^{-1}(c_i)|\} \text{ for each } i = 1,2,\ldots,N,\\ m_j &= \max\{|\beta_1^{-1}(d_j)|, |\beta_2^{-1}(d_j)|\} \text{ for each } j = 1,2,\ldots,M.\end{aligned}$$

Taking disjoint sets $C_1, C_2, \ldots, C_N$ and $D_1, D_2, \ldots, D_M$ with their sizes $|C_i| = n_i (i = 1,2,\ldots,N)$ and $|D_j| = m_j (j = 1,2,\ldots,M)$, we define a Petri net $\mathcal{P} = (P, T.W, \mu)$, where $P = \bigcup_{1 \le i \le N} C_i, T = \bigcup_{1 \le j \le M} D_j$, and for any $p \in P$ and $t \in T$,

$$\begin{aligned}W(p,t) &= W_3(c_i, d_j) && \text{if } (p,t) \in C_i \times D_j,\\ W(t,p) &= W_3(d_j, c_i) && \text{if } (t,p) \in D_j \times C_i,\\ \mu(p) &= \mu_3(c_i) && \text{if } p \in C_i,\end{aligned}$$

Then, $z = (1_{\otimes_P}, (\gamma, \delta)) : \mathcal{P} \to \mathcal{P}_3$, where $\gamma : C_i \ni p \mapsto c_i$ and $\delta : D_j \ni t \mapsto d_j$ are surjective morphisms. Moreover, $\mathcal{S}^1x \cap \mathcal{S}^1y = \mathcal{S}^1z$.

Proof) Let We can easily check that $z = ux = vy$ for some $u, v \in \mathcal{S}^1$. Therefore $z \in \mathcal{S}^1x \cap \mathcal{S}^1y$.

Conversely we show that $w = (h, (\xi, \eta)) \in \mathcal{S}^1x \cap \mathcal{S}^1y$ implies $w \in \mathcal{S}^1z$. We can write $w = u'x = v'y$ for some $u', v' \in \mathcal{S}^1$. Let $p \in P_3$. In our construction, $|\gamma^{-1}(p)| = \max\{|\alpha_1^{-1}(p)|, |\alpha_2^{-1}(p)|\}$. Since $w = u'x = v'y$ holds, we have $|\alpha_1^{-1}(p)| \le |\xi^{-1}(p)|$ and $|\alpha_2^{-1}(p)| \le |\xi^{-1}(p)|$ and thus $|\gamma^{-1}(p)| \le |\xi^{-1}(p)|$. Similarily, $|\delta^{-1}(p)| \le |\eta^{-1}(p)|$. By LEMMA 3.2, we conclude $\mathcal{S}^1x \cap \mathcal{S}^1y = \mathcal{S}^1z$. □

COROLLARY 3.1 (Diamond Property I) *Let* $\mathcal{P}_i =$ (P_i, T_i, W_i, μ_i) $(i = 1, 2, 3)$ *be Petri nets with* $\mathcal{P}_1 \sqsupseteq \mathcal{P}_3$ *and* $\mathcal{P}_2 \sqsupseteq \mathcal{P}_3$. *Then there exists a Petri net* $\mathcal{P}_0$ *such that* $\mathcal{P}_0 \sqsupseteq \mathcal{P}_1$ *and* $\mathcal{P}_0 \sqsupseteq \mathcal{P}_2$.

We consider the intersection of two pricipal right ideals. The case of principal right ideals is rather difficult compared to that of principal left ideals. We begin with an introduction of the relation $=_f$.

Let P be a set and f, g maps whose domain is P. The relation $=_f$ on P defined by $(\forall x, y \in P)\{x =_f y \overset{\text{def}}{\Longleftrightarrow} f(x) = f(y)\}$. Then $(=_f \cup =_g)^*$ is the smallest equivalence relation on P which includes both $=_f$ and $=_g$, where $(=_f \cup =_g)^*$ is the reflexive and transitive closure of $=_f \cup =_g$.

PROPOSITION 3.4 (Intersection of Principal Right Ideals) *Let* $\mathcal{P}_i = (P_i, T_i.W_i, \mu_i)(i = 0, 1, 2)$ *be Petri nets,* $x = (f_1, (\alpha_1, \beta_1)) : \mathcal{P}_1 \to \mathcal{P}_3$, $y = (f_2, (\alpha_2, \beta_2)) : \mathcal{P}_2 \to \mathcal{P}_3$ *be elements of* $\mathcal{S}^1$. *Let* $C_1, C_2, \ldots, C_N$ *be all the* $(=_{\alpha_1} \cup =_{\alpha_2})^*$*-classes in* P_0 *and* $D_1, D_2, \ldots, D_M$ *be all the* $(=_{\beta_1} \cup =_{\beta_2})^*$*-classes in* T_0.

$f \in {Q_+}^P$ *is defined by if* p *is 0-isolated then* $f(p) = 1$ *and otherwise*

$$f(p) = 1/\gcd(\{\mu(p), W_0(p, t_i), W_0(t_i, p) \mid 1 \le i \le n\})$$

where $n = |T_0|$ *and* $T_0 = \{t_1, t_2, \ldots, t_n\}$ *and* $\gcd(S)$ *denotes the greatest common divisor of all integers in a set* S.

(1) A Petri net $\mathcal{P}_3 = (P_3, T_3.W_3, \mu_3)$ *can be constructed in the following way:*

$$\begin{aligned} P_3 &= P_0/(=_{\alpha_1} \cup =_{\alpha_2})^* = \{C_1, C_2, \ldots, C_N\}, \\ T_3 &= T_0/(=_{\beta_1} \cup =_{\beta_2})^* = \{D_1, D_2, \ldots, D_M\}. \end{aligned}$$

For $i \in \{1, 2, \ldots, N\}, j \in \{1, 2, \ldots, M\}$,

$$\begin{aligned} \mu_3(C_i) &= f(p)\mu_0(p) && \text{for any } p \in C_i, \\ W_3(C_i, D_j) &= f(p)W_0(p, t) && \text{for any } p \in C_i, t \in D_j, \\ W_3(D_j, C_i) &= f(p)W_0(t, q) && \text{for any } p \in C_i, t \in D_j \end{aligned}$$

are well-defined.

(2) Let $z = (f, (\alpha, \beta)) : \mathcal{P}_0 \to \mathcal{P}_3$, *where* α *is the canonical surjection from* P_0 *onto* P_3, β *is the canonical surjection from* T_0 *onto* T_3. *Then,* z *is a surjective morphism and* $x\mathcal{S}^1 \cap y\mathcal{S}^1 = z\mathcal{S}^1$.

Proof) Omitted.

The above-mentioned proposition immediately leads the following corollary.,

COROLLARY 3.2 (Diamond Property II) Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(i = 0, 1, 2)$ be Petri nets with $\mathcal{P}_0 \sqsupseteq \mathcal{P}_1$ and $\mathcal{P}_0 \sqsupseteq \mathcal{P}_2$. Then there exists a Petri net $\mathcal{P}_3$ such that $\mathcal{P}_1 \sqsupseteq \mathcal{P}_3$ and $\mathcal{P}_2 \sqsupseteq \mathcal{P}_3$.

We define the concept of irreducible forms of a Petri net with respect to $\sqsupseteq$.

DEFINITION 3.1 (Irreducible) *A Petri net* $\mathcal{P}$ *is called a* $\sqsupseteq$*-irreducible if* $\mathcal{P} \sqsupseteq \mathcal{P}'$ *implies* $\mathcal{P} \simeq \mathcal{P}'$ *for any Petri net* $\mathcal{P}'$. *Then* $\mathcal{P}$ *is called an* $\sqsupseteq$*-irreducible form.* □

COROLLARY 3.3 Let $\mathcal{P}, \mathcal{P}'$ and $\mathcal{P}''$ be Petri nets with $\mathcal{P} \sqsupseteq \mathcal{P}'$ and $\mathcal{P} \sqsupseteq \mathcal{P}''$. Then one has: If $\mathcal{P}'$ and $\mathcal{P}''$ are $\sqsupseteq$-irreducible, then $\mathcal{P}' \simeq \mathcal{P}''$.

Proof) Trivial by COROLLARY 3.2 and the definition of $\sqsupseteq$-irreducibility. □

4 Surjective Morphisms and Petri Net Languages

4.1 Behavior of Petri Nets

The behavior of many systems can be described in terms of system states and their changes. In order to simulate the dynamic behavior of a system, a state or marking in a Petri net $\mathcal{P} = (P, T, W, \mu)$ is changed according to the following transition (firing) rule:
(1) A transition $t \in T$ is said to be *enabled* (under the marking μ or under the Petri net $\mathcal{P}$) if $W(p,t) \leq \mu(p)$ for every place $p \in P$, where $W(p,t)$ is the weight of the arc from p to t. Then each input place p of t is marked with at least $W(p,t)$ tokens. An enabled transition may or may not fire (depending on whether or not the event actually takes place).
(2) A *firing* of an enabled transition t removes $W(p,t)$ tokens from each input place p of t, and adds $W(t,p)$ tokens to each output place p of t. As a consequence of the firing, the current marking μ is replaced with the following new marking μ':

$$\mu'(p) = \mu(p) - W(p,\, t) + W(t,\, p) \;\text{ for } \forall p \in P. \tag{4.1}$$

Then we define the transition function $\delta_{\mathcal{P}}$ by $\delta_{\mathcal{P}}(\mu,\, t) = \mu'$.
(3) A sequence $w = t_1 t_2 \ldots t_n$ of transitions is said to be a *firing sequence* in a Petri net $\mathcal{P} = (P, T, W, \mu)$ if $\mu_0 = \mu$, $\mu_n = \mu'$, and $\mu_i = \delta_{\mathcal{P}}(\mu_{i-1},\, t_i)$ for each $i\,(1 \leq i \leq n)$. Then μ' is called a *reachable* from $\mathcal{P}$, and we extend $\delta_{\mathcal{P}}$ from T to T^* by $\delta_{\mathcal{P}}(\mu,\, w) = \mu'$. By assuming that $\delta_{\mathcal{P}}(\mu,\, w) = \perp$ if w is not a firing sequence from $\mathcal{P}$ or $\mu = \perp$, the transition function $\delta_{\mathcal{P}} : ({N_0}^P \cup \{\perp\}) \times T^* \longrightarrow ({N_0}^P \cup \{\perp\})$ is regarded as a total function. The set of all reachable markings from $\mathcal{P}$ is called the *reachability set* of $\mathcal{P}$, denoted by $R(\mathcal{P})$.

LEMMA 4.1 Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)\,(i = 1, 2)$ be Petri nets. $(f,\, (\alpha, \beta))$ be a surjective morphism from $\mathcal{P}_1$ onto $\mathcal{P}_2$. Then,
(1) $t \in T_1$ is enable in $\mathcal{P}_1 \iff \beta(t) \in T_2$ is enable in $\mathcal{P}_2$. More precisely,

$$\mu_1' = \delta_{\mathcal{P}_1}(\mu_1,\, t)\,(\neq \perp) \iff \mu_2' = \delta_{\mathcal{P}_2}(\mu_2,\, \beta(t))\,(\neq \perp),$$

$f \otimes \mu_1 = \alpha\mu_2$ and $f \otimes \mu_1' = \alpha\mu_2'$ hold.

(2) w is a firing sequence in $\mathcal{P}_1 \iff \beta(w)$ is a firing sequence in $\mathcal{P}_2$. More precisely,

$$\mu_1' = \delta_{\mathcal{P}_1}(\mu_1,\, w)\,(\neq \perp) \iff \mu_2' = \delta_{\mathcal{P}_2}(\mu_2,\, \beta(w))\,(\neq \perp), \tag{4.2}$$

$f \otimes \mu_1 = \alpha\mu_2$ and $f \otimes \mu_1' = \alpha\mu_2'$ hold.

Proof) (1) For each $p \in P_1$,

$$\mu_2(\alpha(p)) - W_2(\alpha(p), \beta(t)) = f(p)\{\mu_1(p) - W_1(p,t)\} \quad \text{and} \quad f(p) > 0.$$

Therefore, if $\beta(t)$ is enabled in $\mathcal{P}_2$, then t is enabled in $\mathcal{P}_1$. Conversely, since α is surjective, $\beta(t)$ is enabled in $\mathcal{P}_2$ if t is enabled in $\mathcal{P}_1$.
In addition, the equation $\mu_2'(\alpha(p)) = \mu_2(\alpha(p)) - W_2(\alpha(p), \beta(t)) + W_2(\beta(p), \alpha(t)) = f(p)\{\mu_1(p) - W_1(p,t) + W_1(t,p)\} = f(p)\mu_1'(p)$ leads the equivalence of the two firing rules shown in (1).
(2) It is trivial by (1) and the definition of a firing sequence. □

LEMMA 4.2 Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)\,(i = 1, 2)$ be Petri nets. $(f,\, (\alpha, \beta))$ be a surjective morphism from $\mathcal{P}_1$ onto $\mathcal{P}_2$. Then,

(1) $\varphi : R(\mathcal{P}_1) \longrightarrow R(\mathcal{P}_2),\, \mu_1' \mapsto \mu_2'$, where μ_1' and μ_2' are markings satisfying (4.2), is a bijection.

(2) Let $R_i \subset R(\mathcal{P}_i)$ with $\varphi(R_1) = R_2$ and $K_i = \{w \in {T_i}^* \,|\, \delta_{\mathcal{P}_i}(\mu_i,\, w) \in R_i\}\,(i = 1, 2)$. Then $K_2 = \beta(K_1)$.

Proof) (1) φ is well-defined. Indeed, for any $\mu_1' \in R(\mathcal{P}_1)$, there exists at least one marking $\mu_2' \in R(\mathcal{P}_2)$ such that $f \otimes \mu_1' = \alpha\mu_2'$ by LEMMA 4.1 (2). Moreover if any two marking $\mu_2', \mu_2'' \in R(\mathcal{P}_2)$ satisfy $f \otimes \mu_1' = \alpha\mu_2' = \alpha\mu_2''$, then we have $\mu_2' = \mu_2''$ because α is surjective.

Next we show that φ is surjective. Let $\mu_2' \in R(\mathcal{P}_2)$. Since β is surjective, by LEMMA 4.1 (2), there exists $w \in T_1{}^*$ such that $\mu_1' = \delta_{\mathcal{P}_1}(\mu_1, w)$ and $\mu_2' = \delta_{\mathcal{P}_2}(\mu_2, \beta(w))$. Then $\varphi(\mu_1') = \mu_2'$.

Finally we show that φ is injective. Suppose that $\varphi(\mu_1') = \varphi(\mu_1'') = \mu_2'$. $f \otimes \mu_1' = f \otimes \mu_1'' = \alpha\mu_2'$. By LEMMA 2.1, $\mu_1' = (f^{-1}f) \otimes \mu_1' = (f^{-1}f) \otimes \mu_1'' = \mu_1''$.

(2) Let $w \in K_1$ with $\delta_{\mathcal{P}_1}(\mu_1, w) = \mu_1' \in R_1$. Then $\delta_{\mathcal{P}_2}(\mu_2, \beta(w)) = \mu_2' = \varphi(\mu_1') \in R_2$. Therefore $\beta(w) \in K_2$.

Conversely let $w \in K_2$ with $\delta_{\mathcal{P}_2}(\mu_2, w) = \mu_2' \in R_2$. Since β is surjective, $w = \beta(u)$ for some $u \in T_1{}^*$. $\delta_{\mathcal{P}_1}(\mu_1, u) = \mu_1' = \varphi^{-1}(\mu_2') \in R_1$. Therefore $w = \beta(u) \in \beta(K_1)$.

□

4.2 Petri net Languages

Let $\mathcal{P} = (P, T, W, \mu_0)$ be a Petri net, Σ be an alphabet, $\sigma : T \to \Sigma$ be a labeling of the transitions and $F \subseteq N_0{}^P$ be a finite set of final markings. Then we define the languages $\mathcal{L}_{\mathrm{L}}(\mathcal{P}, \sigma, F), \mathcal{L}_{\mathrm{G}}(\mathcal{P}, \sigma, F)$, $\mathcal{L}_{\mathrm{T}}(\mathcal{P}, \sigma)$ and $\mathcal{L}_{\mathrm{P}}(\mathcal{P}, \sigma)$ as follows:

$$\begin{aligned}
&\mathcal{L}_{\mathrm{L}}(\mathcal{P}, \sigma, F) \stackrel{\mathrm{def}}{=} \{\sigma(w) \mid w \in T^*, \mu = \delta_{\mathcal{P}}(\mu_0, w) \text{ and } \mu \in F\},\\
&\mathcal{L}_{\mathrm{G}}(\mathcal{P}, \sigma, F) \stackrel{\mathrm{def}}{=} \{\sigma(w) \mid w \in T^* \text{ and } \delta_{\mathcal{P}}(\mu_0, w) \geq \mu_f \text{ for some } \mu_f \in F\},\\
&\mathcal{L}_{\mathrm{T}}(\mathcal{P}, \sigma) \stackrel{\mathrm{def}}{=} \{\sigma(w) \mid w \in T^* \text{ and } \delta_{\mathcal{P}}(\mu_0, w) \neq \perp \text{ but for all } t \in T, \delta_{\mathcal{P}}(\mu, wt) = \perp\},\\
&\mathcal{L}_{\mathrm{P}}(\mathcal{P}, \sigma) \stackrel{\mathrm{def}}{=} \{\sigma(w) \mid w \in T^* \text{ and } \delta_{\mathcal{P}}(\mu_0, w) \neq \perp\}.
\end{aligned}$$

Languages $\mathcal{L}_{\mathrm{L}}(\mathcal{P}, \sigma, F)$, $\mathcal{L}_{\mathrm{G}}(\mathcal{P}, \sigma, F)$, $\mathcal{L}_{\mathrm{T}}(\mathcal{P}, \sigma)$ and $\mathcal{L}_{\mathrm{P}}(\mathcal{P}, \sigma)$ for some Petri net $\mathcal{P}$, some labeling σ and some set F of markings are called *L-type, G-type,T-type* and *P-type Petri net languages* respectively.

PROPOSITION 4.1 Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)\,(i = 1, 2)$ be Petri nets. $(f, (\alpha, \beta))$ be a surjective morphism from $\mathcal{P}_1$ onto $\mathcal{P}_2$.

For any $L_1 = \mathcal{L}_X(\mathcal{P}_1, \sigma_1, F_1)$, $X \in \{\mathrm{L}, \mathrm{G}\}$ (resp. $L_1 = \mathcal{L}_X(\mathcal{P}_1, \sigma_1)$, $X \in \{\mathrm{T}, \mathrm{P}\}$), there exists some $L_2 = \mathcal{L}_X(\mathcal{P}_2, \sigma_2, F_2)$ (resp. $L_2 = \mathcal{L}_X(\mathcal{P}_2, \sigma_2)$) such that $L_1 = \sigma_1(\beta^{-1}(\sigma_2{}^{-1}(L_2)))$. Then L_1 is regular (resp. linear, context-free) if and only if L_2 is regular (resp. linear, context-free).

Proof) We only show the case of $X = \mathrm{L}$. The remainder of proof is done in a similar way.

Putting $\sigma_2 = \mathbf{1}_{T_2}$, $R_1 = F_1 \cap R(\mathcal{P}_1)$, $F_2 = R_2 = \varphi(R_1)$ and $K_i = \{w \in T_i \mid \delta_{\mathcal{P}_i}(\mu_i, w) \in R_i\}$ $(i = 1, 2)$, where $\mathbf{1}_{T_2}$ is the identity map on T_2 and φ is the bijection defined in LEMMA 4.2. Then we have $L_1 = \sigma_1(K_1)$, $L_2 = \mathcal{L}_X(\mathcal{P}_2, \sigma_2, F_2) = \sigma_2(K_2)$, and by LEMMA 4.2 (2) $K_2 = \beta(K_1)$. Therefore $L_1 = \sigma_1(\beta^{-1}(\sigma_2{}^{-1}(L_2)))$.

Regarding operations with languages, the families of regular, linear and context-free languages are closed under the morphism and inverse morphism operations respectively. This leads to the equivalence condition.

□

5 Conclusions

In this paper we introduced Petri net morphisms/automorphism based on similarity of places and trasition. Some algebraic properties related to them were investigeted. We first considered Green's relations and ideals in the monoids $\mathcal{S}^1$ of morphisms of Petri nets, which is adjoined the extra zero 0 and the extra identity 1. For two given monoids, the principal left (resp. right) ideal of them is also a principal left (resp. right) ideal. This implies two kinds of diamond properties (confluencies) with respect to that the pre-order induced by surjective morphisms. It is technically interesting to construct such two kinds of synthesis of Petri nets. Next, the automorphism group $G = \mathbf{Aut}(\mathcal{P})$ of a ginve Petri net $\mathcal{P}$ was investigated. It is closely related to the symmetric groups preserves the partition determined by the equivalence relation of simirality on $\mathcal{P}$. By using this property, we can achieve the decomposition of G into a redundant part N and the other

K. The similarity can be described in term of a surjective morphism onto an irreducible Petri net Finally two Petri nets ordered by a surjective morphism have isomorphic reachability sets. Thus, the languages generated by them have a close correspondence.

Here we did not investigete problems, for example, whether the principal (two-sided) ideal of them is also a principal ideal in $\mathcal{S}^1$, whether an arbitrary left(resp. right, two-sided) ideal is principal in $\mathcal{S}^1$. Also we wonder whether the Petri nets with the same irreducible form constitute a lattice with respect to the order or not. In addition to these problems, we started investigating the application of Petri net morphism/automorphism to formal languages and codes. We will apply these results to famous and basic decision problems related to Petri nets.

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