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A note on the universal sl_2 invariant of Brunnian bottom tangles

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This note is based on the author's talk "On the universal sl_2 invariant of Brunnian bottom tangles" given in the conference "Intelligence of Low-dimensional Topology" held at the RIMS, Kyoto University, during May 16– May 18, 2012. The details appear in [11]

1 Introduction

The universal sl_2 invariant has a universality property for the colored Jones polynomial of links [4, 5, 7, 8]. We are interested in the relationship between topological properties of tangles and links and algebraic properties of the universal sl_2 invariant and the colored Jones polynomials.

A *bottom tangle* is a tangle in a cube consisting of only arc components such that each boundary point is on the bottom and the two boundary points of each arc are adjacent to each other, see Figure 1 (a) for example. The closure of a bottom tangle is defined as in Figure 1 (b).

The universal sl_2 invariant of n-component bottom tangles takes values in the completed n-fold tensor power $U_h(sl_2)^{\hat{\otimes}n}$ of the quantized enveloping algebra $U_h(sl_2)$. The colored Jones polynomial of a link L is obtained from the universal sl_2 invariant of a bottom tangle whose closure is L, by taking the quantum traces associated with the representations attached to the components of links [2].

A bottom tangle is called *ribbon* if its closure is a ribbon link (cf. [3, 9]). A bottom tangle is called *boundary* if its components admit mutually disjoint Seifert surfaces of bottom tangles (cf. [3, 10]). A bottom tangle T is called *Brunnian* if every proper subtangle of T is *trivial*, i.e., looks like $\cap\cdots\cap$.

Habiro [3] proved that the universal sl_2 invariant of n-component, algebraically-split, 0-framed bottom tangles takes values in a certain small subalgebra of $U_h(sl_2)^{\hat{\otimes}n}$. The present author proved improvements of Habiro's result in the special cases of ribbon bottom tangles [9], boundary bottom tangles [10], and Brunnian bottom tangles [11]. In [9, 10, 11], she also proved that the colored Jones polynomials of ribbon links, boundary links, and Brunnian links take values in certain small ideals of $\mathbb{Z}[q, q^{-1}]$.

In this note, we give a survey on the paper [11].

This note is organized as follows. In Section 2, we recall the definitions of $U_h(sl_2)$ and the universal sl_2 invariant of bottom tangles. In Section 3, we give the main result for the the universal sl_2 invariant of Brunnian bottom tangles (Theorem 3.5), and in Section 4, we give an application of Theorem 3.5 to the colored Jones polynomial of Brunnian links (Theorem 4.2).

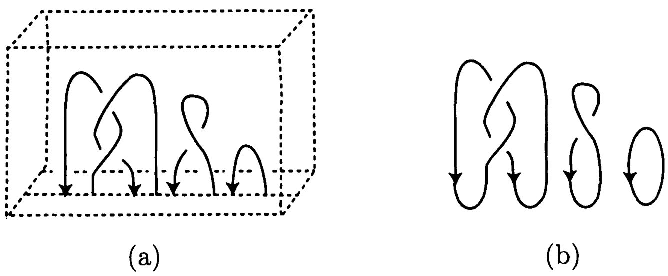


Figure 1: (a) A bottom tangle T, (b) The closure link of T

2 Universal sl_2 invariant of bottom tangles

In this section, we recall the definition of of $U_h(sl_2)$ and the universal sl_2 invariant of bottom tangles.

In what follows, we use the following q-integer notations.

$$\{i\}_q = q^i - 1, \quad \{i\}_{q,n} = \{i\}_q\{i-1\}_q \cdots \{i-n+1\}_q, \quad \{n\}_q! = \{n\}_{q,n},$$
$$[i]_q = \{i\}_q/\{1\}_q, \quad [n]_q! = [n]_q[n-1]_q \cdots [1]_q, \quad \begin{bmatrix} i \\ n \end{bmatrix}_q = \{i\}_{q,n}/\{n\}_q!,$$

for $i \in \mathbb{Z}, n \geq 0$.

2.1 Quantized enveloping algebra $U_h(sl_2)$ and universal R matrix

We recall the definition of the universal enveloping algebra $U_h(sl_2)$.

We denote by $U_h = U_h(sl_2)$ the h-adically complete $\mathbb{Q}[[h]]$-algebra, topologically generated by H, E, and F, defined by the relations

$$HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}},$$

where we set

$$q = \exp h, \quad K = q^{H/2} = \exp \frac{hH}{2}.$$

Set

$$\tilde{E}^{(n)} = (q^{-1/2}E)^n/[n]_q!, \quad \tilde{F}^{(n)} = F^n K^n/[n]_q! \quad \in U_h,$$
$$e = (q^{1/2} - q^{-1/2})E, \quad f = (q-1)FK \quad \in U_h,$$
$$D = q^{\frac{1}{4}H \otimes H} = \exp\left(\frac{h}{4} H \otimes H\right) \in U_h^{\hat{\otimes} 2}.$$

for $n \geq 0$.

We use the following *universal R-matrix* of U_h,

$$R^{\pm 1} = \sum_{n \geq 0} \alpha_n^{\pm} \otimes \beta_n^{\pm} \in U_h^{\hat{\otimes} 2},$$

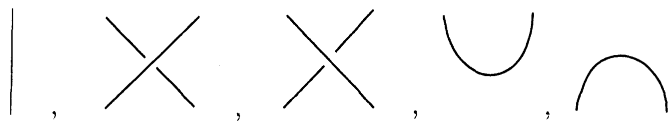
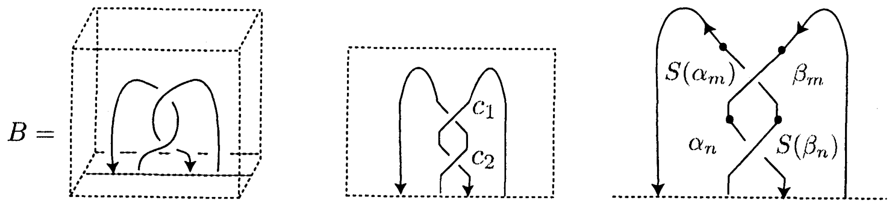


Figure 2: Fundamental tangles, where the orientations of the strands are arbitrary



(a) (b) (c)

Figure 3: (a) A bottom tangle B, (b) A diagram $\tilde{B}$ of B, (c) The labels associated to a state $t \in \mathcal{S}(B)$

where we set formally

$$\alpha_n \otimes \beta_n (= \alpha_n^+ \otimes \beta_n^+) = D\Big(q^{\frac{1}{2}n(n-1)} \tilde{F}^{(n)} K^{-n} \otimes e^n\Big),$$
$$\alpha_n^- \otimes \beta_n^- = D^{-1}\Big((-1)^n \tilde{F}^{(n)} \otimes K^{-n} e^n\Big).$$

2.2 Universal sl_2 invariant of bottom tangles

For an n-component bottom tangle $T = T_1 \cup \cdots \cup T_n$, we define the universal sl_2 invariant $J_T \in U_h^{\hat{\otimes} n}$ as follows. We follow the notation in [10].

We choose a diagram $\tilde{T}$ of T obtained from the copies of the fundamental tangles depicted in Figure 2, by pasting horizontally and vertically. We denote by $C(\tilde{T})$ the set of the crossings of $\tilde{T}$. For example, for the bottom tangle B depicted in Figure 3 (a), we can take a diagram $\tilde{B}$ with $C(\tilde{B}) = \{c_1, c_2\}$ as depicted in Figure 3 (b). We call a map

$$s \colon C(\tilde{T}) \rightarrow \{0, 1, 2, \ldots\}$$

a *state*. We denote by $\mathcal{S}(\tilde{T})$ the set of states of the diagram $\tilde{T}$.

Given a state $s \in \mathcal{S}(\tilde{T})$, we attach labels on the copies of the fundamental tangles in the diagram following the rule described in Figure 4, where "S'" should be replaced with the identity if the string is oriented downward, and with S otherwise. For example, for a state $t \in \mathcal{S}(\tilde{B})$, we put labels on $\tilde{B}$ as in Figure 3 (c), where we set $m = t(c_1)$ and $n = t(c_2)$.

We read the labels we have just put on $\tilde{T}$ and define an element $J_{\tilde{T},s} \in U_h^{\hat{\otimes} n}$ as follows. Let $\tilde{T} = \tilde{T}_1 \cup \cdots \cup \tilde{T}_n$, where $\tilde{T}_i$ corresponds to T_i. We define the ith tensorand of $J_{\tilde{T},s}$ as the product of the labels on $\tilde{T}_i$, where the labels are read off along T_i reversing the orientation, and written from left to right. For example, for the bottom tangle B and the

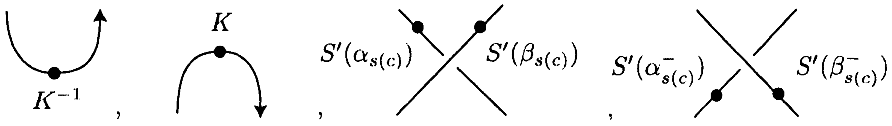


Figure 4: How to place labels on the fundamental tangles

state $t \in \mathcal{S}(\tilde{B})$ in Figure 3, we have

$$J_{\tilde{B},t} = S(\alpha_m)S(\beta_n) \otimes \alpha_n\beta_m.$$

Here, we identify the labels $S'(\alpha_i^{\pm})$ and $S'(\beta_i^{\pm})$ with the first and the second tensorands, respectively, of the element $S'(\alpha_i^{\pm}) \otimes S'(\beta_i^{\pm}) \in U_h^{\hat{\otimes}2}$. Also we identify the label $K^{\pm 1}$ with the element $K^{\pm 1} \in U_h$. Thus $J_{\tilde{T},s}$ is a well-defined element in $U_h^{\hat{\otimes}n}$. For example, we have

$$\begin{aligned} J_{\tilde{B},t} &= S(\alpha_m)S(\beta_n) \otimes \alpha_n\beta_m \\ &= \sum q^{\frac{1}{2}m(m-1)} q^{\frac{1}{2}n(n-1)} S(D_1'\tilde{F}^{(m)}K^{-m})S(D_2''e^n) \otimes D_2'\tilde{F}^{(n)}K^{-n}D_1''e^m \\ &= (-1)^{m+n}q^{-n+2mn}D^{-2}(\tilde{F}^{(m)}K^{-2n}e^n \otimes \tilde{F}^{(n)}K^{-2m}e^m) \in U_h^{\hat{\otimes}2}, \end{aligned}$$

where $D = \sum D_1' \otimes D_1'' = \sum D_2' \otimes D_2''$. Note that $J_{\tilde{T},s}$ *depends* on the choice of the diagram.

Set

$$J_T = \sum_{s \in \mathcal{S}(\tilde{T})} J_{\tilde{T},s}.$$

As is well known [7], J_T does not depend on the choice of the diagram, and defines an isotopy invariant of bottom tangles.

3 Result for the universal sl_2 invariant

In this section, we give the main result for the universal sl_2 invariant of Brunnian bottom tangles. Before that, we recall $\mathbb{Z}[q, q^{-1}]$-subalgebras of U_h and several results for the universal sl_2 invariant of algebraically-split bottom tangles.

3.1 $\mathbb{Z}[q, q^{-1}]$-subalgebras of U_h

We recall $\mathbb{Z}[q, q^{-1}]$-subalgebras of U_h.

Let $U_{\mathbb{Z},q} \subset U_h$ denote the $\mathbb{Z}[q, q^{-1}]$-subalgebra generated by $K, K^{-1}, \tilde{E}^{(n)}$, and $\tilde{F}^{(n)}$ for $n \geq 1$, which is a $\mathbb{Z}[q, q^{-1}]$-version of Lusztig's integral form (cf. [6, 9]).

Let $\mathcal{U}_q \subset U_{\mathbb{Z},q}$ denote the $\mathbb{Z}[q, q^{-1}]$-subalgebra generated by K, K^{-1}, e, and $\tilde{F}^{(n)}$ for $n \geq 1$.

Let $\bar{U}_q \subset \mathcal{U}_q$ denote the $\mathbb{Z}[q, q^{-1}]$-subalgebra generated by K, K^{-1}, e and f, which is a $\mathbb{Z}[q, q^{-1}]$-version of the integral form defined by De Concini and Procesi (cf. [1, 9]).

For $X = U_{\mathbb{Z},q}, \mathcal{U}_q, \bar{U}_q$, let X^{ev} denote the $\mathbb{Z}[q, q^{-1}]$-subalgebra of U_h defined by the same generators as X except that $K^{\pm 2}$ replaces $K^{\pm 1}$.

To summarize, we have the following inclusions of the subalgebras of U_h.

$$\begin{array}{ccccccc} \bar{U}_q^{\mathrm{ev}} & \subset & \mathcal{U}_q^{\mathrm{ev}} & \subset & U_{\mathbb{Z},q}^{\mathrm{ev}} & & \\ \cap & & \cap & & \cap & & \\ \bar{U}_q & \subset & \mathcal{U}_q & \subset & U_{\mathbb{Z},q} & \subset & U_h \end{array}$$

We recall the completion $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$ of $\mathcal{U}_q^{\mathrm{ev}}$ in U_h and its completed tensor powers $(\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$ for $n \geq 0$.

For $p \geq 0$, let $\mathcal{F}_p(\mathcal{U}_q^{\mathrm{ev}})$ be the two-sided ideal in $\mathcal{U}_q^{\mathrm{ev}}$ generated by e^p. Let $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$ be the completion of $\mathcal{U}_q^{\mathrm{ev}}$ in U_h with respect to the decreasing filtration $\{\mathcal{F}_p(\mathcal{U}_q^{\mathrm{ev}})\}_{p\geq 0}$, i.e., we define $\tilde{\mathcal{U}}_q^{\mathrm{ev}}$ as the image of the homomorphism

$$\varprojlim_{p\geq 0} \mathcal{U}_q^{\mathrm{ev}}/\mathcal{F}_p(\mathcal{U}_q^{\mathrm{ev}}) \to U_h$$

induced by $\mathcal{U}_q^{\mathrm{ev}} \subset U_h$.

For $n \geq 1$ and $p \geq 0$, set

$$\mathcal{F}_p\big((\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}\big) = \sum_{i=1}^{n} (\mathcal{U}_q^{\mathrm{ev}})^{\otimes (i-1)} \otimes \mathcal{F}_p(\mathcal{U}_q^{\mathrm{ev}}) \otimes (\mathcal{U}_q^{\mathrm{ev}})^{\otimes (n-i)}.$$

For $n \geq 1$, we define $(\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$ as the completion of $(\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}$ in $U_h^{\hat{\otimes} n}$ with respect to the decreasing filtration $\{\mathcal{F}_p\big((\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}\big)\}_{p\geq 0}$.

For a $\mathbb{Z}[q,q^{-1}]$-subalgebra A of $(\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}$, we denote by $\{A\}^{\sim}$ the *closure* of A in $(\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$, i.e., we set

$$\{A\}^{\sim} = \mathrm{Im}\Big(\varprojlim_{p\geq 0}(A/(\mathcal{F}_p\big((\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}\big) \cap A) \to U_h^{\hat{\otimes} n}\Big).$$

For $n = 0$, we define $(\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} 0} = \mathbb{Z}[q,q^{-1}]$.

3.2 Universal sl_2 invariant of algebraically-split bottom tangles, ribbon bottom tangles and boundary bottom tangles

We recall several results for the value of the universal sl_2 invariant of algebraically-split bottom tangles. In what follows, we assume that bottom tangles are 0-framed.

Theorem 3.1 ([9, Proposition 4.2, Remark 4.7]). *Let T be an n-component algebraically-split bottom tangle. For every diagram $\tilde{T}$ of T and every state $s \in \mathcal{S}(\tilde{T})$, we have*

$$J_{\tilde{T},s} \in (\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}.$$

More precisely, the proof of [9, Proposition 4.2] implies the following proposition.

Proposition 3.2. *Let T be an n-component algebraically-split bottom tangle. For any diagram $\tilde{T}$ and any state $s \in \mathcal{S}(\tilde{T})$, we have*

$$J_{\tilde{T},s} \in \mathcal{F}_{|s|}\big((\mathcal{U}_q^{\mathrm{ev}})^{\otimes n}\big),$$

where we set $|s| = \max\{s(c) \mid c \in C(\tilde{T})\}$.

Theorem 3.1 and Proposition 3.2 imply the following theorem, which was first proved by Habiro [3] in a different way.

Theorem 3.3 (Habiro [3]). *For an n-component algebraically-split bottom tangle T, we have*

$$J_T \in (\tilde{\mathcal{U}}_q^{ev})^{\tilde{\otimes} n}.$$

In [3], Habiro denoted by $(\bar{U}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$ the closure $\{(\bar{U}_q^{\mathrm{ev}})^{\otimes n}\}^{\sim}$ of $(\bar{U}_q^{\mathrm{ev}})^{\otimes n}$ in $(\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$. In [9] and [10], we defined a refined completion $(\bar{U}_q^{\mathrm{ev}})^{\hat{\otimes} n} \subset (\bar{U}_q^{\mathrm{ev}})^{\tilde{\otimes} n}$, and proved the following theorem, which is an improvement of Theorem 3.3 in the case of ribbon bottom tangles and boundary bottom tangles.

Theorem 3.4 ([9, 10]). *Let T be an n-component ribbon or boundary bottom tangle. Then we have*

$$J_T \in (\bar{U}_q^{\mathrm{ev}})^{\hat{\otimes} n}.$$

3.3 Results for the universal sl_2 invariant of Brunnian bottom tangles

The following theorem is the main result of the paper [11], which is an improvement of Theorems 3.1 and 3.3 in the case of Brunnian bottom tangles.

Theorem 3.5. *Let T be an n-component algebraically-split Brunnian bottom tangle with $n \geq 2$.*

(i) *For each $i = 1, \ldots, n$, there is a diagram $\tilde{T}^{(i)}$ of T such that*

$$J_{\tilde{T}^{(i)},s} \in (\bar{U}_q^{\mathrm{ev}})^{\otimes i-1} \otimes U_{\mathbb{Z},q}^{\mathrm{ev}} \otimes (\bar{U}_q^{\mathrm{ev}})^{\otimes n-i}$$

for any state $s \in \mathcal{S}(\tilde{T}^{(i)})$.

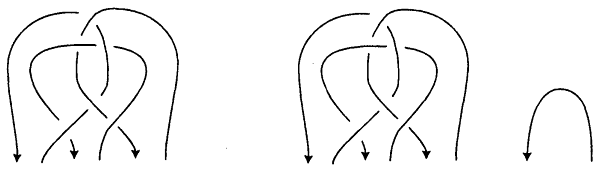
(ii) *We have $J_T \in U_{Br}^{(n)}$, where we set*

$$U_{Br}^{(n)} = \bigcap_{i=1}^{n} \Big\{ \Big((\bar{U}_q^{\mathrm{ev}})^{\otimes i-1} \otimes U_{\mathbb{Z},q}^{\mathrm{ev}} \otimes (\bar{U}_q^{\mathrm{ev}})^{\otimes n-i} \Big) \cap (\mathcal{U}_q^{\mathrm{ev}})^{\otimes n} \Big\}^{\wedge}.$$

Note that the condition "algebraically-split" in Theorem 3.5 is not necessary when $n \geq 3$.

To compare Theorem 3.5 (ii) with Theorems 3.3 and 3.4 for $n \geq 2$, we have the following diagram.

$$\begin{array}{ccc} \{n\text{-comp. alg. split bottom tangles}\} & \xrightarrow{J} & (\tilde{\mathcal{U}}_q^{\mathrm{ev}})^{\tilde{\otimes} n} \\ & & \cup \\ \{n\text{-comp. alg. split Brunnian bottom tangles}\} & \xrightarrow{J} & U_{Br}^{(n)} \\ & & \cup \\ \{n\text{-comp. ribbon or boundary bottom tangles}\} & \xrightarrow{J} & (\bar{U}_q^{\mathrm{ev}})^{\hat{\otimes} n} \end{array}$$



(a) (b)

Figure 5: (a) The Borromean bottom tangle T_B, (b) A bottom tangle T'_B

Example 3.6. *Let T_B be the Borromean bottom tangle, which is the Brunnian bottom tangle depicted in Figure 5 (a). Let T'_B be the bottom tangle as in Figure 5 (b). Note that the bottom tangle T'_B is not Brunnian but algebraically-split. We have*

$$J_{T'_B} = J_{T_B} \otimes 1 \notin \left\{ \left((\bar{U}_q^{\mathrm{ev}})^{\otimes 3} \otimes U^{\mathrm{ev}}_{\mathbb{Z},q} \right) \cap (\mathcal{U}_q^{\mathrm{ev}})^{\otimes 4} \right\}^{\widehat{}}.$$

4 Application to the colored Jones polynomial

In this section, we give an application of Theorem 3.5 to the colored Jones polynomial of Brunnian links (Theorem 4.2). In what follows, we assume that links are 0-framed.

4.1 Colored Jones polynomials of algebraically-split links, ribbon links and boundary links

We recall results for the colored Jones polynomials of algebraically-split links.

For $m \geq 1$, let V_m denote the m-dimensional irreducible representation of U_h. Let $\mathcal{R}$ denote the representation ring of U_h over $\mathbb{Q}(q^{\frac{1}{2}})$, i.e., $\mathcal{R}$ is the $\mathbb{Q}(q^{\frac{1}{2}})$-algebra

$$\mathcal{R} = \mathrm{Span}_{\mathbb{Q}(q^{\frac{1}{2}})}\{V_m \mid m \geq 1\}$$

with the multiplication induced by the tensor product.

For an n-component link L, take a bottom tangle T whose closure is L. For $X_1, \ldots, X_n \in \mathcal{R}$, the colored Jones polynomial $J_{L;X_1,\ldots,X_n}$ of L with the ith component L_i colored by X_i is given by

$$J_{L;X_1,\ldots,X_n} = (\mathrm{tr}_q^{X_1} \otimes \cdots \otimes \mathrm{tr}_q^{X_n})(J_T) \in \mathbb{Q}(q^{\frac{1}{2}}).$$

Habiro [3] studied the following element in $\mathcal{R}$

$$\tilde{P}'_l = \frac{q^{\frac{1}{2}l}}{\{l\}_q!} \prod_{i=0}^{l-1} (V_2 - q^{i+\frac{1}{2}} - q^{-i-\frac{1}{2}}) \in \mathcal{R}, \tag{1}$$

for $l \geq 0$.

Recall the notation $\{l\}_{q,i} = \{l\}_q\{l-1\}_q \cdots \{l-i+1\}_q$ for $l \in \mathbb{Z}$, $i \geq 0$. Theorem 3.3 implies the following result.

Theorem 4.1 (Habiro [3]). *Let L be an n-component algebraically-split link. For $l_1, \ldots, l_n \geq 0$, we have*

$$J_{L;\tilde{P}'_{l_1},\ldots,\tilde{P}'_{l_n}} \in Z_a^{(l_1,\ldots,l_n)}. \tag{2}$$

Here we set

$$Z_a^{(l_1,\ldots,l_n)} = \frac{\{2l_{\max}+1\}_{q,l_{\max}+1}}{\{1\}_q}\mathbb{Z}[q,q^{-1}],$$

where $l_{\max} = \max(l_1, \ldots, l_n)$.

For $l \geq 0$, let I_l denote the ideal in $\mathbb{Z}[q,q^{-1}]$ generated by $\{l-k\}_q!\{k\}_q!$ for $k = 0, \ldots, l$. Theorem 3.4 implies the following result.

Theorem 4.2 ([9, 10]). *Let L be an n-component ribbon link or boundary link. For $l_1, \ldots, l_n \geq 0$, we have*

$$J_{L;\tilde{P}'_{l_1},\ldots,\tilde{P}'_{l_n}} \in Z_{r,b}^{(l_1,\ldots,l_n)}. \tag{3}$$

Here we set

$$\begin{aligned} Z_{r,b}^{(l_1,\ldots,l_n)} &= \Big(\prod_{1\leq i\leq n, i\neq i_M} I_{l_i}\Big)\cdot Z_a^{(l_1,\ldots,l_n)} \\ &= \frac{\{2l_{\max}+1\}_{q,l_{\max}+1}}{\{1\}_q}\prod_{1\leq i\leq n, i\neq i_M} I_{l_i}, \end{aligned}$$

where $l_{\max} = \max(l_1, \ldots, l_n)$ and i_M is an integer such that $l_{i_M} = l_{\max}$.

4.2 Result for the colored Jones polynomial of Brunnian links

The following theorem is an application of Theorem 3.5 to the colored Jones polynomial of Brunnian links.

Theorem 4.3. *Let L be an n-component algebraically-split Brunnian link with $n \geq 2$. For $l_1, \ldots, l_n \geq 0$, we have*

$$J_{L;\tilde{P}'_{l_1},\ldots,\tilde{P}'_{l_n}} \in Z_{Br}^{(l_1,\ldots,l_n)}. \tag{4}$$

Here we set

$$Z_{Br}^{(l_1,\ldots,l_n)} = \frac{\{2l_{\max}+1\}_{q,l_{\max}+1}}{\{1\}_q\{l_{\min}\}_q!}\prod_{1\leq i\leq n, i\neq i_M, i_m} I_{l_i},$$

where $l_{\max} = \max(l_1, \ldots, l_n)$, $l_{\min} = \min(l_1, \ldots, l_n)$ and i_M, i_m, $i_M \neq i_m$, are two integers such that $l_{i_M} = l_{\max}$, $l_{i_m} = l_{\min}$, respectively.

Note that an algebraically-split Brunnian link satisfies both (2) and (4). In fact, there is no inclusion which satisfies for all $l_1, \ldots, l_n \geq 0$ between $Z_a^{(l_1,\ldots,l_n)}$ and $Z_{Br}^{(l_1,\ldots,l_n)}$. Thus we have a refinement of Theorem as follows.

Theorem 4.4. *Let L be an n-component algebraically-split Brunnian link with $n \geq 2$. For $l_1, \ldots, l_n \geq 0$, we have*

$$J_{L;\tilde{P}'_{l_1},\ldots,\tilde{P}'_{l_n}} \in \tilde{Z}_{Br}^{(l_1,\ldots,l_n)}.$$

Here we set

$$\tilde{Z}_{Br}^{(l_1,\ldots,l_n)} = Z_a^{(l_1,\ldots,l_n)} \cap Z_{Br}^{(l_1,\ldots,l_n)}.$$

Comparing Theorem 4.4 with Theorems 4.1 and 4.2 for $n \geq 2$, we have the following diagram.

$$\begin{array}{rcc} \{n\text{-comp. alg. split links}\} & \xrightarrow{J_{*;\tilde{P}'_{l_1},\ldots,\tilde{P}'_{l_n}}} & Z_a^{(l_1,\ldots,l_n)} \\ & & \cup \\ \{n\text{-comp. alg. split Brunnian links}\} & \xrightarrow{J_{*;\tilde{P}'_{l_1},\ldots,\tilde{P}'_{l_n}}} & \tilde{Z}_{Br}^{(l_1,\ldots,l_n)} \\ & & \cup \\ \{n\text{-comp. ribbon or boundary links}\} & \xrightarrow{J_{*;\tilde{P}'_{l_1},\ldots,\tilde{P}'_{l_n}}} & Z_{r,b}^{(l_1,\ldots,l_n)} \end{array}$$

Remark 4.5. In fact, the ideals $Z_a^{(l_1,\ldots,l_n)}$, $Z_{r,b}^{(l_1,\ldots,l_n)}$, $Z_{Br}^{(l_1,\ldots,l_n)}$ and $\tilde{Z}_{Br}^{(l_1,\ldots,l_n)}$ are principal, each generated by a product of cyclotomic polynomials. See [12] for details and examples.

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