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INTERPRETATION OF RACK COLORING KNOT INVARIANTS IN TERMS OF QUANDLES

KOKORO TANAKA AND YUMA TANIGUCHI

1. Introduction

This note is a survey of [15]. It is known that racks give us invariants of oriented framed knots [6] and quandles give us that of oriented knots [11, 13]. Considering an oriented knot with an integer as the oriented framed knot, Nelson [14] constructed an invariant of (unframed) oriented knots by using rack coloring invariants. It is natural to consider whether there is some relationship between his invariant and an invariant of oriented knots derived from quandle theory. In this note, we give two interpretation of his invariant in terms of quandles.

This note is organized as follows. We review basics of racks and quandles in Section 2. In Section 3, we introduce Nelson's polynomial rack counting invariant. In Section 4, we give a first interpretation of Nelson's invariant in terms of quandle colorings with a kink map. In Section 5, we give a second interpretation of Nelson's invariant in terms of quandle cocycle invariants. We give a byproduct of this study in Section 6.

2. Preliminaries

2.1. **Racks and quandles.** For a non-empty set X and a binary operation $*$ on X, we consider the following three conditions:

(Q1) For any $a\in X$, $a*a=a$.
(Q2) For any $a\in X$, the map $*a:X\to X$, defined by $\bullet\mapsto\bullet*a$, is bijective.
(Q3) For any $a, b, c\in X$, $(a*b)*c=(a*c)*(b*c)$.

These three conditions correspond to the Reidemeister moves of type I, II and III respectively.

A pair $(X, *)$ is called a *rack* if it satisfies conditions (Q2) and (Q3). Hence racks are useful for studying oriented framed knots. A pair $(X, *)$ is called a *quandle* if it satisfies conditions (Q1), (Q2) and (Q3). Hence quandles are useful for studying oriented knots. We remark that a quandle is a rack by definition. Racks and quandles have been studied in, for example, [6, 11, 13].

For racks X and Y, a *rack homomorphism* $f:X\to Y$ is a map such that $f(a*b)=f(a)*f(b)$ for any $a, b\in X$. If both X and Y are quandles, we call it a *quandle homomorphism*.

2.2. **Rack colorings and quandle colorings.** We define an invariant of oriented framed knots by using racks. Let R be a finite rack. Let (D, w) be a diagram of an oriented knot K whose writhe is an integer w. We can think of (D, w) as a diagram of (K, w) by blackboard framing, where (K, w) is an oriented framed knot whose underlying oriented knot is K and whose framing is w. Let $\mathcal{A}(D, w)$ be the set of arcs of (D, w). A map $c : \mathcal{A}(D, w) \to R$ is a *rack coloring* if it satisfies the following relation at every crossing. Let x_j be the over-arc at a crossing, and x_i, x_k be under-arcs at the crossing such that the normal direction of x_j points from x_i to x_k. Then it is required that $c(x_k) = c(x_i) * c(x_j)$. See Figure 1. Let $\mathrm{Col}_R(D, w)$ be the set of rack colorings of a diagram (D, w) with respect to R. Then the cardinality $|\mathrm{Col}_R(D, w)|$ is an invariant of the framed knot (K, w). More precisely, it is invariant under Reidemeister moves of type II and III (and is invariant under framed Reidemeister move of type I). Thus we denote the value $|\mathrm{Col}_R(D, w)|$ by $|\mathrm{Col}_R(K, w)|$. We note that $|\mathrm{Col}_R(K, w)|$ is finite, since R is finite.

Similarly, we define an invariant of oriented knots by using quandles. Let Q be a finite quandle. Let D be a diagram of an oriented knot K and $\mathcal{A}(D)$ the set of arcs of D. A map $c : \mathcal{A}(D) \to Q$ is a *quandle coloring* if it satisfies the same relation at every crossing as that in rack colorings. Let $\mathrm{Col}_Q(D)$ be the set of quandle colorings of a diagram D with respect to Q. Then the cardinality $|\mathrm{Col}_Q(D)|$ is a invariant of the knot K. More precisely, it is invariant under Reidemeister moves of type I, II and III. Thus we denote the value $|\mathrm{Col}_Q(D)|$ by $|\mathrm{Col}_Q(K)|$ We note that $|\mathrm{Col}_Q(K)|$ is finite, since Q is finite.

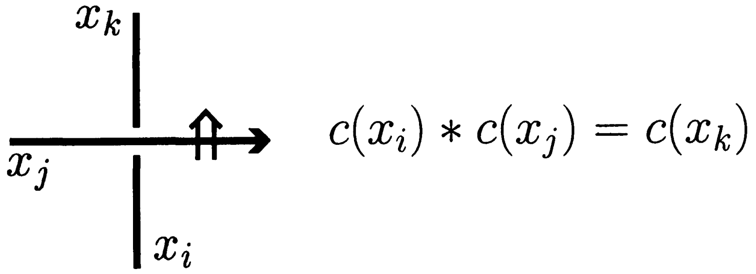


FIGURE 1. Coloring relation at a crossing

3. Nelson's polynomial rack counting invariant

3.1. **Rack rank.** For a rack $R = (R, *)$, let $\iota_R : R \to R$ be the map characterized by $\iota_R(a) * a = a$ for any $a \in R$. The map ι_R is well-defined by the condition (Q2). It is easy to see that ι_R is bijective. We remark that ι_R corresponds to a negative kink as in the left most dotted box of Figure 2, where we denote the map ι_R by ι for simplicity.

The *rack rank* of a rack R, denoted by N_R, is defined to be the minimum natural number, say n, such that ι_R^n is the identity map on R. If there exists no such n, then N_R is defined to be ∞. The rack rank corresponds to a diagram consisting of N_R copies of negative kinks as in Figure 2, where we denote the rack rank N_R by N for simplicity. For a finite rack R, we have $N_R \neq \infty$, since ι_R is bijective. We remark that the rack rank of a quandle is 1, since the map ι_Q is the identity map for any quandle Q.

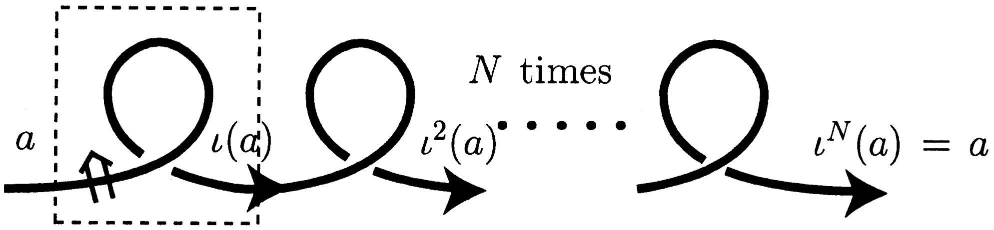


FIGURE 2. Diagrammatic meaning of the rack rank

3.2. **Polynomial rack counting invariant.** Nelson [14] found a periodicity of rack coloring invariants of oriented framed knots with respect to their framings.

Proposition 3.1. *Let R be a finite rack with rack rank N. Let (K, w) be an oriented framed knot whose underlying oriented knot is K and whose framing is an integer w. Then we have $|\mathrm{Col}_R(K, w)| = |\mathrm{Col}_R(K, w - N)|$.*

With the above proposition in hand, Nelson [14] constructed an invariant of (unframed) oriented knots by using rack coloring invariants.

Definition 3.2. Let R be a finite rack with rack rank N, K an oriented knot. The *polynomial rack counting invariant* of K with respect to R is given by

$$PR(K, R) := \sum_{w=0}^{N-1} |\mathrm{Col}_R(K, w)| t^w \in \mathbb{Z}[t, t^{-1}]/(t^N - 1),$$

where t is a formal variable.

4. FIRST INTERPRETATION

4.1. **Kink map.** For a rack $R = (R, *)$, a map $\varphi : R \to R$ is said to be a *kink map* of R if it satisfies the following three conditions:

(K1) The map φ is bijective.
(K2) For any $a, b \in R$, $\varphi(a) * b = \varphi(a * b)$.
(K3) For any $a, b \in R$, $a * \varphi(b) = a * b$.

The conditions (K2) and (K3) correspond to Figure 3 and Figure 4 respectively. It is easy to check that the map ι_R is a kink map of R. This is the most important example among kink maps of R. We note that the notion of "a kink map of a quandle" does make sense, since a quandle is a rack.

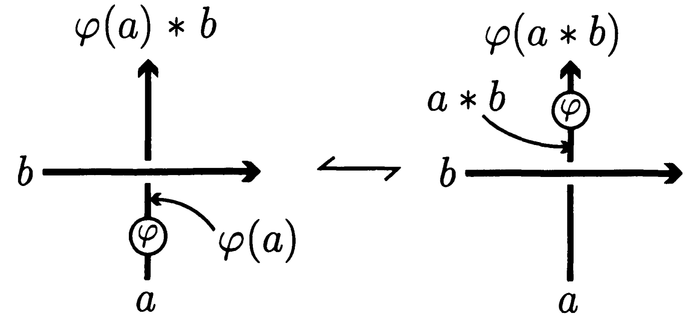


FIGURE 3. Diagrammatic meaning of the condition (K2)

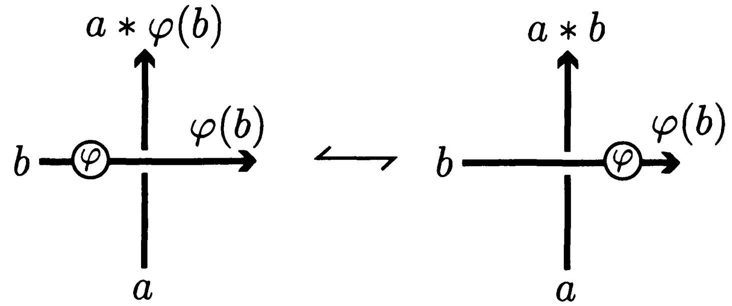


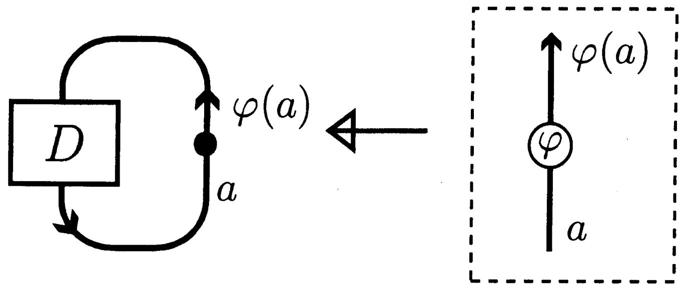
FIGURE 4. Diagrammatic meaning of the condition (K3)

4.2. **Quandle coloring with a kink map.** Using a kink map of a quandle, we can extend the notion of quandle coloring. Let Q be a finite quandle and φ a kink map of Q. Let D be a diagram of an oriented knot K, and denote $D_\bullet$ the diagram D with a base point. At the base point, we cut the arc of $D_\bullet$ into two arcs x_{in} and x_{out}, where the orientation points from x_{in} to x_{out}. Let $\mathcal{A}(D_\bullet)$ be the set of arcs of $D_\bullet$. A map $c : \mathcal{A}(D_\bullet) \to Q$ is a *quandle coloring with a kink map* if it satisfies the same relation at every crossing as that in rack colorings and quandle colorings, and the relation at the base point such that $\varphi(c(x_{\text{in}})) = c(x_{\text{out}})$. See Figure 5. For a diagram $D_\bullet$, let $\mathrm{Col}_{Q,\varphi}(D_\bullet)$ be the set of quandle colorings with a kink map with respect to Q and φ. Then the cardinality $|\mathrm{Col}_{Q,\varphi}(D_\bullet)|$ is a invariant of the knot K. More precisely, it is invariant under Reidemeister moves of type I, II and III, and it does not depend on the choice of a base point. Thus we denote the value $|\mathrm{Col}_{Q,\varphi}(D_\bullet)|$ by $|\mathrm{Col}_{Q,\varphi}(K)|$. We note that $|\mathrm{Col}_{Q,\varphi}(K)|$ is finite, since Q is finite.

4.3. **Associated quandle.** For a rack $R = (R, *)$, we denote the map ι_R by ι for simplicity, and define a new binary operation $*^\iota$ on the set R by $a *^\iota b := \iota(a) * b$. Then we have the following.

Proposition 4.1. *The pair* $(R, *^\iota)$ *is a quandle.*

The quandle $(R, *^\iota)$ is called the *associated quandle* of R and is denoted by R_Q. We note that this construction has essentially appeared in [1].



$$a = c(x_{\mathrm{in}}), \quad \varphi(a) = \varphi\big(c(x_{\mathrm{in}})\big) = c(x_{\mathrm{out}})$$

FIGURE 5. Coloring relation at the base point

4.4. **First interpretation.** Let $R = (R, *)$ be a finite rack with rack rank N. We denote the map ι_R by ι for simplicity. Let $R_Q = (R, *^{\iota})$ be the associated quandle of R.

Proposition 4.2. *For any integer n, a map ι^n is a kink map of R_Q.*

Remark 4.3. In the above proposition, the finiteness of R is not needed.

Theorem 4.4. *Let K be an oriented knot and w be an integer. Let (K, w) be the oriented framed knot whose underlying oriented knot is K and whose framing is w. Then we have the following.*

(1) *For any integer w, $|\mathrm{Col}_R(K, w)| = |\mathrm{Col}_{R_Q,\iota^{-w}}(K)|$.*

(2) $PR(K, R) = \displaystyle\sum_{w=0}^{N-1} |\mathrm{Col}_{R_Q,\iota^{-w}}(K)| t^w.$

5. SECOND INTERPRETATION

5.1. **Rack 2-cocycles and quandle 2-cocycles.** Let R be a rack and N a natural number. A *rack 2-cocycle* [7, 8, 9, 10] is a map $\theta : R \times R \to \mathbb{Z}/N\mathbb{Z}$ such that

$$\theta(a, b) + \theta(a * b, c) = \theta(a, c) + \theta(a * c, b * c)$$

for any $a, b, c \in R$.

Let Q be a quandle and N a natural number. A rack 2-cocycle $\theta : Q \times Q \to \mathbb{Z}/N\mathbb{Z}$ is said to be a *quandle 2-cocycle* [2] if $\theta(a, a) = \bar{0}$ for any $a \in Q$.

5.2. **Quandle cocycle invariant.** Let Q be a quandle, N a natural number, and $\theta : Q \times Q \to \mathbb{Z}/N\mathbb{Z}$ a quandle 2-cocycle. Let D be a diagram of an oriented knot K.

For each $c \in \mathrm{Col}_Q(D)$ and each crossing τ, we assign an element $W_\theta(\tau, c)$ in $\mathbb{Z}/N\mathbb{Z}$ as follows. Let x_j be the over-arc at the crossing τ, and x_i, x_k be under-arcs such that the normal direction of x_j points from x_i to x_k. Then we define $W_\theta(\tau, c)$ by

$$W_\theta(\tau, c) := \varepsilon(\tau) \cdot \theta(c(x_i), c(x_j)) \in \mathbb{Z}/N\mathbb{Z},$$

where $\varepsilon(\tau) = 1$ or -1 if the sign of the crossing τ is positive or negative respectively. See Figure 6.

For each $c \in \mathrm{Col}_Q(D)$, the element $W_\theta(c)$ in $\mathbb{Z}/N\mathbb{Z}$ is then defined by

$$W_\theta(c) := \sum_\tau W_\theta(\tau, c) \in \mathbb{Z}/N\mathbb{Z},$$

where τ runs over all crossings of D.

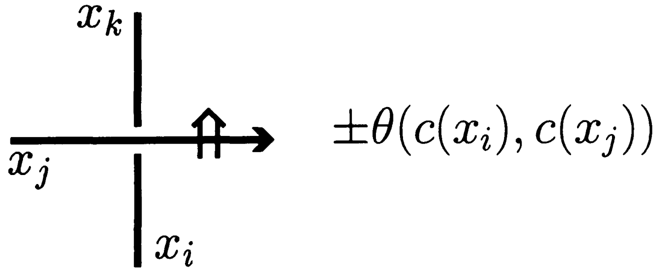


FIGURE 6. Weight at a crossing

The *quandle cocycle invariant* [2] of D with respect to the 2-cocycle θ, denoted by $\Phi_\theta(D)$, is defined by

$$\Phi_\theta(D) := \sum_{c \in \mathrm{Col}_Q(D)} t^{W_\theta(c)} \in \mathbb{Z}[t, t^{-1}]/(t^N - 1).$$

Then $\Phi_\theta(D)$ is invariant of K, that is, it is invariant under Reidemeister moves of type I, II and III. Thus we denote the value $\Phi_\theta(D)$ by $\Phi_\theta(K)$. We remark that the quandle cocycle invariant $\Phi_\theta(K)$ is a refinement of the number of quandle colorings $|\mathrm{Col}_Q(K)|$. More precisely, for a map $\varepsilon : \mathbb{Z}[t, t^{-1}]/(t^N - 1) \to \mathbb{Z}$ defined by $\varepsilon(t) = 1$, we have $\varepsilon(\Phi_\theta(K)) = |\mathrm{Col}_Q(K)|$.

5.3. **Quotient quandle.** For a rack $R = (R, *)$, we denote the map ι_R by ι for simplicity, and define the relation $a \overset{\iota}{\sim} b$ on R if there exists an integer n such that $b = \iota^n(a)$ for $a, b \in R$. It is easy to check that the relation $\overset{\iota}{\sim}$ is an equivalence relation on R. Moreover, we have the following.

Proposition 5.1. *The quotient set $R/\overset{\iota}{\sim}$ has a natural binary operation, which we denote by the same symbol $*$ by abuse of notation, induced from $R = (R, *)$. And the pair $(R/\overset{\iota}{\sim}, *)$ is a quandle.*

The quandle $(R/\overset{\iota}{\sim}, *)$ is called the *quotient quandle* of R and is denoted by Q. For any $a \in R$, we denote its equivalence class by $[a] \in Q$.

5.4. **Second interpretation.** Let $R = (R, *)$ be a finite rack with rack rank N, and $\pi : R \to Q$ a natural projection from R to its associated quandle $Q = (R/\overset{\iota}{\sim}, *)$. Using extension theory for racks and quandles developed in [3, 4, 5, 12], we can prove the following.

Proposition 5.2. *If the number of elements in* $\pi^{-1}([a])$ *is* N *for any* $[a] \in Q$, *then*

(1) *there exists a rack 2-cocycle* $\theta_R : Q \times Q \to \mathbb{N}/N\mathbb{Z}$ *such that*

$$\theta_R([a],[a]) = -\bar{1} \text{ for any } [a] \in Q,$$

and

(2) *the map* $\theta : Q \times Q \to \mathbb{Z}/N\mathbb{Z}$, *defined by*

$$\theta([a],[b]) := \theta_R([a],[b]) + \bar{1} \text{ for any } [a],[b] \in Q,$$

is a quandle 2-cocycle.

Remark 5.3. In the above proposition, the finiteness of R is not needed.

Theorem 5.4. *Let* K *be an oriented knot and* w *be an integer. Let* (K, w) *be the framed knot whose underlying knot is* K *and whose framing is* w. *Then we have the following.*

(1) $\displaystyle\sum_{w=0}^{N-1} |\mathrm{Col}_R(K,w)| = N \cdot |\mathrm{Col}_Q(K)|.$

(2) *If the number of elements in* $\pi^{-1}([a])$ *is* N *for any* $[a] \in Q$, *then*

$$PR(K,R) = N \cdot \Phi_\theta(K)$$

for the quandle 2-cocycle θ *as in Proposition* 5.2(2).

6. Byproduct

As a byproduct of two interpretations of Nelson's polynomial rack counting invariants, we can interpret quandle cocycle invariants in terms of quandle colorings with a kink map. Let $Q = (Q, *)$ be a finite quandle, N a natural integer, and $\theta : Q \times Q \to \mathbb{Z}/N\mathbb{Z}$ a quandle 2-cocycle. Let $\widetilde{Q}$ be a set given by $Q \times \mathbb{Z}/N\mathbb{Z}$.

Proposition 6.1. *Define a binary operation* $\tilde{*}$ *on* $\widetilde{Q}$ *by*

$$(a, \bar{m}) \tilde{*} (b, \bar{n}) := (a * b, \bar{m} + \theta(a,b)).$$

Then the pair $(\widetilde{Q}, \tilde{*})$ *is a quandle.*

Let $\varphi : \widetilde{Q} \to \widetilde{Q}$ be a map defined by $\varphi(a, \bar{m}) := (a, \bar{m} - \bar{1})$ for all $(a, \bar{m}) \in \widetilde{Q}$.

Proposition 6.2. *For any integer* n, *a map* φ^n *is a kink map of* $\widetilde{Q}$.

Theorem 6.3. *Let* K *be an oriented knot and* D *a diagram of* K. *Then the following hold.*

(1) *For any integer* w, *we have*

$$|\mathrm{Col}_{\widetilde{Q},\varphi^w}(K)| = N \cdot |\{c \in \mathrm{Col}_Q(D) \mid W_\theta(c) = w\}|.$$

$$(2)\ \Phi_\theta(K) = \frac{1}{N}\sum_{w=0}^{N-1} |\mathrm{Col}_{\widetilde{Q},\varphi^w}(K)| t^w.$$

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DEPARTMENT OF MATHEMATICS, TOKYO GAKUGEI UNIVERSITY, NUKUIKITA 4-1-1, KOGANEI, TOKYO 184-8501, JAPAN

E-mail address: kotanaka@u-gakugei.ac.jp

DEPARTMENT OF MATHEMATICS, TOKYO GAKUGEI UNIVERSITY, NUKUIKITA 4-1-1, KOGANEI, TOKYO 184-8501, JAPAN