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On the problem of Goldberg for the rational maps

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Abstract

In this paper, we solve a problem of Goldberg that determine the number of equivalence classes of rational maps corresponding to each critical set, when the degree is small and ∞ is critical.

1 Introduction

In [3], Goldberg suggested a problem that determine the number of equivalence classes of rational maps corresponding to each critical set. This problem is based on her theorem (Theorem 1.3 in [3]), and it is known that the theorem deeply concern with B. and M. Shapiro conjecture (see [1]).

By using algebraic computation system, we solve a problem of Goldberg when the degree is small and ∞ is critical, and this gives a complete answer to this problem together with our results in [2]. This work is joint work with M. Karima and M. Taniguchi (Nara Women's Univ.).

A rational map of degree d is a map with the following form,

$$R(z)=\frac{P(z)}{Q(z)},$$

where P and Q are coprime polynomials with $\max\{\deg P, \deg Q\}=d$.

Definition 1. Two rational maps R_1 and R_2 are said to be *Möbius equivalent* if there is a Möbius transformation $M:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ such that $R_2=M\circ R_1$.

Let X_d be the set of all equivalence classes of rational maps of degree d, and $X_d^{(k)}$ be the subset of X_d consisting of all equivalence classes of rational maps with k-hold critical point at ∞, where $k=0$ means the rational maps that ∞ is non-critical.

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Remark 1. A rational map R of degree d has $2d-2$ critical points counted including multiplicity. The set of critical points of R is invariant under taking a Möbius conjugate.

Every set of critical points of R is admissible, i.e., every critical point has multiplicity at most $d-1$. Therefore, the space X_d is the disjoint union of $X_d^{(0)}, X_d^{(1)}, \cdots$, and $X_d^{(d-1)}$.

Goldberg showed the following theorem.

Theorem (Goldberg [3]). A $(2d-2)$-tuple B is the critical set of at most $C(d)$ classes in X_d, where $C(d)$ means the d-th *Catalan number* $\frac{1}{d}\begin{pmatrix} 2d-2 \\ d-1 \end{pmatrix}$. The maximal is attained by a Zariski open subset of the space $\hat{\mathbb{C}}^{2d-2}$ of all B.

The map $\Phi_d : X_d \to \widehat{\mathbb{C}}^{2d-2}$ is defined by sending a equivalence class to the set of critical points, and the restriction of Φ_d to $X_d^{(k)}$ is denoted by $\Phi_d^{(k)}$.

Then Goldberg's problem (see [3]) is written as follows.

Problem

- Describe in detail the ramification sets of the maps Φ_d.
- For every point $\boldsymbol{c} \in \widehat{\mathbb{C}}^{2d-2}$, determine the number of points in the preimage $\Phi_d^{-1}(\boldsymbol{c})$.

We give the complete answer to this problem for the case of $d = 3$ and 4.

2 The case that ∞ is non-critical

Theorem 2 (Fujimura, Karima and Taniguchi [2]).

For each class in $X_d^{(0)}$, there is a unique representative R of the form

$$R(z) = \frac{P(z)}{Q(z)} = z + \frac{a_{d-2}z^{d-2} + \cdots + a_0}{z^{d-1} + b_{d-2}z^{d-2} + \cdots + b_0}.$$

For each $R = \dfrac{P}{Q}$ in the above form, the critical points of R is obtained by the equation

$$P'(z)Q(z) - P(z)Q'(z) = z^{2d-2} + c_{2d-3}z^{2d-3} + \cdots + c_0 = 0.$$

Then, the map $\Phi_d^{(0)}$ is defined as follows,

$$\begin{array}{cccc} \Phi_d^{(0)} : & \mathbb{C}^{2d-2} & \longrightarrow & \mathbb{C}^{2d-2} \\ & \cup\!\!\!| & & \cup\!\!\!| \\ & (a_{d-2}, \cdots, a_0, b_{d-2}, \cdots, b_0) & \mapsto & (c_{2d-3}, \cdots, c_0). \end{array}$$

The defining equation of the ramification locus of $\Phi_d^{(0)}$ gives the answer to a problem of Goldberg for the case that ∞ is non-critical. For the details, see [2].

Thereafter, we consider the case that ∞ is critical.

3 The case that ∞ is critical

3.1 The case of degree 3

Proposition 3.

1. *For each class in $X_3^{(1)}$, there is a unique representative in $CB_3^{(1)}$, where*

$$CB_3^{(1)} = \left\{ R(z) = z^2 + az + \frac{c}{z+b} \quad (c \neq 0) \right\}.$$

2. *For each class in $X_3^{(2)}$, there is a unique representative in $CB_3^{(2)}$, where*

$$CB_3^{(2)} = \left\{ R(z) = z^3 + az^2 + bz \right\}.$$

3.1.1 The case that ∞ is simple critical point

Let $R = \dfrac{P}{Q}$ be a rational map in $CB_3^{(1)}$, and $z^3 + c_2z^2 + c_1z + c_0 = 0$ be the equation defined by $P'(z)Q(z) - P(z)Q'(z) = 0$.

Then, the map $\Phi_3^{(1)} : CB_3^{(1)} \to \mathbb{C}^3$ is defined by sending (a, b, c) to (c_0, c_1, c_2).

Proposition 4. *The ramification locus of $\Phi_3^{(1)}$ is given by $a = 0$, $\Phi_3^{(1)}(CB_3^{(1)}) = \mathbb{C}^3 \setminus E^{(1)}(3)$ and $\Phi_3^{(1)}$ is 2-valent on the the set of the points in $\mathbb{C}^3 \setminus E^{(1)}(3)$ satisfying that*

$$4c_0c_2^3 - c_1^2c_2^2 - 4c_0c_1c_2 + c_1^3 + c_0^2 \neq 0 \quad \textit{or} \quad 2c_2^3 - 2c_1c_2 + c_0 \neq 0.$$

Proof. The map $\Phi_3^{(1)}$ is defined by

$$(a, b, c) \mapsto (c_0, c_1, c_2) = \left(\frac{ab^2 - c}{2},\ ab + b^2,\ b + \frac{a}{2}\right).$$

For $\boldsymbol{c} = (c_0, c_1, c_2) \in \mathbb{C}^3 \setminus E^{(1)}(3)$, every $(\Phi_3^{(1)})^{-1}(\boldsymbol{c})$ is given by

$$\begin{cases} B = b^2 - 2c_2b + c_1 = 0 \\ C = (4c_2^2 - 2c_1)b + c - 2c_1c_2 + 2c_0 = 0 \\ A = a + 2b - 2c_2 = 0, \end{cases} \tag{1}$$

which has exactly 2 solutions except for $\mathrm{discriminant}_b(B) = c_2^2 - c_1 = 0$.

The map $\Phi_3^{(1)}$ is not defined on $\{(a,b,c)\,|\,c=0\}$ where

$$\text{resultant}_z(\text{numerator}(R), \text{denominator}(R)) = c = 0.$$

From (1), for each (c_0, c_1, c_2), the coefficient c is determined by

$$-c^2 + (-8c_2^3 + 8c_1c_2 - 4c_0)c - 16c_0c_2^3 + 4c_1^2c_2^2 + 16c_0c_1c_2 - 4c_1^3 - 4c_0^2 = 0. \quad (2)$$

Therefore, the exceptional set $E^{(1)}(3)$ corresponds to the condition that the equation (2) has 0 as a unique solution. Thus we have

$$E^{(1)}(3) = \left\{4c_0c_2^3 - c_1^2c_2^2 - 4c_0c_1c_2 + c_1^3 + c_0^2 = 0 \text{ and } 2c_2^3 - 2c_1c_2 + c_0 = 0\right\}.$$

□

3.1.2 The case that ∞ is double critical point

Let $R(z) = z^3 + az + b$ be a polynomial map in $CB_3^{(2)}$, and $z^2 + c_1z + c_0 = 0$ be the equation defined by $R'(z) = 0$.

Then, the map $\Phi_3^{(2)} : CB_3^{(2)} \to \mathbb{C}^2$ is defined by sending (a,b) to (c_0, c_1).

Proposition 5. *The map $\Phi_3^{(2)}$ is bijective.*

Proof. Since the map $\Phi_3^{(2)}$ is given by $(a,b) \mapsto (c_0, c_1) = \left(\frac{2a}{3}, \frac{b}{3}\right)$, the assertion follows. □

3.2 The case of degree 4

Proposition 6.

1. *For each class in $X_4^{(1)}$, there is a unique representative in $CB_4^{(1)}$, where*

$$CB_4^{(1)} = \left\{R(z) = z^2 + cz + \frac{a_1z + a_0}{z^2 + b_1z + b_0} \quad (a_0a_1b_1 - b_0a_1^2 - a_0^2 \neq 0)\right\}.$$

2. *For each class in $X_4^{(2)}$, there is a unique representative in $CB_4^{(2)}$, where*

$$CB_4^{(2)} = \left\{R(z) = z^3 + a_2z^2 + a_1z + \frac{c}{z+b} \quad (c \neq 0)\right\}.$$

3. *For each class in $X_4^{(3)}$, there is a unique representative in $CB_4^{(3)}$, where*

$$CB_4^{(3)} = \left\{R(z) = z^4 + a_3z^3 + a_2z^2 + a_1z\right\}.$$

3.2.1 The case that ∞ is simple critical point

Let $R = \dfrac{P}{Q}$ be a rational map in $CB_4^{(1)}$, and $z^5 + c_4 z^4 + \cdots + c_0 = 0$ be the equation defined by $P'(z)Q(z) - P(z)Q'(z) = 0$.

Then, the map $\Phi_4^{(1)} : CB_4^{(1)} \to \mathbb{C}^5$ is defined by sending (a_0, a_1, b_0, b_1, c) to $(c_0, \cdots, c_4)$.

Proposition 7. *The ramification locus of the map $\Phi_4^{(1)}$ is given by*

$$(b_1^2 - 4b_0)c^2 + (-2b_1^3 + 8b_0 b_1 - a_1)c + 4b_0 b_1^2 + 2a_1 b_1 - 2a_0 - 16b_0^2 = 0,$$

$\Phi_4^{(1)}(CB_4^{(1)}) = \mathbb{C}^5 \setminus E^{(1)}(4)$, and $\Phi_4^{(1)}$ is 5-valent on the set of points in $\mathbb{C}^5 \setminus E^{(1)}(4)$, where defining equation of $E^{(1)}(4)$ is given in the proof.

Proof. The five critical points of R is given as the solution of the following equation,

$$\begin{aligned} 2z^5 + (c + 4b_1)z^4 + (2b_1 c + 2b_1^2 + 4b_0)z^3 + ((b_1^2 + 2b_0)c + 4b_0 b_1 - a_1)z^2 \\ + (2b_0 b_1 c - 2a_0 + 2b_0^2)z + b_0^2 c - a_0 b_1 + b_0 a_1 = 0. \quad (3) \end{aligned}$$

Therefore, the map $\Phi_4^{(1)}$ is defined by $(a_0, a_1, b_0, b_1, c) \mapsto (c_0, \cdots, c_4)$, where

$$\begin{aligned} c_0 &= (b_0^2 c - a_0 b_1 + b_0 a_1)/2, \\ c_1 &= (2b_0 b_1 c - 2a_0 + 2b_0^2)/2, \\ c_2 &= ((b_1^2 + 2b_0)c + 4b_0 b_1 - a_1)/2, \\ c_3 &= (2b_1 c + 2b_1^2 + 4b_0)/2, \\ c_4 &= (c + 4b_1)/2. \end{aligned} \quad (4)$$

The ramification locus is obtained from the Jacobian of the map $\Phi_4^{(1)}$,

$$(b_1^2 - 4b_0)c^2 + (-2b_1^3 + 8b_0 b_1 - a_1)c + 4b_0 b_1^2 + 2a_1 b_1 - 2a_0 - 16b_0^2 = 0.$$

For $\boldsymbol{c} \in \mathbb{C}^5 \setminus E^{(1)}(4)$, every $(\Phi_4^{(1)})^{-1}(\boldsymbol{c})$ is given by,

$$\begin{cases} B_1 = 81b_1^5 - 162c_4 b_1^4 + (108c_4^2 + 54c_3)b_1^3 + (-24c_4^3 - 72c_3 c_4 + 12c_2)b_1^2 \\ \qquad +(24c_3 c_4^2 - 8c_2 c_4 + 9c_3^2 - 4c_1)b_1 - 6c_3^2 c_4 + 4c_2 c_3 + 8c_0 \\ B_0 = -3b_1^2 + 2c_4 b_1 + 2b_0 - c_3 \\ A_1 = -10b_1^3 + 12c_4 b_1^2 + (-4c_4^2 - 2c_3)b_1 - a_1 + 2c_3 c_4 - 2c_2 \\ A_0 = 15b_1^4 - 16c_4 b_1^3 + (4c_4^2 + 2c_3)b_1^2 + 4a_0 - c_3^2 + 4c_1. \\ C = c + 4b_1 - 2c_4, \end{cases}$$

which has exactly 5 solutions except for $\mathrm{discriminant}_{b_1}(B_1) = 0$,

$1296c_0c_1^2c_4^7 + ((-1296c_0c_1c_2 - 324c_1^3)c_3 + 384c_0c_2^3 + 108c_1^2c_2^2 - 7776c_0^2c_1)c_4^6 + (324c_0c_1c_3^3 + (-108c_0c_2^2 + 324c_1^2c_2)c_3^2 + (-204c_1c_2^3 + 3888c_0^2c_2 - 7452c_0c_1^2)c_3 + 32c_2^5 - 936c_0c_1c_2^2 + 108c_1^3c_2 + 11664c_0^3)c_4^5 + (-81c_1^2c_3^4 + (54c_1c_2^2 - 972c_0^2)c_3^3 + (-9c_2^4 + 8316c_0c_1c_2 + 2106c_1^3)c_3^2 + (-2412c_0c_2^3 - 738c_1^2c_2^2 + 49572c_0^2c_1)c_3 + 8c_1c_2^4 + 108c_0^2c_2^2 + 4284c_0c_1^2c_2 + 27c_1^4)c_4^4 + (-1944c_0c_1c_3^4 + (648c_0c_2^2 - 2052c_1^2c_2)c_3^3 + (1296c_1c_2^3 - 24624c_0^2c_2 + 9288c_0c_1^2)c_3^2 + (-204c_2^5 + 1512c_0c_1c_2^2 - 1800c_1^3c_2 - 72900c_0^3)c_3 + 1320c_0c_2^4 + 368c_1^2c_2^3 - 26460c_0^2c_1c_2 + 3396c_0c_1^3)c_4^3 + (486c_1^2c_3^5 + (-324c_1c_2^2 + 5832c_0^2)c_3^4 + (54c_2^4 - 13608c_0c_1c_2 - 3834c_1^3)c_3^3 + (3672c_0c_2^3 + 2592c_1^2c_2^2 - 86670c_0^2c_1)c_3^2 + (-738c_1c_2^4 + 12690c_0^2c_2^2 - 13284c_0c_1^2c_2 - 984c_1^4)c_3 + 108c_2^6 - 2124c_0c_1c_2^3 + 634c_1^3c_2^2 + 40500c_0^3c_2 - 49950c_0^2c_1^2)c_4^2 + (2916c_0c_1c_3^5 + (-972c_0c_2^2 + 3240c_1^2c_2)c_3^4 + (-2052c_1c_2^3 + 38880c_0^2c_2 + 6156c_0c_1^2)c_3^3 + (324c_2^5 + 3024c_0c_1c_2^2 + 5544c_1^3c_2 + 121500c_0^3)c_3^2 + (-3888c_0c_2^4 - 1800c_1^2c_2^3 + 118800c_0^2c_1c_2 + 12240c_0c_1^3)c_3 + 108c_1c_2^5 - 8100c_0^2c_2^3 - 5220c_0c_1^2c_2^2 + 352c_1^4c_2 + 202500c_0^3c_1)c_4 - 729c_1^2c_3^6 + (486c_1c_2^2 - 8748c_0^2)c_3^5 + (-81c_2^4 + 972c_0c_1c_2 + 972c_1^3)c_3^4 + (324c_0c_2^3 - 3834c_1^2c_2^2 + 12150c_0^2c_1)c_3^3 + (2106c_1c_2^4 - 36450c_0^2c_2^2 - 18360c_0c_1^2c_2 - 432c_1^4)c_3^2 + (-324c_2^6 + 14580c_0c_1c_2^3 - 984c_1^3c_2^2 - 202500c_0^3c_2 - 27000c_0^2c_1^2)c_3 - 648c_0c_2^5 + 27c_1^2c_2^4 + 20250c_0^2c_1c_2^2 - 2400c_0c_1^3c_2 + 64c_1^5 - 253125c_0^4 = 0.$

The map $\Phi_4^{(1)}$ is not defined on

$$r := \text{resultant}_z\,(\text{numerator}(R), \text{denominator}(R)) = -a_0a_1b_1 + b_0a_1^2 + a_0^2 = 0.$$

From (4), for each $(c_0, \cdots, c_4)$, r is determined by the equation of the form,

$$8503056r^5 + P_4r^4 + P_3r^3 + P_2r^2 + P_1r + P_0 = 0$$
$$(P_k \in \mathbb{C}[c_0, c_1, c_2, c_3, c_4],\ k = 0, 1, 2, 3, 4). \quad (5)$$

Therefore, the exceptional set $E^{(1)}(4)$ corresponds to the condition that this equation has 0 as a unique solution. Thus we have

$$E^{(1)}(4) = \{P_0 = P_1 = P_2 = P_3 = P_4 = 0\},$$

where

$P_0 = -256(256c_0^3c_4^5 + (-192c_0^2c_1c_3 - 128c_0^2c_2^2 + 144c_0c_1^2c_2 - 27c_1^4)c_4^4 + ((144c_0^2c_2 - 6c_0c_1^2)c_3^2 + (-80c_0c_1c_2^2 + 18c_1^3c_2 - 1600c_0^3)c_3 + 16c_0c_2^4 - 4c_1^2c_2^3 + 160c_0^2c_1c_2 - 36c_0c_1^3)c_4^3 + (-27c_0^2c_3^4 + (18c_0c_1c_2 - 4c_1^3)c_3^3 + (-4c_0c_2^3 + c_1^2c_2^2 + 1020c_0^2c_1)c_3^2 + (560c_0^2c_2^2 - 746c_0c_1^2c_2 + 144c_1^4)c_3 + 24c_0c_1c_2^3 - 6c_1^3c_2^2 + 2000c_0^3c_2 - 50c_0^2c_1^2)c_4^2 + ((-630c_0^2c_2 + 24c_0c_1^2)c_3^3 + (356c_0c_1c_2^2 - 80c_1^3c_2 + 2250c_0^3)c_3^2 + (-72c_0c_2^4 + 18c_1^2c_2^3 - 2050c_0^2c_1c_2 + 160c_0c_1^3)c_3 - 900c_0^2c_2^3 + 1020c_0c_1^2c_2^2 - 192c_1^4c_2 - 2500c_0^3c_1)c_4 + 108c_0^2c_3^5 + (-72c_0c_1c_2 + 16c_1^3)c_3^4 + (16c_0c_2^3 - 4c_1^2c_2^2 - 900c_0^2c_1)c_3^3 + (825c_0^2c_2^2 + 560c_0c_1^2c_2 - 128c_1^4)c_3^2 + (-630c_0c_1c_2^3 + 144c_1^3c_2^2 - 3750c_0^3c_2 + 2000c_0^2c_1^2)c_3 + 108c_0c_2^5 - 27c_1^2c_2^4 + 2250c_0^2c_1c_2^2 - 1600c_0c_1^3c_2 + 256c_1^5 + 3125c_0^4)^2.$

$$P_1 = 256((144c_0c_2-54c_1^2)c_4^4+(-54c_0c_3^2+18c_1c_2c_3-4c_2^3-36c_0c_1)c_4^3+((-702c_0c_2+279c_1^2)c_3-6c_1c_2^2-1350c_0^2)c_4^2+(243c_0c_3^3-81c_1c_2c_3^2+(18c_2^3+810c_0c_1)c_3+1440c_0c_2^2-624c_1^2c_2)c_4+(-405c_0c_2-216c_1^2)c_3^2+(279c_1c_2^2+3375c_0^2)c_3-54c_2^4-3600c_0c_1c_2+1120c_1^3, 1], [256c_0^3c_4^5+(-192c_0^2c_1c_3-128c_0^2c_2^2+144c_0c_1^2c_2-27c_1^4)c_4^4+((144c_0^2c_2-6c_0c_1^2)c_3^2+(-80c_0c_1c_2^2+18c_1^3c_2-1600c_0^3)c_3+16c_0c_2^4-4c_1^2c_2^3+160c_0^2c_1c_2-36c_0c_1^3)c_4^3+(-27c_0^2c_3^4+(18c_0c_1c_2-4c_1^3)c_3^3+(-4c_0c_2^3+c_1^2c_2^2+1020c_0^2c_1)c_3^2+(560c_0^2c_2^2-746c_0c_1^2c_2+144c_1^4)c_3+24c_0c_1c_2^3-6c_1^3c_2^2+2000c_0^3c_2-50c_0^2c_1^2)c_4^2+((-630c_0^2c_2+24c_0c_1^2)c_3^3+(356c_0c_1c_2^2-80c_1^3c_2+2250c_0^3)c_3^2+(-72c_0c_2^4+18c_1^2c_2^3-2050c_0^2c_1c_2+160c_0c_1^3)c_3-900c_0^2c_2^3+1020c_0c_1^2c_2^2-192c_1^4c_2-2500c_0^3c_1)c_4+108c_0^2c_3^5+(-72c_0c_1c_2+16c_1^3)c_3^4+(16c_0c_2^3-4c_1^2c_2^2-900c_0^2c_1)c_3^3+(825c_0^2c_2^2+560c_0c_1^2c_2-128c_1^4)c_3^2+(-630c_0c_1c_2^3+144c_1^3c_2^2-3750c_0^3c_2+2000c_0^2c_1^2)c_3+108c_0c_2^5-27c_1^2c_2^4+2250c_0^2c_1c_2^2-1600c_0c_1^3c_2+256c_1^5+3125c_0^4),$$

$$P_2 = 864(2048c_0^3c_4^9+(-1536c_0^2c_1c_3-1024c_0^2c_2^2+1152c_0c_1^2c_2-216c_1^4)c_4^8+((1152c_0^2c_2-48c_0c_1^2)c_3^2+(-640c_0c_1c_2^2+144c_1^3c_2-23040c_0^3)c_3+128c_0c_2^4-32c_1^2c_2^3+1280c_0^2c_1c_2-288c_0c_1^3)c_4^7+(-216c_0^2c_3^4+(144c_0c_1c_2-32c_1^3)c_3^3+(-32c_0c_2^3+8c_1^2c_2^2+15840c_0^2c_1)c_3^2+(9600c_0^2c_2^2-11728c_0c_1^2c_2+2232c_1^4)c_3+192c_0c_1c_2^3-48c_1^3c_2^2+85632c_0^3c_2-21136c_0^2c_1^2)c_4^6+((-10800c_0^2c_2+432c_0c_1^2)c_3^3+(6048c_0c_1c_2^2-1360c_1^3c_2+68688c_0^3)c_3^2+(-1216c_0c_2^4+304c_1^2c_2^3-54288c_0^2c_1c_2+13088c_0c_1^3)c_3-34336c_0^2c_2^3+33504c_0c_1^2c_2^2-6936c_1^4c_2-53280c_0^3c_1)c_4^5+(1944c_0^2c_3^5+(-1296c_0c_1c_2+288c_1^3)c_3^4+(288c_0c_2^3-72c_1^2c_2^2-43200c_0^2c_1)c_3^3+(16200c_0^2c_2^2+22608c_0c_1^2c_2-7405c_1^4)c_3^2+(-17776c_0c_1c_2^3+5856c_1^3c_2^2-615600c_0^3c_2+198480c_0^2c_1^2)c_3+3424c_0c_2^5-1160c_1^2c_2^4+124240c_0^2c_1c_2^2-86240c_0c_1^3c_2+13820c_1^5-135000c_0^4)c_4^4+((16200c_0^2c_2+486c_0c_1^2)c_3^4+(-9504c_0c_1c_2^2+2958c_1^3c_2+10800c_0^3)c_3^3+(1824c_0c_2^4-1380c_1^2c_2^3+208080c_0^2c_1c_2-58180c_0c_1^3)c_3^2+(304c_1c_2^5+152480c_0^2c_2^3-142160c_0c_1^2c_2^2+30360c_1^4c_2+468000c_0^3c_1)c_3-32c_2^7-1760c_0c_1c_2^4+256c_1^3c_2^3+672000c_0^3c_2^2-304400c_0^2c_1^2c_2+29520c_0c_1^4)c_4^3+(-3645c_0^2c_3^6+(2430c_0c_1c_2-756c_1^3)c_3^5+(-540c_0c_2^3+351c_1^2c_2^2+12960c_0^2c_1)c_3^4+(-72c_1c_2^4-176040c_0^2c_2^2+27810c_0c_1^2c_2+9600c_1^4)c_3^3+(8c_2^6+80040c_0c_1c_2^3-27518c_1^3c_2^2+918000c_0^3c_2-557550c_0^2c_1^2)c_3^2+(-16240c_0c_2^5+5856c_1^2c_2^4-820800c_0^2c_1c_2^2+524360c_0c_1^3c_2-76480c_1^5+675000c_0^4)c_3-48c_1c_2^6-209200c_0^2c_2^4+188640c_0c_1^2c_2^3-39240c_1^4c_2^2-1260000c_0^3c_1c_2+233000c_0^2c_1^3)c_4^2+((39366c_0^2c_2-5832c_0c_1^2)c_3^5+(-21060c_0c_1c_2^2+1008c_1^3c_2-182250c_0^3)c_3^4+(4680c_0c_2^4+2958c_1^2c_2^3+106650c_0^2c_1c_2+34560c_0c_1^3)c_3^3+(-1360c_1c_2^5+233100c_0^2c_2^3-133740c_0c_1^2c_2^2+2560c_1^4c_2-742500c_0^3c_1)c_3^2+(144c_2^7-77440c_0c_1c_2^4+30360c_1^3c_2^3-1890000c_0^3c_2^2+1737000c_0^2c_1^2c_2-227200c_0c_1^4)c_3+19296c_0c_2^6-6936c_1^2c_2^5+1112000c_0^2c_1c_2^3-881200c_0c_1^3c_2^2+(157952c_1^5+225000c_0^4)c_2+450000c_0^3c_1^2)c_4-2916c_0^2c_3^7+(1944c_0c_1c_2+432c_1^3)c_3^6+(-432c_0c_2^3-756c_1^2c_2^2+4860c_0^2c_1)c_3^5+(288c_1c_2^4-34425c_0^2c_2^2+6480c_0c_1^2c_2-8640c_1^4)c_3^4+(-32c_2^6+16470c_0c_1c_2^3+9600c_1^3c_2^2+303750c_0^3c_2+216000c_0^2c_1^2)c_3^3+(-3756c_0c_2^5-7405c_1^2c_2^4-384750c_0^2c_1c_2^2-201600c_0c_1^3c_2+64768c_1^5-928125c_0^4)c_3^2+(2232c_1c_2^6-33000c_0^2c_2^4+294200c_0c_1^2c_2^3-76480c_1^4c_2^2+2475000c_0^3c_1c_2-920000c_0^2c_1^3)c_3-216c_2^8-48240c_0c_1c_2^5+13820c_1^3c_2^4+500000c_0^3c_2^3-1765000c_0^2c_1^2c_2^2+992000c_0c_1^4c_2-148480c_1^6-562500c_0^4c_1),$$

$P_3 = 11664(768c_0^2c_4^6 + (-384c_0c_1c_3 - 256c_0c_2^2 + 144c_1^2c_2)c_4^5 + ((288c_0c_2 - 6c_1^2)c_3^2 + (-80c_1c_2^2 - 5760c_0^2)c_3 + 16c_2^4 + 1184c_0c_1c_2 - 360c_1^3)c_4^4 + (-54c_0c_3^4 + 18c_1c_2c_3^3 + (-4c_2^3 + 2196c_0c_1)c_3^2 + (1008c_0c_2^2 - 656c_1^2c_2)c_3 + 13200c_0^2c_2 - 2800c_0c_1^2)c_4^3 + ((-1350c_0c_2 - 9c_1^2)c_3^3 + (402c_1c_2^2 + 9450c_0^2)c_3^2 + (-80c_2^4 - 9720c_0c_1c_2 + 2500c_1^3)c_3 - 3040c_0c_2^3 + 1416c_1^2c_2^2 - 9000c_0^2c_1)c_4^2 + (243c_0c_3^5 - 81c_1c_2c_3^4 + (18c_2^3 - 1890c_0c_1)c_3^3 + (3960c_0c_2^2 - 492c_1^2c_2)c_3^2 + (-656c_1c_2^3 - 45000c_0^2c_2 + 15000c_0c_1^2)c_3 + 144c_2^5 + 20000c_0c_1c_2^2 - 7360c_1^3c_2)c_4 + (-405c_0c_2 + 216c_1^2)c_3^4 + (-9c_1c_2^2 + 3375c_0^2)c_3^3 + (-6c_2^4 + 900c_0c_1c_2 - 2240c_1^3)c_3^2 + (-2400c_0c_2^3 + 2500c_1^2c_2^2 + 22500c_0^2c_1)c_3 - 360c_1c_2^4 + 30000c_0^2c_2^2 - 40000c_0c_1^2c_2 + 9600c_1^4),$

$P_4 = 19683(768c_0c_4^3 + (-192c_1c_3 - 128c_2^2)c_4^2 + (144c_2c_3^2 - 2880c_0c_3 + 1024c_1c_2)c_4 - 27c_3^4 + 216c_1c_3^2 - 192c_2^2c_3 + 4800c_0c_2 - 2480c_1^2).$

□

3.2.2 The case that ∞ is double critical point

Let $R = \dfrac{P}{Q}$ be a rational map in $CB_4^{(2)}$, and $z^4 + c_3z^3 + \cdots + c_0 = 0$ be the equation defined by $P'(z)Q(z) - P(z)Q'(z) = 0$.

Then, the map $\Phi_4^{(2)} : CB_4^{(2)} \to \mathbb{C}^4$ is defined by sending (a_1, a_2, b, c) to $(c_0, \cdots, c_3)$.

Proposition 8. *The ramification locus of* $\Phi_4^{(2)}$ *is given by* $3b^2 - 2a_2b + a_1 = 0$, $\Phi_4^{(2)}(CB_4^{(2)}) = \mathbb{C}^4 \setminus E^{(2)}(4)$, *and* $\Phi_4^{(2)}$ *is 3-valent on the set of the points in* $\mathbb{C}^4 \setminus E^{(2)}(4)$, *where the defining equation of* $E^{(2)}(4)$ *is given in the proof.*

Proof. The four critical points of R in $\mathbb{C}$ is given as the solution of

$$3z^4 + (6b + 2a_2)z^3 + (3b^2 + 4a_2b + a_1)z^2 + (2a_2b^2 + 2a_1b)z + a_1b^2 - c = 0.$$

Therefore, the map $\Phi_4^{(2)}$ is defined by $(a_1, a_2, b, c) \mapsto (c_0, \cdots, c_3)$, where

$$\begin{aligned} c_0 &= (a_1b^2 - c)/3, \\ c_1 &= (2a_2b^2 + 2a_1b)/3, \\ c_2 &= (3b^2 + 4a_2b + a_1)/3, \\ c_3 &= (6b + 2a_2)/3. \end{aligned} \tag{6}$$

The ramification locus is obtained from the Jacobian of the map $\Phi_4^{(2)}$,

$$3b^2 - 2a_2b + a_1 = 0.$$

For $\boldsymbol{c} \in \mathbb{C}^4 \setminus E^{(2)}(4)$, every $(\Phi_4^{(2)})^{-1}(\boldsymbol{c})$ is given by,

$$\begin{cases} B = 4b^3 - 3c_3b^2 + 2c_2b - c_1 \\ A_1 = -9b^2 + 6c_3b + a_1 - 3c_2 \\ A_2 = 6b + 2a_2 - 3c_3 \\ C = (9c_3^2 - 24c_2)b^2 + (-6c_2c_3 + 36c_1)b - 16c + 3c_1c_3 - 48c_0, \end{cases}$$

which has exactly 2 solutions except for $\text{discriminant}_b(B) = 0$.

The map $\Phi_4^{(2)}$ is not defined on

$$\text{resultant}_z(\text{numerator}(R), \text{denominator}(R)) = c = 0.$$

From (6), for each $(c_0, \cdots, c_3)$, c is determined by the equation,

$$\begin{aligned}
&256c^3 - 3(27c_3^4 - 144c_2c_3^2 + 192c_1c_3 + 128c_2^2 - 768c_0)c^2 \\
&\quad - 18(27c_0c_3^4 - 9c_1c_2c_3^3 + (2c_2^3 - 144c_0c_2 + 3c_1^2)c_3^2 + (40c_1c_2^2 + 192c_0c_1)c_3 \\
&\quad - 8c_2^4 + 128c_0c_2^2 - 72c_1^2c_2 - 384c_0^2)c \\
&\quad - 27(27c_0^2c_3^4 + (-18c_0c_1c_2 + 4c_1^3)c_3^3 + (4c_0c_2^3 - c_1^2c_2^2 - 144c_0^2c_2 + 6c_0c_1^2)c_3^2 \\
&\quad + (80c_0c_1c_2^2 - 18c_1^3c_2 + 192c_0^2c_1)c_3 - 16c_0c_2^4 + 4c_1^2c_2^3 + 128c_0^2c_2^2 - 144c_0c_1^2c_2 \\
&\quad + 27c_1^4 - 256c_0^3 = 0.
\end{aligned}$$

Therefore, the exceptional set $E^{(2)}(4)$ corresponds to the condition that this equation has 0 as a unique solution.

Hence, the defining equation of $E^{(2)}(4)$ is

$$P_0 = P_1 = P_2 = 0,$$

where

$$\begin{aligned}
P_0 &= -729c_0^2c_3^4 + (486c_0c_1c_2 - 108c_1^3)c_3^3 \\
&\quad + (-108c_0c_2^3 + 27c_1^2c_2^2 + 3888c_0^2c_2 - 162c_0c_1^2)c_3^2 \\
&\quad + (-2160c_0c_1c_2^2 + 486c_1^3c_2 - 5184c_0^2c_1)c_3 + 432c_0c_2^4 - 108c_1^2c_2^3 \\
&\quad - 3456c_0^2c_2^2 + 3888c_0c_1^2c_2 - 729c_1^4 + 6912c_0^3, \\
P_1 &= -486c_0c_3^4 + 162c_1c_2c_3^3 + (-36c_2^3 + 2592c_0c_2 - 54c_1^2)c_3^2 \\
&\quad + (-720c_1c_2^2 - 3456c_0c_1)c_3 + 144c_2^4 - 2304c_0c_2^2 + 1296c_1^2c_2 + 6912c_0^2, \\
P_2 &= -81c_3^4 + 432c_2c_3^2 - 576c_1c_3 - 384c_2^2 + 2304c_0.
\end{aligned}$$

□

3.2.3 The case that ∞ is triple critical point

Let R be a polynomial map in $CB_4^{(3)}$, $z^3 + c_2z^2 + c_1z + c_0 = 0$ be the equation defined by $R'(z) = 0$.

Then, the map $\Phi_4^{(3)} : CB_4^{(3)} \to \mathbb{C}^3$ is defined by sending (a_1, a_2, a_3) to (c_0, c_1, c_2).

Proposition 9. *The map $\Phi_4^{(3)}$ is bijective.*

Proof. The three critical points of R in $\mathbb{C}$ is given as the solution of the following equation

$$4z^3 + 3a_3z^2 + 2a_2z + a_1 = 0.$$

Therefore, the map $\Phi_4^{(3)}$ is defined by

$$(a_1, a_2, a_3) \mapsto (c_0, c_1, c_2) = (\frac{a_1}{4}, \frac{2a_2}{4}, \frac{3a_3}{4}),$$

and the assertion follows. □

For $d = 3,\ 4$, the complete answer for the problem of Goldberg is obtained.

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