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The Relation between Normal Play Chocolate Games and Misere Play Chocolate Games that Satisfy Inequality $y \leq \lfloor\frac{x+z}{k}\rfloor$

宮寺良平
関西学院高等部*
RYOHEI MIYADERA
KWANSEI GAKUIN

1 Introduction

In this article the author presents the result of a high school mathematics research project. The author and his students studied chocolate games that are variants of the game of Nim.
The game of Nim is well known combinatorial game, and it is a very good topic for the research by computer algebra systems such as Mathematica.
By using Mathematica program and graphics ability of Mathematica the author and his students discovered interesting facts.
They studied the relation between normal play chocolate games and misere play chocolate games. In [1], [2], [3] and [4] the authors studied normal play chocolate games that satisfy inequality $y \leq \lfloor\frac{x+z}{k}\rfloor$, and they discovered formulas for L states when $k = 1, 3$, but they did not find any formula when $k \neq 1, 3$. It seems that the research on the chocolate problem that satisfies the inequality $y \leq \lfloor\frac{x+z}{k}\rfloor$ for arbitrary positive number k is quite difficult, but the authors discovered that the L states of normal play chocolate games and misere play chocolate games are almost the same when $k > 2$.
The authors used the computer algebra system Mathematica a lot in their research on chocolate games, because it is easy to make a Mathematica program to calculate L states of the games.

2 The chocolate problems that satisfy inequality $y \leq \lfloor\frac{x+z}{k}\rfloor$

In this article we study two types of games. Normal play chocolate games are defined by Definition 1 and misere type of chocolate games are defined by Definition 2.

Definition 1
Pieces of chocolate are of two types, sweet and very bitter. Two players in turn break the chocolate in a straight line along the grooves and eat the piece broken off.

*miyadera1272000@yahoo.co.jp

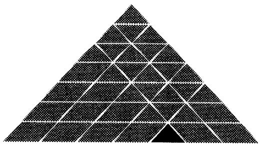
The winner is the player who leaves their opponent with the single bitter part.
In this article the graphs of the chocolate problems use the convention that light gray parts are sweet, dark gray parts are bitter.
This is a normal play chocolate game, and the player who makes the last move wins.

Definition 2
Pieces of chocolate are of two types, sweet and very bitter. The players cannot eat the very bitter part. Two players in turn break the chocolate in a straight line along the grooves and eat the piece broken off. The player who makes the last move loses, and hence this is a misere chocolate game
In this article the graphs of the chocolate problems use the convention that light gray parts are sweet, dark gray parts are bitter.

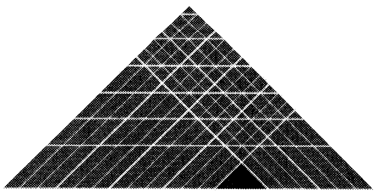
Definition 3
Here we define two important states of chocolates.
(a) W States, from which we can force a win, as long as we play correctly at every stage.
(b) L States, from which we will lose however well we play, but we may end up winning if our opponents make a mistake.

In this section we deal with only the normal play games of Definition 1.

Example 1
We can study many kinds of chocolate problems.



Graph 2.1
{4,6,2}



Graph 2.2
{12,6,6}

Definition 4
Suppose that a chocolate can be cut end-to-end from the upper right to the lower left at most x times, horizontally at most y times, and from the upper left to the lower right at most z times for some non-negative numbers x, y, z.
Then we represent this chocolate with three numbers $\{x, y, z\}$, and we call these numbers the coordinates of the chocolate.

Remark 1
In Example 1 we attached coordinates to each chocolate.
In a normal play chocolate game the player who makes the last move (i.e., who moves to $\{0,0,0\}$) wins, and in misere play game the player who makes the last move (i.e., who moves to $\{0,0,0\}$) loses.
The chocolate problems in Graph 2.1 and Graph 2.2 satisfy the inequalities $y \leq x+z$ and $y \leq \lfloor \frac{x+z}{3} \rfloor$ respectively, and in this article we only study the chocolate games that satisfy inequality $y \leq \lfloor \frac{x+z}{k} \rfloor$ for some positive number k.

Here we define the function $move1k(\{x,y,z\})$. $move1k(\{x,y,z\})$ is the list of all the states that can be reached from the state $\{x,y,z\}$ in one step (directly).

Definition 5
We define $move1k(\{x,y,z\}) = \{\{x', Min(y, \lfloor \frac{x'+z}{k} \rfloor), z\}; x' < x\} \cup \{\{x, Min(y, \lfloor \frac{x+z'}{k} \rfloor), z'\}; z' < z\}$ $\cup \{\{x, y', z\}; y' < y\}$.

Example 2
This is a Mathematica program to calculate L states of the chocolate game for $k = 3$.
The program for chocolate game is easy to make once we define the function movek( ) properly.

```
k =3; ss = 20; al =
 Flatten[Table[{a, b, c}, {a, 0, Floor[1.5 ss]}, {b, 0, 2 ss},
{c,0,Floor[1.5 ss]}], 2];
allcases = Select[al, (1/k) (#[[1]] + #[[3]]) >= #[[2]] &];
move[z_] := Block[{p}, p = z;
Union[Table[{t1, Min[Floor[(1/k) (t1 + p[[3]])], p[[2]]],
p[[3]]}, {t1, 0, p[[1]] - 1}],
Table[{p[[1]], t2, p[[3]]}, {t2, 0, p[[2]] - 1}],
Table[{p[[1]], Min[Floor[(1/k) (t3 + p[[1]])], p[[2]]], t3},
{t3,0, p[[3]] - 1}]]];
Mex[L_] := Min[Complement[Range[0, Length[L]], L]];
Gr[pos_] := Gr[pos] = Mex[Map[Gr, move[pos]]];
ppositionxy = Select[allcases, Gr[#] == 0 &];
```

To more easily describe the set of L states we utilize the nim-sum notation $x \oplus y$ for any non-negative integers x, y.

Definition 6
Let x, y be non-negative integers, and write them in base 2, so $x = \sum_{i=0}^{n} x_i 2^i$ and $y = \sum_{i=0}^{n} y_i 2^i$ with $x_i, y_i \in \{0,1\}$. We define the nim-sum $x \oplus y$ by

$$x \oplus y = \sum_{i=0}^{n} w_i 2^i, where w_i = x_i + y_i \ (mod\ 2). \tag{1}$$

Theorem 7
In the game of Graph 2.1 we have
(1) The state $\{x, 0, z\}$ is a L state if and only if $x = z$.
(2) The state $\{x+1, y, z+1\}$ is a L state if and only if $x \oplus y \oplus z = 0$.

Proof For a proof see [1]. ∎

Theorem 8
In the game of Graph 2.2
$\{\{x,y,z\}; x, y, z \in Z_{\geq 0}, x \oplus y \oplus z = 0\}$ is the set of L states.

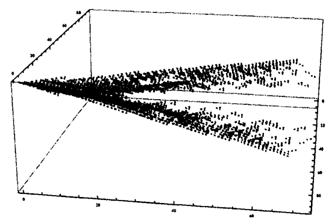
Proof For a proof see [4]. ∎

Conjecture 2.1
In the game that satisfies inequality $y \leq \lfloor \frac{x+z}{k} \rfloor$ *for an odd number* $k \geq 3$
$\{\{x,y,z\}; x,y,z \in Z_{\geq 0}, x \oplus y \oplus z = 0\}$ *is the set of L states.*

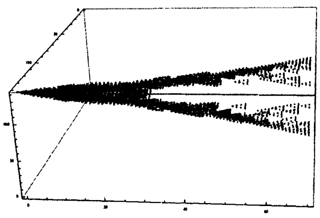
By the calculation of Mathematica Conjecture 2.1 seems true, but we have only proved the case of $k = 3$.

3 The relation between the normal games and misere play games that satisfy inequality $y \leq \lfloor \frac{x+z}{k} \rfloor$

Example 3
Graph 3.3 and Graph 3.4 are the 3D graph of the set of L states of the normal play game and the misere play game with $k = 2$ *respectively. Although there is a difference between them, the difference is not clear.*



Graph 3.1



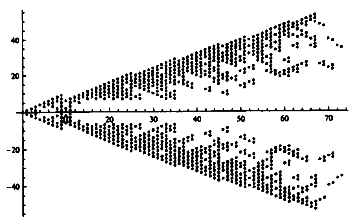
Graph 3.2

We define two transformations that are useful in studying the difference between Graph 3.3 and Graph 3.4.

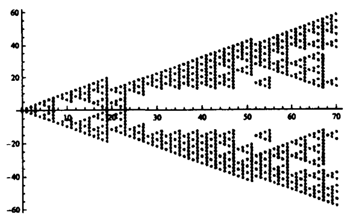
Definition 9
Under the following transformations (1) *and* (2) *we can map the point* (a,b,c) *to* $(b, \frac{c-a}{\sqrt{2}})$.
(1) *The rotation of the point by* $\frac{\pi}{4}$ *radians around the second coordinate (or the vector* $(0,1,0)$ *).*
(2) *The projection of the point to the plane made by the second and the third coordinates.*

Remark 2
We discovered transformations in Definition 9 by rotating the Graph 3.3 and Graph 3.4 using Mathematica.

Example 4
We project the sets of L states of the normal play game and the misere play game with $k = 2$ *onto* R^2 *by using transformations* (1) *and* (2), *then the projected images are Graph 3.3 and Graph 3.4 respectively. There is a clear difference between them.*



Graph 3.3



Graph 3.4

In the remainder of this section we assume that $k > 2$. Note that k can be arbitrary positive number.

In this section we study chocolate problems that satisfy inequality $y \leq \lfloor \frac{x+z}{k} \rfloor$, and when we treat $\{x, y, z\}$ for any non-negative numbers x, y, z, then we assume that $y \leq \lfloor \frac{x+z}{k} \rfloor$.

Definition 10
Let C and D be the sets of L states and W states of the game respectively. Clearly $\{0,0,0\}, \{1,0,1\} \in C$ and $\{1,0,0\}, \{0,0,1\} \in D$. Let $C_0 = C - \{\{0,0,0\}, \{1,0,1\}\}$ and $D_0 = D - \{\{1,0,0\}, \{0,0,1\}\}$.

Theorem 11
$\{x, y, z\} \in C$ if and only if

$$movelk(\{x, y, z\}) \subset D. \tag{2}$$

$\{x, y, z\} \in D$ if and only if

$$movelk(\{x, y, z\}) \cap C \neq \emptyset. \tag{3}$$

Proof From a L state any move leads to a W state, and from a W state there always exists a move that leads to a L state. Therefore this is direct from the definition of C and D. ∎

Lemma 12
Let $k > 2$. If $x, z \geq 1$, then $\{0, y, z\}, \{x, y, 0\} \in D$. If $x, z \geq 2$, then $\{x, y, 1\}, \{1, y, z\} \in D$.

Proof By Definition 5 it is clear that $\{0,0,0\} \in movelk(\{0, y, z\})$. Since $\{0,0,0\} \in C$, by Theorem 11 $\{0, y, z\} \in D$. Similarly $\{x, y, 0\} \in D$.
Since $k > 2$, by Definition 5 it is clear that $\{1,0,1\} \in movelk(\{1, y, z\})$. Since $\{1,0,1\} \in C$, by Theorem 11 $\{1, y, z\} \in D$. Similarly $\{x, y, 1\} \in D$. ∎

Lemma 13
If $\{u, v, w\} \in C_0$, then $u, w \geq 2$.

Proof If $u \leq 1$ or $w \leq 1$, then $\{u, v, w\} \in \{\{0, y, z\}; z \geq 1\} \cup \{\{x, y, 0\}; x \geq 1\} \cup \{\{x, y, 1\}; x \geq 2\} \cup \{\{1, y, z\}; z \geq 2\} \cup \{\{0,0,0\}, \{1,0,1\}\}$.
Therefore by the fact that $D \cap C_0 = \emptyset$, $\{0,0,0\}, \{1,0,1\} \notin C_0$, and Lemma 12 we get this lemma. ∎

Lemma 14
Let $\{x, y, z\} \in D_0$. If $\{0,0,0\}$ or $\{1,0,1\} \in movelk(\{x, y, z\})$, then we have $\{1,0,0\}$ or $\{0,0,1\} \in movelk(\{x, y, z\})$.

Proof If $\{0,0,0\} \in move1k(\{x,y,z\})$, then we have $\{x,y,z\} = \{0,y,z\}$ or $\{x,y,0\}$.
Since $\{1,0,0\},\{0,0,1\} \notin D_0$, $\{x,y,z\} = \{0,y,z\}$ with $z \geq 2$ or $\{x,y,0\}$ with $x \geq 2$. Then $\{0,0,1\}$ or $\{1,0,0\} \in move1k(\{x,y,z\})$.
If $\{1,0,1\} \in move1k(\{x,y,z\})$, then $\{x,y,z\} = \{1,y,z\}$ with $z \geq 2$ or $\{x,y,1\}$ with $x \geq 2$. Clearly we have $\{1,0,0\}$ or $\{0,0,1\} \in move1k(\{x,y,z\})$. ∎

Let $CM = C_0 \cup \{\{1,0,0\},\{0,0,1\}\}$ and $DM = D_0 \cup \{\{0,0,0\},\{1,0,1\}\}$.

Theorem 15
CM and DM are the sets of L states and W states of misere play chocolate game of Definition 2.

Proof [1] Let $\{x,y,z\} \in CM$ and $\{p,q,r\} \in move1k(\{x,y,z\})$.
[1.1] If $\{x,y,z\} \in C_0$, by Definition 10 and Theorem 11 we have $\{p,q,r\} \in D$. Since $\{x,y,z\} \in C_0$, by Lemma 13 $x,z \geq 2$. Therefore $\{p,q,r\} \neq \{1,0,0\},\{0,0,1\}$ and $\{p,q,r\} \in D_0 \subset DM$.
[1.2] If $\{x,y,z\} = \{1,0,0\}$ or $\{0,0,1\}$, then $\{p,q,r\} = \{0,0,0\} \in DM$. We proved that

$$move1k(\{x,y,z\}) \subset DM. \quad (4)$$

[2] Let $\{x,y,z\} \in DM$.
[2.1] If $\{x,y,z\} \in D_0$, by Definition 10 and Theorem 11 there exists $\{p,q,r\} \in move1k(\{x,y,z\}) \cap C$.
[2.1.1] If $\{p,q,r\} = \{0,0,0\}$ or $\{1,0,1\}$. Then by Lemma 14 $\{1,0,0\}$ or $\{0,0,1\} \in move1k(\{x,y,z\}) \cap CM$.
[2.1.2] If $\{p,q,r\} \neq \{0,0,0\}$ or $\{1,0,1\}$, then $\{p,q,r\} \in move1k(\{x,y,z\}) \cap C_0 \subset move1k(\{x,y,z\}) \cap CM$.
[2.2] If $\{x,y,z\} = \{0,0,0\}$, then the player does not have to move. This state $\{x,y,z\}$ is W-state.
[2.3] If $\{x,y,z\} = \{1,0,1\}$, then $\{1,0,0\}$ and $\{0,0,1\} \in move1k(\{x,y,z\}) \cap CM$.
We proved that

$$move1k(\{x,y,z\}) \cap CM \neq \emptyset. \quad (5)$$

By (4) and (5) we finish the proof of this theorem. ∎

Remark 3
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