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Description of a mean curvature sphere of a surface by quaternionic holomorphic geometry

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1 Introduction

In this paper, we collect definitions and propositions from the surface theory in terms of quaternions. These are selected so that they complement the paper [7]. Proofs are omitted. The details are described in [2], [3] and [5].

2 Mean curvature spheres

We explain the notion of a mean curvature sphere of a conformal map.

2.1 Sphere congruences

We model S^4 on the quaternionic projective line $\mathbb{H}P^1$. Set

$$\mathcal{Z} := \{C \in \operatorname{End}(\mathbb{H}^2) \mid C^2 = -\operatorname{Id}\}.$$

This is the set of all quatenionic linear complex structures of $\mathbb{H}^2$. Then two-spheres are parametrized by $\mathcal{Z}$:

Lemma 1 ([2], Proposition 2).

$$\{\text{oriented two-spheres in } \mathbb{H}P^1\} = \mathcal{Z}.$$

In a classical terminology, a sphere congruence is a smooth family of two-spheres. Hence a map from a Riemann surface M to $\mathcal{Z}$ is a sphere congruence in $\mathbb{H}P^1$ parametrized by M.

2.2 Mean curvature spheres

Let M be a Riemann surface with complex structure J and $f\colon M \to \mathbb{R}^4$ a conformal map.

Definition 1. At a point $p \in M$, a two-sphere in M is called the mean curvature sphere of f at p if

- the sphere is tangent to $f(M)$ at p,
- the sphere is centered in the direction of the mean curvature vector at p, and
- the radius of the sphere is equal to the reciprocal of the norm of the mean curvature vector at p.

A sphere congruence parametrized by M which consists of the mean curvature spheres of f is called the mean curvature sphere of f.

We see that f is the envelop of the mean curvature sphere of f. The mean curvature of f at $p \in M$ is equal to the mean curvature of the mean curvature sphere of f at p.

Let $\mathcal{S}$ be the mean curvature sphere of f and τ a conformal transformation of $\mathbb{R}^4$. Then $\tau \circ \mathcal{S}$ is the mean curvature sphere of $\tau \circ f$. Hence the mean curvature sphere is a concept for conformal geometry of surfaces in S^4. For a conformal map $f\colon M \to S^4 \cong \mathbb{H}P^1$, the mean curvature sphere is a map from M to $\mathcal{Z}$.

2.3 Conformal Gauss maps

A mean curvature sphere is called a conformal Gauss map in [1]. This terminology is valid as follows. For $C \in \mathrm{End}(\mathbb{H}^2)$, we set $\langle C \rangle := \frac{1}{8} \operatorname{tr}_{\mathbb{R}} C$. Then an indefinite scalar product $\langle\ ,\ \rangle$ of $\mathrm{End}(\mathbb{H}^2)$ is defined by setting $\langle C_1, C_2 \rangle := \langle C_1 C_2 \rangle$ for C_1, $C_2 \in \mathrm{End}(\mathbb{H}^2)$.

Lemma 2 ([1], [2], Proposition 4)**.** The mean curvature sphere $\mathcal{S}$ of a conformal map $f\colon M \to S^4$ is conformal with respect to $\langle\ ,\ \rangle$.

2.4 Energy of a sphere congruence

Let $\mathcal{C}\colon M \to \mathcal{Z}$ be a sphere congruence. For a one-form ω on M, we set $*\omega := \omega \circ J$.

Definition 2 ([2], Definition 7)**.**

$$E(\mathcal{C}) := \int_M \langle d\mathcal{C} \wedge * d\mathcal{C} \rangle$$

is called the energy of a sphere congruence.

Because $\langle\ ,\ \rangle$ is indefinite, the functional E might take negative values. Set $A_{\mathcal{C}} := \frac{1}{4}(* d\mathcal{C} + \mathcal{C}\, d\mathcal{C})$. The Euler-Lagrange equation of $E(\mathcal{C})$ is written by the one-form $A_{\mathcal{C}}$.

Proposition 1 ([2], Proposition 5)**.** A sphere congruence $\mathcal{C}$ is harmonic if and only if $d{*}A_{\mathcal{C}} = 0$.

3 Associated vector bundles

We explain a conformal map in terms of vector bundles.

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3.1 Conformal maps

Let $\underline{\mathbb{H}}^2$ be the trivial right quaternionic vector bundle over M of rank two. We consider a standard basis e_1, e_2 of $\mathbb{H}^2$ as a section of $\underline{\mathbb{H}}^2$. Then $de_1 = de_2 = 0$. A conformal map $f\colon M \to \mathbb{H}P^1$ with mean curvature sphere $\mathcal{S}$ is translated in terms of vector bundles as Table 1 (See [2], Section 4, Section 5).

map	vector bundle
$f\colon M \to \mathbb{H}P^1$: map	$L \subset \underline{\mathbb{H}}^2$: quaternionic line subbundle $L_p = f(p)$
$df\colon TM \to T\mathbb{H}P^1$	$\pi\colon \underline{\mathbb{H}}^2 \to \underline{\mathbb{H}}^2/L$: projection $\delta := \pi d\|_{\Gamma(L)}$
f: conformal $\mathcal{S}$: the mean curvature sphere	$\mathcal{S}L = L$ $*\,\delta = \mathcal{S}\,\delta = \delta\,\mathcal{S}\|_{\Gamma(L)}$

Table 1: Vector bundles

3.2 The Willmore functional

Let L be a conformal map with mean curvature sphere $\mathcal{S}$.

Definition 3 ([2], Definition 8).

$$W(L) := \frac{1}{\pi}\int_M \langle A_{\mathcal{S}} \wedge *A_{\mathcal{S}} \rangle$$

is called the Willmore energy of L.

Lemma 3 ([2], Lemma 8). For any conformal map L, the functional W takes non-negative values.

A cirtical conformal map of the Willmore functional is called a Willmore conformal map.

Theorem 1 ([4], [8], [2]). A conformal map with mean curvature sphere $\mathcal{S}$ is Willmore if and only if $\mathcal{S}$ is harmonic.

By Proposition 1, the mean curvature sphere $\mathcal{S}$ is harmonic if and only if $d * A_{\mathcal{S}} = 0$.

We connect the above discussion with the classical terminology. Let L be a conformal map and $f\colon M \to \mathbb{H}$ a stereographic projection of S^4 followed by L. We induce a (singular) metric on M by a conformal map $f\colon M \to \mathbb{H}$. Let K be the Gauss curvature, $K^{\perp}$ the normal curvature, and $\mathcal{H}$ the mean curvature vector of f.

Lemma 4 ([2], Example 19).

$$W(L) = \frac{1}{4\pi}\int_M (|\mathcal{H}|^2 - K - K^{\perp})|df|^2.$$

4 Transforms

We explain transforms of conformal maps and sphere congruences.

4.1 Darboux transforms

Let L be a conformal map with mean curvature sphere $\mathcal{S}$. For $\phi \in \Gamma(\underline{\mathbb{H}}^2/L)$, we denote by $\hat{\phi} \in \Gamma(\underline{\mathbb{H}}^2)$ a lift of ϕ, that is $\pi\hat{\phi} = \phi$. Set

$$D(\phi) := \frac{1}{2}(\pi\, d\hat{\phi} + \mathcal{S} * \pi\, d\hat{\phi}).$$

We denote by $\widetilde{M}$ the universal covering of M. Similarly, for an object B defined on M, we denote by $\widetilde{B}$ for the object induced from B by the universal covering map of M.

Theorem 2 ([3], Lemma 2.1). Let $\phi \in \Gamma(\widetilde{\underline{\mathbb{H}}^2/L})$. If $\widetilde{D}(\phi) = 0$, then there exists $\widehat{\phi} \in \Gamma(\widetilde{\underline{\mathbb{H}}^2})$ uniquely such that $\widetilde{\pi}\widetilde{d\phi} = 0$. The line bundle $\widehat{\widetilde{L}} := \underline{\widehat{\phi}\mathbb{H}}$ is conformal

Definition 4 ([3], Definition 2.2). The line bundle $\widehat{\widetilde{L}}$ in the above theorem is called the Darboux transform of L.

4.2 μ-Darboux transforms

Let $\mathcal{C}\colon M \to \mathcal{Z}$. We set $I\phi := \phi i$. We identify $\mathbb{H}^2$ with $\mathbb{C}^4$ by taking I as a complex structure.

Theorem 3 ([5], Theorem 4.1). The sphere congruence $\mathcal{C}$ is harmonic if and only if $d_\lambda := d + (\lambda - 1)A_{\mathcal{C}}^{(1,0)} + (\lambda^{-1} - 1)A_{\mathcal{C}}^{(0,1)}$ is flat for all $\lambda \in \mathbb{C} \setminus \{0\}$

Definition 5. We call d_λ the associated family of d.

Theorem 4 ([5], Theorem 4.2). We assume that $\mathcal{C}\colon M \to \mathcal{Z}$ is harmonic, $A_{\mathcal{C}} \neq 0$, $\mu \in \mathbb{C}\setminus\{0\}$, ψ_1, $\psi_2 \in \Gamma(\underline{\mathbb{H}}^2)$ are linearly independent over $\mathbb{C}$, $d_\mu\psi_1 = d_\mu\psi_2 = 0$, $W_\mu := \mathrm{span}\{\psi_1, \psi_2\}$, and $\Gamma(\underline{\mathbb{H}}^2) = W_\mu \oplus jW_\mu$. Then for $G := (\psi_1, \psi_2)\colon M \to \mathrm{GL}(2, \mathbb{H})$, $a = G\left(\frac{\mu+\mu^{-1}}{2}E_2\right)G^{-1}$, $b = G\left(I(\frac{\mu^{-1}-\mu}{2}E_2)\right)G^{-1}$, and $T := \mathcal{C}(a-1)+b$, the sphere congruence $\widehat{\mathcal{C}} := T^{-1}\mathcal{C}T\colon M \to \mathcal{Z}$ is harmonic.

Definition 6 ([5]). The sphere congruence $\widehat{\mathcal{C}}$ is called the μ-Darboux transform of $\mathcal{C}$.

It is known that a μ-Darboux transform is a Darboux transform.

Let $\mathcal{S}$ be a mean curvature sphere of a Willmore conformal map L. Then $\mathcal{S}$ is harmonic by Theorem 1. Hence a harmonic sphere congruence $\widehat{\mathcal{S}}$ is defined.

Theorem 5 ([5], Theorem 4.4). Let L be a Willmore conformal map with harmonic mean cuvature sphere $\mathcal{S}$ such that $A_{\mathcal{S}} \neq 0$. Then, $\widehat{L} := T(a-1)^{-1}L$ is a Willmore conformal map and $\widehat{\mathcal{S}}$ is the mean curvature sphere of $\widehat{L}$.

Hence a μ-Darboux transform of a mean curvature sphere induces a transform of a Willmore conformal map.

4.3 Simple factor dressing

Let L be a conformal map with the mean curvature sphere $\mathcal{S}$. Because $\mathcal{S}$ is a harmonic sphere congruence, the associated family d_λ is defined. We assume that $r_\lambda\colon M \to \mathrm{GL}(4,\mathbb{C})$ is a map parametrized by $\lambda \in \mathbb{C}\setminus\{0\}$ such that, with respect to λ, it is meromorphic with the only simple pole on $\mathbb{C}\setminus\{0\}$ and holomorphic at 0 and ∞.

Definition 7 ([6]). If $\hat{d}_\lambda := r_\lambda \circ d_\mu \circ r_\lambda^{-1}$ is an associated family of a harmonic map $\hat{\mathcal{C}}$, then $\hat{\mathcal{C}}$ is called a simple factor dressing of $\mathcal{C}$.

A simple factor dressing is a harmonic map.

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