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Spherical functions on the space of p-adic unitary hermitian matrices

Yumiko Hironaka
Department of Mathematics,
Faculty of Education and Integrated Sciences, Waseda University
Nishi-Waseda, Tokyo, 169-8050, JAPAN,
and
Yasushi Komori
Department of Mathematics,
Faculty of Science, Rikkyo University
Nishi-Ikebukururo, Tokyo, 171-8501, JAPAN

§0 Introduction

Let k' be an unramified quadratic extension of a $\mathfrak{p}$-adic field k, and we consider hermitian and unitary matrices with respect to k'/k. For a matrix $A = (a_{ij}) \in M_{mn}(k')$, we denote by $A^* \in M_{nm}(k')$ the conjugate transpose with respect to k'/k, and say A is *hermitian* if $A^* = A$. We introduce the unitary group and the space of unitary hermitian matrices:

$$G = U(j_{2n}) = \{g \in GL_{2n}(k') \mid g^* j_{2n} g = j_{2n}\}, \quad j_{2n} = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix} \in GL_{2n}(k'),$$

$$X = \left\{ x \in G \;\middle|\; x^* = x,\ \Phi_{xj_{2n}}(t) = (t^2-1)^n \right\},$$

where $\Phi_y(t)$ is the characteristic function of the matrix y. The group G acts on X by

$$g \cdot x = gxg^*, \quad (g \in G,\ x \in X).$$

We take the maximal compact subgroup $K = G \cap GL_{2n}(\mathcal{O}_{k'})$ and the Borel subgroup B consisting of all upper triangular matrices in G, then $G = BK = KB$. In the following,

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E-mail: hironaka@waseda.jp, komori@rikkyo.ac.jp
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we fix a prime element π in k and the absolute value $|\ |$ on k normalized by $|\pi|^{-1} = q = \sharp(\mathcal{O}_k/(\pi))$.

In §1, we study K-orbits and G-orbits in X and obtain (cf. Theorem 1.2, Theorem 1.3, and Theorem 1.4)

Theorem 1 (1) If k has odd residual characteristic, one has

$$X = \bigsqcup_{\lambda \in \Lambda_n^+} K \cdot x_\lambda, \quad x_\lambda = Diag(\pi^{\lambda_1}, \dots, \pi^{\lambda_n}, \pi^{-\lambda_n}, \dots, \pi^{-\lambda_1}) \in X, \tag{0.1}$$

where

$$\Lambda_n^+ = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}$$

(2) If k has even residual characteristic, there are K-orbits not of type $K \cdot x_\lambda,\ \lambda \in \Lambda_n^+$.
(3) There are precisely two G-orbits in X, independent of the residual characteristic of k.

In §2, we introduce a spherical function $\omega(x; s)$ for $x \in X$ and $s \in \mathbb{C}^n$:

$$\omega(x; s) = \int_K \prod_{i=1}^n |d_i(k \cdot x)|^{s_i + \varepsilon_i}\, dk, \tag{0.2}$$

where $d_i(y)$ is the determinant of the lower right i by i block of y, $\varepsilon \in \mathbb{C}^n$ is a certain fixed number, dk is the Haar measure on K. The above integral is absolutely convergent if $\mathrm{Re}(s_i) \geq \mathrm{Re}(\varepsilon_i),\ 1 \leq i \leq n$, continued to a rational function of $q^{s_1}, \dots, q^{s_n}$, and becomes a K-invariant function on X, hence $\omega(x; s) \in \mathcal{C}^\infty(K\backslash X)$ for each $s \in \mathbb{C}^n$. It is convenient to introduce a new variable $z \in \mathbb{C}^n$ which is related to s by

$$s_i = -z_i + z_{i+1},\ (1 \leq i \leq n), \quad s_n = -z_n, \tag{0.3}$$

and we write $\omega(x; z) = \omega(x; s)$.

Hereafter, we assume that k has odd residual characteristic, i.e. q is odd. Let W be the Weyl group of G with respect to the maximal k-split torus in B. Then W acts on rational characters on B so does on s and z, and we obtain (cf. Theorem 2.5, Theorem 2.6)

Theorem 2 (1) *For every $\sigma \in W$, one has*

$$\omega(x; z) = \Gamma_\sigma(z) \cdot \omega(x; \sigma(z)),$$

where

$$\Gamma_\sigma(z) = \prod_{\alpha \in \Sigma_s^+(\sigma)} \frac{1 - q^{\langle \alpha, z\rangle - 1}}{q^{\langle \alpha, z\rangle} - q^{-1}},$$

Σ_s^+ is the set of positive roots and $\Sigma_s^+(\sigma) = \Sigma_s^+ \cap \sigma(-\Sigma_s^+)$.

(2) *One has*

$$\prod_{\alpha \in \Sigma_s^+} \frac{1 + q^{\langle \alpha, z\rangle}}{1 - q^{\langle \alpha, z\rangle - 1}} \times \omega(x; z) \in \mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W (= \mathcal{R},\ say).$$

In §3, we will give the explicit formula for $\omega(x_\lambda; z)$ for each $\lambda \in \Lambda_n^+$ (Theorem 3.1) by a method introduced in [H4], which is based on functional equations of $\omega(x; z)$ and some data depending only on the group G. Since $\omega(x; z)$ is K-invariant for x, it is enough to consider the explicit formula for x_λ, $\lambda \in \Lambda_n^+$ by Theorem 1.

Theorem 3 *For each $\lambda \in \Lambda_n^+$, one has*

$$\omega(x_\lambda; z) = \frac{(1-q^{-2})^n}{w_{2n}(-q^{-1})} \cdot \prod_{\alpha \in \Sigma_s^+} \frac{1-q^{\langle \alpha, z\rangle - 1}}{1+q^{\langle \alpha, z\rangle}} \cdot c_\lambda \cdot Q_\lambda(z), \tag{0.4}$$

where $G(z)$ is the same as in Theorem 2,

$$c_\lambda = (-1)^{\sum_i \lambda_i(n-i+1)} q^{-\sum_i \lambda_i (n-i+\frac{1}{2})}, \quad w_m(t) = \prod_{i=1}^{m}(1-t^i),$$

$$Q_\lambda(z) = \sum_{\sigma \in W} \sigma\left(q^{-\langle \lambda, z\rangle} c(z)\right), \quad c(z) = \prod_{\alpha \in \Sigma_s^+} \frac{1+q^{\langle \alpha, z\rangle - 1}}{1-q^{\langle \alpha, z\rangle}} \prod_{\alpha \in \Sigma_\ell^+} \frac{1-q^{\langle \alpha, z\rangle - 1}}{1-q^{\langle \alpha, z\rangle}}.$$

By Theorem 2, we see $Q_\lambda(z)$ is a polynomial in $\mathcal{R}$. On the other hand, this is a specialization of Macdonald polynomial P_λ, and it is known that the set $\{Q_\lambda(z) \mid \lambda \in \Lambda_n^+\}$ forms a $\mathbb{C}$-basis for $\mathcal{R}$ and $Q_0(z)$ is a constant. Hence, we have

$$\omega(1_{2n}; z) = \frac{(1-q^{-1})^n w_n(-q^{-1})^2}{w_{2n}(-q^{-1})} \times \prod_{\alpha \in \Sigma_s^+} \frac{1-q^{\langle \alpha, z\rangle - 1}}{1+q^{\langle \alpha, z\rangle}}. \tag{0.5}$$

In §4, we consider the spherical Fourier transform on the Schwartz space $\mathcal{S}(K\backslash X)$:

$$\begin{array}{rccl} F: & \mathcal{S}(K\backslash X) & \longrightarrow & \mathcal{R} \\ & \varphi & \longmapsto & F(\varphi) = \int_X \varphi(x)\Psi(x; z)dx, \end{array}$$

where $\Psi(x; z) = \omega(x; z)/\omega(1_{2n}; z)$ and dx is a G-invariant measure on X. We obtain the following (cf. Theorem 4.1, Theorem 4.2, Theorem 4.5).

Theorem 4 (1) The spherical Fourier transform F is an $\mathcal{H}(G, K)$-module isomorphism, in particular, $\mathcal{S}(K\backslash X)$ is a free $\mathcal{H}(G, K)$-module of rank 2^n.

(2) For each $z \in \mathbb{C}^n$, the set $\left\{ \Psi(x; z+u) \;\middle|\; u \in \{0, \frac{\pi\sqrt{-1}}{\log q}\}^n \right\}$ forms a basis for the spherical functions on X corresponding to λ_z.

(3) (Plancherel formula) We give explicitly the normalization of dx on X and a measure $d\mu(z)$ on

$$\mathfrak{a}^* = \left\{\sqrt{-1}\left(\mathbb{R}/\frac{2\pi}{\log q}\mathbb{Z}\right)\right\}^n,$$

for which

$$\int_X \varphi(x)\overline{\psi(x)}dx = \int_{\mathfrak{a}^*} F(\varphi)(z)\overline{F(\psi)(z)}d\mu(z), \quad (\varphi,\ \psi \in \mathcal{S}(K\backslash X)).$$

In [H5], we have investigated spherical functions on a similar space X_T associated to each nondegenerate hermitian matrix T, and obtained functional equations of hermitian Siegel series as an application. Both spaces, X_T and the present X, are isomorphic to $U(2n)/U(n) \times U(n)$ over the algebraic closure of k, and the former realization was useful for the application to hermitian Siegel series. But it was not easily understandable, and we could not obtain its Cartan decomposition, nor complete parametrization of spherical functions. For the present space X, we give an explicit Cartan decomposition in §2, and complete parametrization for spherical functions in §4, using explicit formulas of particular spherical functions given in §4. We discuss the correspondence between both spaces in Appendix.

Throughout of this article, we denote by k a non-archimedean local field of characteristic 0, fix an unramified quadratic extension k' and consider unitary and hermitian matrices with respect to k'/k. We fix a prime element π of k, denote by $v_\pi(\ )$ the additive value on k, and normalize the absolute value $|\ |$ on $k^\times$ by $|\pi|^{-1} = q = \sharp(\mathcal{O}_k/(\pi))$. We also fix a unit $\epsilon \in \mathcal{O}_k^\times$ for which $k' = k(\sqrt{\epsilon})$. We may take ϵ such as $\epsilon - 1 \in 4\mathcal{O}_k^\times$, then $\{1,\ \frac{1+\sqrt{\epsilon}}{2}\}$ forms an $\mathcal{O}_k$-basis for $\mathcal{O}_{k'}$ (cf. [Om], 64.3 and 64.4). From §2 to §4, we assume that q is odd.

§1 The space X and its K-orbit decomposition and G-orbit decomposition

Let k' be an unramified quadratic extension of a $\mathfrak{p}$-adic field k and consider hermitian matrices and unitary matrices with respect to k'/k. For a matrix $A \in M_{mn}(k')$, we denote by $A^* \in M_{nm}(k')$ its conjugate transpose with respect to k'/k, and say A is *hermitian* if $A^* = A$.

We consider the unitary group

$$G = G_n = \{g \in GL_{2n}(k') \mid g^* j_{2n} g = j_{2n}\}, \qquad j_{2n} = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix} \in M_{2n}, \tag{1.1}$$

the space X of unitary hermitian matrices in G

$$X = X_n = \left\{x \in G \;\middle|\; x^* = x,\ \Phi_{xj_{2n}}(t) = (t^2-1)^n\right\}, \tag{1.2}$$

and a supplementary space $\widetilde{X}$ containing X

$$\widetilde{X} = \widetilde{X}_n = \{x \in G \mid x = x^*\}, \tag{1.3}$$

where $\Phi_y(t)$ is the characteristic polynomial of the matrix y. The group G acts on X and $\widetilde{X}$ by

$$g \cdot x = gxg^* = x[g^*] = gxj_{2n}g^{-1}j_{2n}, \quad g \in G, \ x \in \widetilde{X}.$$

Over the algebraic closure $\overline{k}$ of k, we may understand as follows.

$$G(\overline{k}) = GL_{2n}(\overline{k}), \quad \widetilde{X}(\overline{k}) = \left\{ x \in \mathbb{G}(\overline{k}) \mid (xj_{2n})^2 = 1_{2n} \right\}, \tag{1.4}$$

with action given by

$$g \star x = gxj_{2n}g^{-1}j_{2n}, \qquad (g \in G(\overline{k}), \ x \in \widetilde{X}(\overline{k})).$$

Then $\widetilde{X}(\overline{k})$ is decomposed into $2n+1$ $G(\overline{k})$-orbits according to the shape of $\Phi_{xj_{2n}}(t)$, and we take

$$X(\overline{k}) = G(\overline{k}) \star 1_{2n} = \left\{ x \in \widetilde{X}(\overline{k}) \;\middle|\; \Phi_{xj_{2n}}(t) = (t^2-1)^n \right\}. \tag{1.5}$$

Then

$$X = X(\overline{k}) \cap G = X(\overline{k}) \cap \widetilde{X}.$$

We fix the maximal compact subgroup K of G by

$$K = K_n = G \cap M_{2n}(\mathcal{O}_{k'}). \tag{1.6}$$

The main purpose of this section is to give the Cartan decomposition of X, i.e., the K-orbit decomposition of X for odd q (Theorem 1.2), and G-orbit decomposition of X (Theorem 1.4).

To start with, we recall the case of unramified hermitian matrices. The group $G_0 = GL_n(k')$ acts on the space $\mathcal{H}_n(k') = \{y \in G_0 \mid y^* = y\}$ by $g \cdot y = gyg^*$, and there are two G_0-orbits in $\mathcal{H}_n(k')$ determined by the parity of $v_\pi(\det(y))$. Setting $K_0 = GL_n(\mathcal{O}_{k'})$, the Cartan decomposition is known(cf. [Jac]) as follows:

$$\mathcal{H}_n(k') = \bigsqcup_{\lambda \in \Lambda_n} K_0 \cdot \pi^\lambda, \tag{1.7}$$

where

$$\pi^\lambda = Diag(\pi^{\lambda_1}, \ldots, \pi^{\lambda_n}), \quad \Lambda_n = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}.$$

Proposition 1.1 *Let $n = 1$. Then*

$$X_1 = \bigsqcup_{\ell \geq 0} K_1 \cdot \begin{pmatrix} \pi^\ell & 0 \\ 0 & \pi^{-\ell} \end{pmatrix} \sqcup \bigsqcup_{1 \leq r \leq v_\pi(2)} \begin{pmatrix} \pi^{-r}(1-\epsilon) & -\sqrt{\epsilon} \\ \sqrt{\epsilon} & \pi^r \end{pmatrix},$$

where the latter union is empty if q is odd.

For general n, we have the following.

Theorem 1.2 *Assume that k has odd residual characteristic. Then, the K-orbit decomposition of X_n is given as follows:*

$$X_n = \bigsqcup_{\lambda \in \Lambda_n^+} K \cdot x_\lambda, \tag{1.8}$$

where

$$\Lambda_n^+ = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n \geq 0\},$$
$$x_\lambda = \begin{pmatrix} \pi^\lambda & 0 \\ 0 & \pi^{-\lambda} \end{pmatrix} = Diag(\pi^{\lambda_1}, \ldots, \pi^{\lambda_n}, \pi^{-\lambda_n}, \cdots, \pi^{-\lambda_1}).$$

For dyadic case, we show the following.

Theorem 1.3 *Assume that q is even. Then*

$$X_n = \bigcup_{r=0}^{n} \bigcup_{\substack{\lambda \in \Lambda_r^+ \\ \mu \in \Lambda_{n-r}^{(2)}}} K \cdot x_{\lambda,\mu}, \qquad x_{\lambda,\mu} = \begin{pmatrix} D_r(\lambda) & 0 & 0 \\ 0 & E_{n-r}(\mu) & 0 \\ 0 & 0 & D_r(-\lambda) \end{pmatrix}.$$

Here

$$x_\lambda = \begin{pmatrix} D_r(\lambda) & 0 \\ 0 & D_r(-\lambda) \end{pmatrix} \in X_r, \qquad (\lambda \in \Lambda_r^+),$$
$$\Lambda_m^{(2)} = \left\{\mu \in \Lambda_m^+ \mid v_\pi(2) \geq \mu_1 \geq \cdots \geq \mu_m \geq 1\right\}, \quad (m = n - r),$$

$$E_m(\mu) = \begin{pmatrix} \pi^{-\mu_m}(1-\epsilon) & & & & & -\sqrt{\epsilon} \\ & \ddots & & & \iddots & \\ & & \pi^{-\mu_1}(1-\epsilon) & -\sqrt{\epsilon} & & \\ & & \sqrt{\epsilon} & \pi^{\mu_1} & & \\ & \iddots & & & \ddots & \\ \sqrt{\epsilon} & & & & & \pi^{\mu_m} \end{pmatrix} \in X_m,$$

where any entry of $E_m(\mu)$ except in the diagonal or anti-diagonal is 0, and $x_{\lambda,\mu}$ is understood as x_λ (resp. $E_n(\mu)$) if $r = n$ (resp. $r = 0$). Further,

$$\bigcup_{\lambda \in \Lambda_n^+} K \cdot x_\lambda = \bigsqcup_{\lambda \in \Lambda_n^+} K \cdot x_\lambda \not\ni E_n(\mu), \quad \mu \in \Lambda_n^{(2)}.$$

As for G-orbits, we have the following. For $\lambda \in \Lambda_n^+$, we set $|\lambda| = \sum_{i=1}^n \lambda_i$ and call λ to be *even* or *odd* according to the parity of $|\lambda|$.

Theorem 1.4 *There are precisely two G-orbits in X_n:*

$$X_n = G \cdot x_0 \sqcup G \cdot x_1, \quad x_0 = 1_{2n},\ x_1 = Diag(\pi, 1, \ldots, 1, \pi^{-1}). \tag{1.9}$$

If q is odd, then

$$G \cdot x_0 = \bigsqcup_{\substack{\lambda \in \Lambda_n^+ \\ even}} K \cdot x_\lambda, \quad G \cdot x_1 = \bigsqcup_{\substack{\lambda \in \Lambda_n^+ \\ odd}} K \cdot x_\lambda.$$

If q is even, $x_{\lambda,\mu}$ is G-equivalent to x_0 if and only if $|\lambda| + |\mu|$ is even.

§2 Spherical function $\omega(x;s)$ on X

For simplicity, we write $j = j_n$, and take a Borel subgroup B of G by

$$B = \left\{ \begin{pmatrix} b & 0 \\ 0 & jb^{*-1}j \end{pmatrix} \begin{pmatrix} 1_n & aj \\ 0 & 1_n \end{pmatrix} \in G \;\middle|\; \begin{array}{l} b \text{ is upper triangular of size } n \\ a + a^* = 0 \end{array} \right\},$$

where B consists of all the upper triangular matrices in G.

We introduce a spherical function $\omega(x;s)$ on X by Poisson transform from relative B-invariants. For a matrix $g \in G$, denote by $d_i(g)$ the determinant of lower right i by i block of g. Then $d_i(x)$, $1 \le i \le n$ are relative B-invariants on X associated with rational characters ψ_i of B, where

$$d_i(p \cdot x) = \psi_i(p) d_i(x), \quad \psi_i(p) = N_{k'/k}(d_i(p)), \quad (x \in X,\ p \in B). \tag{2.1}$$

We set

$$X^{op} = \{x \in X \mid d_i(x) \neq 0,\ 1 \le i \le n\}. \tag{2.2}$$

For $x \in X$ and $s \in \mathbb{C}^n$, we consider the integral

$$\omega(x;s) = \int_K |\mathbf{d}(k \cdot x)|^{s+\varepsilon}\, dk, \quad |\mathbf{d}(y)|^s = \prod_{i=1}^n |d_i(y)|^{s_i}, \tag{2.3}$$

where dk is the normalized Haar measure on K, k runs over the set $\{k \in K \mid k \cdot x \in X^{op}\}$, and

$$\varepsilon = \varepsilon_0 + \left(\frac{\pi\sqrt{-1}}{\log q}, \ldots, \frac{\pi\sqrt{-1}}{\log q}\right), \quad \varepsilon_0 = (-1, \ldots, -1, -\frac{1}{2}) \in \mathbb{C}^n.$$

The right hand side of (2.3) is absolutely convergent if $\mathrm{Re}(s_i) \ge -\mathrm{Re}(\varepsilon_i) = -\varepsilon_{0,i}$, $1 \le i \le n$, and continued to a rational function of $q^{s_1}, \ldots, q^{s_n}$, and we use the notation $\omega(x;s)$ in such sense. We note here that

$$|\psi(p)|^{\varepsilon} \left(= \prod_{i=1}^n |\psi_i(p)|^{\varepsilon_i} \right) = |\psi(p)|^{\varepsilon_0} = \delta^{\frac{1}{2}}(p), \tag{2.4}$$

where δ is the modulus character on B (i.e., $d(pp') = \delta(p')^{-1} dp$ for the left invariant measure dp on B).

By a general theory, the function $\omega(x;s)$ becomes an $\mathcal{H}(G,K)$-common eigen function on X (cf. [H2]-§1, or [H4]-§1), and we call it a spherical function on X. More precisely, the Hecke algebra $\mathcal{H}(G,K)$ of G with respect to K is the commutative $\mathbb{C}$-algebra consisting of compactly supported two-sided K-invariant functions on G, which acts on the space $\mathcal{C}^\infty(K\backslash X)$ of left K-invariant functions on X by

$$(f * \Psi)(y) = \int_G f(g) \Psi(g^{-1} \cdot y) dg, \quad (f \in \mathcal{H}(G,K),\ \Psi \in \mathcal{C}^\infty(K\backslash X)), \tag{2.5}$$

where dg is the Haar measure on G normalized by $\int_K dk = 1$, and we see

$$(f * \omega(\ ;s))(x) = \lambda_s(f) \omega(x;s), \quad (f \in \mathcal{H}(G,K)), \tag{2.6}$$

where λ_s is the $\mathbb{C}$-algebra homomorphism defined by

$$\lambda_s : \mathcal{H}(G,K) \longrightarrow \mathbb{C}(q^{s_1},\ldots,q^{s_n}),$$
$$f \longmapsto \int_B f(p)\,|\psi(p)|^{-s+\varepsilon}\,dp.$$

We introduce a new variable z which is related to s by

$$s_i = -z_i + z_{i+1} \quad (1 \le i \le n-1), \qquad s_n = -z_n \tag{2.7}$$

and write $\omega(x;z) = \omega(x;s)$. Denote by W the Weyl group of G with respect to the maximal k-split torus in B. Then W acts on rational characters of B as usual (i.e., $\sigma(\psi)(b) = \psi(n_\sigma^{-1} b n_\sigma)$ by taking a representative n_σ of σ), so W acts on $z \in \mathbb{C}^n$ and on $s \in \mathbb{C}^n$ as well. We will determine the functional equations of $\omega(x;s)$ with respect to this Weyl group action. The group W is isomorphic to $S_n \ltimes C_2^n$, S_n acts on z by permutation of indices, and W is generated by S_n and $\tau : (z_1,\ldots,z_n) \longmapsto (z_1,\ldots,z_{n-1},-z_n)$. Keeping the relation (2.7), we also write $\lambda_z(f) = \lambda_s(f)$. Since

$$|\psi(p)|^{-s+\varepsilon} = \prod_{i=1}^{n} |N(p_i)|^{-z_i} \times \delta^{\frac{1}{2}}(p),$$

where p_i is the i-th diagonal component of $p \in B$, the $\mathbb{C}$-algebra map λ_z is an isomorphism (the Satake isomorphism)

$$\lambda_z \;:\; \mathcal{H}(G,K) \xrightarrow{\sim} \mathbb{C}[q^{\pm 2z_1},\ldots,q^{\pm 2z_n}]^W, \tag{2.8}$$

where the ring of the right hand side is the invariant subring of the Laurent polynomial ring $\mathbb{C}[q^{2z_1},q^{-2z_1},\ldots,q^{2z_n},q^{-2z_n}]$ by W.

By using a result on spherical functions on the space of hermitian forms, we obtain the following results.

Theorem 2.1 *The function $G_1(z)\cdot\omega(x;s)$ is invariant under the action of S_n on z, where*

$$G_1(z) = \prod_{1\le i<j\le n} \frac{1+q^{z_i-z_j}}{1-q^{z_i-z_j-1}}. \tag{2.9}$$

Hereafter till the end of §4, we assume k has odd residual characteristic, i.e. k is non dyadic and q is odd.

Next we study the functional equation with respect to τ. To begin with, based on Proposition 1.1, we calculate $\omega^{(1)}(x_\ell;s)$ explicitly and obtain the following.

Proposition 2.2 *For $n = 1$, the spherical function $\omega^{(1)}(x;s)$ is holomorphic for any $s \in \mathbb{C}$ and satisfies the functional equation*

$$\omega^{(1)}(x;s) = \omega^{(1)}(x;-s).$$

Theorem 2.3 *For general size n, the spherical function satisfies the functional equation*

$$\omega(x;z) = \omega(x;\tau(z)).$$

We assume $n \geq 2$ and introduce the following standard parabolic subgroup P attached to τ:

$$P = \left\{ \left(\begin{array}{c|c|c} q' & & \\ \hline & a & b \\ & c & d \\ \hline & & q \end{array}\right) \left(\begin{array}{cc|cc} 1_{n-1} & \alpha & & \\ & 1 & 0 & \\ \hline & 0 & 1 & -\alpha^* j \\ & & & 1_{n-1} \end{array}\right) \left(\begin{array}{cc|cc} 1_{n-1} & & \beta & \gamma j \\ & 1 & 0 & -\beta^* j \\ \hline & 0 & 1 & \\ & & & 1_{n-1} \end{array}\right) \in G \;\middle|\; \begin{array}{l} q \text{ is upper triangular in } GL_{n-1}(k'),\ q' = jq^{*-1}j \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(j_2),\ \alpha, \beta \in M_{n-1,1}(k'), \\ \gamma \in M_{n-1}(k'),\ \gamma + \gamma^* = 0 \end{array} \right\}, \tag{2.10}$$

where $j = j_{n-1}$ and each empty place in the above expression means zero-entry.

The relative B-invariants $d_i(x)$, $1 \leq i \leq n-1$ are relative P-invariants, but $d_n(x)$ is not. So we enlarged X and P as follows: Set $\widetilde{P} = P \times GL_1(k')$, $\widetilde{X} = X \times V$ with $V = M_{21}(k')$, and

$$(p,r) \star (x,v) = (p \cdot x, \rho(p)vr^{-1}), \quad (p,r) \in \widetilde{P},\ (x,v) \in \widetilde{X},$$

where $\rho(p) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for the decomposition of $p \in P$ as in (2.10).

Then we have the following relative $\widetilde{P}$-invariant $g(x;v)$ on $\widetilde{X}$ instead of $d_n(x)$.

Lemma 2.4 *Set*

$$g(x,v) = \det\left[\left(\begin{array}{c|c} -v_2\ v_1 & 0 \\ \hline 0 & 1_{n-1} \end{array}\right) \cdot x_{(n+1)}\right], \qquad (x,v) \in \widetilde{X},\ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

where $x_{(n+1)}$ is the lower right $(n+1)$ by $(n+1)$ block of x. Then

(i) *$g(x,v)$ is a relative $\widetilde{P}$-invariant on $\widetilde{X}$ associated with the $\widetilde{P}$-rational character $\widetilde{\psi}(p,r) = \psi_{n-1}(p)N(r)^{-1}$, and $g(x,v_0) = d_n(x)$ with $v_0 = {}^t(1\,0)$.*

(ii) *$g(x,v)$ is expressed as $g(x,v) = D(x)[v]$ by some hermitian matrix $D(x)$ of size 2. For $x \in X^{op}$, $D_1(x) = d_{n-1}(x)^{-1}D(x)$ belongs to X_1.*

By the embedding

$$K_1 = U(j_2) \hookrightarrow K = K_n,\ h \longmapsto \widetilde{h} = \begin{pmatrix} 1_{n-1} & & \\ & h & \\ & & 1_{n-1} \end{pmatrix}, \tag{2.11}$$

we have

$$\begin{aligned} \omega(x;s) &= \int_{K_1} dh \int_K |\mathbf{d}(k \cdot x)|^{s+\varepsilon}\, dk \\ &= \int_K \prod_{i<n} |d_i(k \cdot x)|^{s_i+\varepsilon_i} \int_{K_1} \left|d_n(\widetilde{h}k \cdot x)\right|^{s_n+\varepsilon_n} dhdk. \end{aligned} \tag{2.12}$$

By Lemma 2.4, we see

$$\begin{aligned} d_n(\widetilde{h}\cdot y) &= g(\widetilde{h}\cdot y, v_0) = g((\widetilde{h}, 1) \star (y, h^{-1}v_0)) \\ &= d_{n-1}(y) d_1((h^{*-1}\cdot D_1(y))^{-1}) = d_{n-1}(y) d_1(h\cdot D_1(y)^{-1}), \end{aligned}$$

and we obtain

$$\omega(x;s) = \int_K \prod_{i<n} |d_i(k\cdot x)|^{s_i+\varepsilon_i} |d_{n-1}(k\cdot x)|^{s_n+\varepsilon_n} \omega^{(1)}(D_1(k\cdot x)^{-1}; s_n) dk.$$

Then, by Proposition 2.2, we obtain

$$\omega(x;s) = \omega(x; s_1, \ldots, s_{n-2}, s_{n-1}+2s_n, -s_n),$$

which shows in z-variable

$$\omega(x;z) = \omega(x;\tau(z)), \quad \tau(z) = (z_1, \ldots, z_{n-1}, -z_n),$$

and we conclude the proof of Theorem 2.3. ∎

In order to describe functional equations of $\omega(x;z)$, we prepare some notations. We denote by Σ the set of roots of G with respect to the maximal k-split torus of G contained in B and by Σ^+ the set of positive roots with respect to B. We may understand Σ as a subset in $\mathbb{Z}^n$, and set

$$\begin{aligned} &\Sigma^+ = \Sigma_s^+ \cup \Sigma_\ell^+, \\ &\Sigma_s^+ = \{e_i - e_j,\ e_i + e_j \mid 1 \le i < j \le n\}, \quad \Sigma_\ell^+ = \{2e_i \mid 1 \le i \le n\}, \end{aligned} \tag{2.13}$$

where e_i is the i-th unit vector in $\mathbb{Z}^n$, $1 \le i \le n$. Then Λ_n^+ can be regarded as the set of dominant weights. We define a pairing on $\mathbb{Z}^n \times \mathbb{C}^n$ by

$$\langle t,\, z\rangle = \sum_{i=1}^n t_i z_i, \qquad (t \in \mathbb{Z}^n,\ z \in \mathbb{C}^n),$$

which satisfies

$$\langle \alpha,\, z\rangle = \langle \sigma(\alpha), \sigma(z)\rangle, \qquad (\alpha \in \Sigma,\ z \in \mathbb{C}^n,\ \sigma \in W).$$

By Theorem 2.1 and Theorem 2.3 and cocycle relations of Gamma factors, we have the following.

Theorem 2.5 *The spherical function $\omega(x;z)$ satisfies the following functional equation for each $\sigma \in W$*

$$\omega(x;z) = \Gamma_\sigma(z)\cdot \omega(x;\sigma(z)), \tag{2.14}$$

where

$$\Gamma_\sigma(z) = \prod_{\alpha\in\Sigma_s^+(\sigma)} \frac{1-q^{\langle\alpha,\, z\rangle - 1}}{q^{\langle\alpha,\, z\rangle} - q^{-1}}, \qquad \Sigma_s^+(\sigma) = \left\{\alpha \in \Sigma_s^+ \mid -\sigma(\alpha) \in \Sigma^+\right\}.$$

and we understand $\Gamma_\sigma(z) = 1$ if $\Sigma_s^+(\sigma) = \emptyset$.

Further we obtain the following in the similar line to the proof of Theorem 2.9 in [H5].

Theorem 2.6 *The function*

$$\prod_{\alpha\in\Sigma_s^+} \frac{1+q^{\langle\alpha,\,z\rangle}}{1-q^{\langle\alpha,\,z\rangle-1}} \cdot \omega(x;z)$$

is holomorphic for all z in $\mathbb{C}^n$ and W-invariant, in particular it is an element in $\mathbb{C}[q^{\pm z_1},\dots,q^{\pm z_n}]^W$.

§3 The explicit formula for $\omega(x;z)$

We give the explicit formula of $\omega(x;z)$. Since $\omega(x;z)$ is stable on each K-orbit, it is enough to show the explicit formula for each x_λ, $\lambda\in\Lambda_n^+$ by Theorem 1.2.

Theorem 3.1 *For $\lambda\in\Lambda_n^+$, one has the explicit formula:*

$$\omega(x_\lambda;z) \;=\; \frac{(1-q^{-2})^n}{w_{2n}(-q^{-1})}\cdot\prod_{\alpha\in\Sigma_s^+}\frac{1-q^{\langle\alpha,\,z\rangle-1}}{1+q^{\langle\alpha,\,z\rangle}}\cdot c_\lambda\cdot Q_\lambda(z), \tag{3.1}$$

where

$$w_m(t)=\prod_{i=1}^{m}(1-t^i), \qquad c_\lambda=(-1)^{\sum_i\lambda_i(n-i+1)}q^{-\sum_i\lambda_i(n-i+\frac{1}{2})},$$

$$Q_\lambda(z)=\sum_{\sigma\in W}\sigma\left(q^{-\langle\lambda,\,z\rangle}c(z)\right),$$

$$c(z)=\prod_{\alpha\in\Sigma_s^+}\frac{1+q^{\langle\alpha,\,z\rangle-1}}{1-q^{\langle\alpha,\,z\rangle}}\prod_{\alpha\in\Sigma_\ell^+}\frac{1-q^{\langle\alpha,\,z\rangle-1}}{1-q^{\langle\alpha,\,z\rangle}}. \tag{3.2}$$

Remark 3.2 We see that the main part $Q_\lambda(z)$ of $\omega(x_\lambda;z)$ belongs to $\mathcal{R}=\mathbb{C}[q^{\pm z_1},\dots,q^{\pm z_n}]^W$ by Theorem 2.6. On the other hand $Q_\lambda(z)$ is a specialization of Hall-Littlewood polynomial P_λ of type C_n up to constant multiple, which is introduced in a general context of orthogonal polynomials associated with root systems ([M2], §10), and $Q_0(z)$ is a specialization of Poincaré polynomial ([M1],Th.2.8.). More precisely,

$$Q_\lambda(z)=\frac{\widetilde{w_\lambda}(-q^{-1})}{(1+q^{-1})^n}\cdot P_\lambda(z), \tag{3.3}$$

$$\widetilde{w_\lambda}(t)=w_{m_0(\lambda)}(t)^2\cdot\prod_{\ell\geq 1}w_{m_\ell(\lambda)}(t), \quad m_\ell(\lambda)=\sharp\{i\mid\lambda_i=\ell\},$$

and it is known that the set $\{Q_\lambda(z)\mid\lambda\in\Lambda_n^+\}$ forms a $\mathbb{C}$-basis for $\mathcal{R}$, and in particular, $Q_0(z)$ is a constant independent of z.

By Theorem 3.1 and Remark 3.2, we have the following corollary.

Corollary 3.3 *For* $x_0 = 1_{2n}$*, one has*

$$\omega(1_{2n}; z) = \frac{(1-q^{-1})^n w_n(-q^{-1})^2}{w_{2n}(-q^{-1})} \times \prod_{\alpha \in \Sigma_s^+} \frac{1 - q^{\langle \alpha, z \rangle - 1}}{1 + q^{\langle \alpha, z \rangle}}.$$

Theorem 3.1 is proved by using a general expression formula given in [H4] (or in [H2]) of spherical functions on homogeneous spaces, which is based on functional equations of finer spherical functions and some data depending only on the group G. We need to check the assumptions there. Let $\mathbb{G}$ be a connected reductive linear algebraic group and $\mathbb{X}$ be an affine algebraic variety which is $\mathbb{G}$-homogeneous, where everything is assumed to be defined over a p-adic field k. For an algebraic set, we use the same ordinary letter to indicate the set of k-rational points. Let K be a special good maximal compact open subgroup of G, and $\mathbb{B}$ a minimal parabolic subgroup of $\mathbb{G}$ defined over k satisfying $G = KB = BK$. We denote by $\mathfrak{X}(\mathbb{B})$ the group of rational character of $\mathbb{B}$ defined over k and by $\mathfrak{X}_0(\mathbb{B})$ the subgroup consisting of those characters associated with some relative $\mathbb{B}$-invariant on $\mathbb{X}$ defined over k. In these situation, the assumptions are the following:

$(A1)$ $\mathbb{X}$ has only a finite number of $\mathbb{B}$-orbits (, hence there is only one open orbit $\mathbb{X}^{op}$).

$(A2)$ A basic set of relative $\mathbb{B}$-invariants on $\mathbb{X}$ defined over k can be taken by regular functions on $\mathbb{X}$.

$(A3)$ For $y \in \mathbb{X} \backslash \mathbb{X}^{op}$, there exists some ψ in $\mathfrak{X}_0(\mathbb{B})$ whose restriction to the identity component of the stabilizer $\mathbb{H}_y$ of $\mathbb{B}$ at y is not trivial.

$(A4)$ The rank of $\mathfrak{X}_0(\mathbb{B})$ coincides with that of $\mathfrak{X}(\mathbb{B})$.

In the present situation, our space X is isomorphic to $U(j_{2n})/U(1_{2n})$ over $\overline{k}$ (cf. (1.5), which is a symmetric space and $(A1)$ is satisfied. $(A2)$ and $(A4)$ are satisfied by our relative B-invariants $\{d_i(x) \mid 1 \le i \le n\}$, where n is the rank of $\mathfrak{X}_0(\mathbb{B}) = \mathfrak{X}_0(\mathbb{B})$ and $\mathbb{X}^{op} = \{x \in \mathbb{X} \mid d_i(x) \neq 0,\ 1 \le i \le n\}$. To check (A3) is crucial and rather complicated. It is proved by showing the existence of ψ as above for each $y \in \mathbb{X} \backslash \mathbb{X}^{op}$.

According to the B-orbit decomposition of X^{op}, we define finer spherical functions as follows

$$\omega_u(x; s) = \int_K |\mathbf{d}(k \cdot x)|_u^{s+\varepsilon}\, dk, \quad |\mathbf{d}(y)|_u^s = \begin{cases} \prod_{i=1}^n |d_i(y)|^{s_i} & \text{if } y \in X_u, \\ 0 & \text{otherwise}. \end{cases}$$

where

$$X^{op} = \bigsqcup_{u \in \mathcal{U}} X_u, \qquad \mathcal{U} = (\mathbb{Z}/2\mathbb{Z})^n,$$
$$X_u = \{x \in X^{op} \mid v_\pi(d_i(x)) \equiv u_1 + \cdots + u_i \pmod 2,\ 1 \le i \le n\}.$$

For each character χ of $\mathcal{U}$, we may represent as follows

$$\sum_{u\in\mathcal{U}} \chi(u)\omega_u(x;s) = \omega(x;z_\chi), \tag{3.4}$$

where z_χ is obtained by adding $\dfrac{\pi\sqrt{-1}}{\log q}$ to z_i for suitable i according to χ, and they are linearly independent (for generic z) as varying characters χ. By Theorem 2.5, we have, for each character χ of $\mathcal{U}$ and $\sigma \in W$,

$$\begin{aligned} \omega(x;z_\chi) &= \Gamma_\sigma(z_\chi)\omega(x;\sigma(z_\chi)) \\ &= \Gamma_\sigma(z_\chi)\omega(x;\sigma(z)_{\sigma(\chi)}), \end{aligned} \tag{3.5}$$

by taking a suitable character $\sigma(\chi)$ of $\mathcal{U}$. If χ is trivial character $\mathbf{1}$, then (3.5) coincides with the original one. Further we obtain vector-wise functional equations as follows

$$(\omega_u(x;z))_{u\in\mathcal{U}} = A^{-1}\cdot G(\sigma,z)\cdot\sigma A\cdot(\omega_u(x;\sigma(z)))_{u\in\mathcal{U}}, \qquad \sigma\in W, \tag{3.6}$$

where

$$A = (\chi(u))_{\chi,u}, \quad \sigma A = (\sigma(\chi)(u))_{\chi,u} \in GL_{2^n}(\mathbb{Z}),$$

χ runs over characters of $\mathcal{U}$, $u \in \mathcal{U}$, and $G(\sigma,z)$ is the diagonal matrix of size 2^n whose (χ,χ)-component is $\Gamma_\sigma(z_\chi)$. Here we fix the first entry of χ to be $\mathbf{1}$. Applying Theorem 2.6 in [H4] to our present case, we obtain for generic z, by virtue of (3.6),

$$(\omega_u(x_\lambda;z))_{u\in\mathcal{U}} = \frac{1}{Q}\sum_{\sigma\in W}\gamma(\sigma(z))\left(A^{-1}\cdot G(\sigma,z)\cdot\sigma A\right)(\delta_u(x_\lambda,\sigma(z)))_{u\in\mathcal{U}}, \tag{3.7}$$

where, taking U as the Iwahori subgroup of K compatible with B,

$$Q = \sum_{\sigma\in W}[U\sigma U:U]^{-1} = \frac{w_{2n}(-q^{-1})}{(1-q^{-2})^n},$$

$$\gamma(z) = \prod_{\alpha\in\Sigma_s^+}\frac{1-q^{2\langle\alpha,z\rangle-2}}{1-q^{2\langle\alpha,z\rangle}}\cdot\prod_{\alpha\in\Sigma_\ell^+}\frac{1-q^{\langle\alpha,z\rangle-1}}{1-q^{\langle\alpha,z\rangle}},$$

$$\delta_u(x_\lambda;z) = \begin{cases} c_\lambda q^{-\langle\lambda,z\rangle} & \text{if } x_\lambda\in X_u \\ 0 & \text{otherwise.}\end{cases}$$

Then, we obtain

$$\begin{aligned} \omega(x_\lambda;z) &= \sum_{u\in\mathcal{U}}\mathbf{1}(u)\omega_u(x_\lambda;z) \\ &= \text{the first entry of } A\,(\omega_u(x_\lambda;z))_{u\in\mathcal{U}} \\ &= \frac{(1-q^{-2})^n}{w_{2n}(-q^{-1})}\times\sum_{\sigma\in W}\gamma(\sigma(z))\Gamma_\sigma(z)\sum_u\delta_u(x_\lambda,\sigma(z)) \\ &= \frac{c_\lambda(1-q^{-2})^n}{w_{2n}(-q^{-1})}\times\sum_{\sigma\in W}\gamma(\sigma(z))\Gamma_\sigma(z)q^{-\langle\lambda,\sigma(z)\rangle}. \end{aligned}$$

Setting

$$G(z)=\prod_{\alpha\in\Sigma_s^+}\frac{1+q^{\langle\alpha,z\rangle}}{1-q^{\langle\alpha,z\rangle-1}},$$

we have

$$\Gamma_\sigma(z)=\frac{G(\sigma(z))}{G(z)}\quad\text{(by Theorem 2.6)},\quad \gamma(z)\cdot G(z)=c(z).$$

Thus we obtain the required explicit formula of $\omega(x_\lambda;z)$ for generic z, and it is valid for every $z\in\mathbb{C}^n$, since $G(z)\cdot\omega(x_\lambda;z)$ is a polynomial in $q^{\pm z_1},\dots,q^{\pm z_n}$. ∎

§4 Spherical Fourier transform and Plancherel formula on $\mathcal{S}(K\backslash X)$

We consider the Schwartz space

$$\mathcal{S}(K\backslash X)\ =\ \{\varphi:X\longrightarrow\mathbb{C}\mid\text{left }K\text{-invariant, compactly supported}\},$$

which is an $\mathcal{H}(G,K)$-submodule of $\mathcal{C}^\infty(K\backslash X)$ by the convolution product and spanned by the characteristic function of $K\cdot x$, $x\in X$. We introduce a modified spherical function

$$\Psi(x;z)=\omega(x;z)/\omega(1_{2n};z)\in\mathbb{C}[q^{\pm z_1},\dots,q^{\pm z_n}]^W(=\mathcal{R},\text{ say}),\tag{4.1}$$

and define the spherical Fourier transform

$$\begin{array}{rccl} F: & \mathcal{S}(K\backslash X) & \longrightarrow & \mathcal{R} \\ & \varphi & \longmapsto & F(\varphi)(z)=\int_X\varphi(x)\Psi(x;z)dx, \end{array}\tag{4.2}$$

where dx is a G-invariant measure on X. There is a G-invariant measure on X, since X is a disjoint union of two G-orbits, and G is reductive. We don't need to fix the normalization of dx at this moment, we will determine suitably afterward(cf. Theorem 4.5). We denote by $v(K\cdot y)$ the volume of $K\cdot y$ by dx. We regard $\mathcal{R}$ as an $\mathcal{H}(G,K)$-module through the Satake isomorphism

$$\lambda_z:\mathcal{H}(G,K)\xrightarrow{\sim}\mathbb{C}[q^{\pm 2z_1},\dots,q^{\pm 2z_n}]^W(=\mathcal{R}_0,\text{ say}).$$

By Theorem 3.1, Corollary 3.3 and (3.3), we see, for any $\lambda\in\Lambda_n^+$,

$$\begin{aligned}&\Psi(x_\lambda;z)=c_\lambda w_\lambda P_\lambda(z),\\ &F(ch_\lambda)(z)=v(K\cdot x)\Psi(x_\lambda;z)=c_\lambda w_\lambda v(K\cdot x_\lambda)P_\lambda(z),\end{aligned}\tag{4.3}$$

where c_λ is the same as in Theorem 3.1, ch_λ is the characteristic function of $K\cdot x_\lambda$, and

$$w_\lambda=\frac{\widetilde{w_\lambda}(-q^{-1})}{w_n(-q^{-1})^2}.$$

Theorem 4.1 *The spherical Fourier transform F is an $\mathcal{H}(G,K)$-module isomorphism, in particular $\mathcal{S}(K\backslash X)$ is a free $\mathcal{H}(G,K)$-module of rank 2^n.*

Proof. Since $\{P_\lambda(z) \mid \lambda \in \Lambda_n^+\}$ forms a $\mathbb{C}$-basis for $\mathcal{R}$(cf. Remark 3.2) and $\{ch_\lambda \mid \lambda \in \Lambda_n^+\}$ forms a $\mathbb{C}$-basis for $\mathcal{S}(K\backslash X)$(cf. Theorem 1.2), F is bijective by (4.3). It is easy to check

$$F(f * \varphi) = \lambda_z(f)F(\varphi), \qquad (f \in \mathcal{H}(G,K),\ \varphi \in \mathcal{S}(K\backslash X)),$$

and we see $\mathcal{S}(K\backslash X)$ is a free $\mathcal{H}(G,K)$-module of rank 2^n, since $\mathcal{R}$ is a free $\mathcal{R}_0$-module of rank 2^n. ∎

As a corollary we have the following.

Theorem 4.2 *All the spherical functions on X are parametrized by eigenvalues $z \in \left(\mathbb{C}/\frac{2\pi\sqrt{-1}}{\log q}\mathbb{Z}\right)^n/W$ through $\lambda_z(f)$. The set $\{\Psi(x; z+u) \mid u \in \{0, \pi\sqrt{-1}/\log q\}^n\}$ forms a basis of the space of spherical functions on X corresponding to z.*

In order to give the Plancherel formula on $\mathcal{S}(K\backslash X)$, we introduce an inner product on $\mathcal{R}$ by

$$\langle P, Q\rangle_{\mathcal{R}} = \int_{\mathfrak{a}^*} P(z)\overline{Q(z)}d\mu(z), \qquad (P, Q \in \mathcal{R}).$$

Here

$$\mathfrak{a}^* = \left\{\sqrt{-1}\left(\mathbb{R}/\frac{2\pi}{\log q}\mathbb{Z}\right)\right\}^n,$$

and the measure $d\mu = d\mu(z)$ on $\mathfrak{a}^*$ is given by

$$d\mu = \frac{1}{n!2^n}\cdot\frac{w_n(-q^{-1})^2}{(1+q^{-1})^n}\cdot\frac{1}{|c(z)|^2}dz,$$

where $c(z)$ is defined in (3.2) and and dz is the Haar measure on $\mathfrak{a}^*$ with $\int_{\mathfrak{a}^*} = 1$. Then, the following lemma is essentially reduced to a result of Macdonald([M2], §10).

Lemma 4.3 *For $\lambda, \mu \in \Lambda_n^+$, one has*

$$\langle P_\lambda, P_\mu\rangle_{\mathcal{R}} = \langle P_\mu, P_\lambda\rangle_{\mathcal{R}} = \delta_{\lambda,\mu}w_\lambda^{-1}.$$

On the other hand, one may obtain

Lemma 4.4 *For $\lambda, \mu \in \Lambda_n^+$ such that $|\lambda| \equiv |\mu| \pmod 2$,*

$$\frac{v(K\cdot x_\lambda)}{v(K\cdot x_\mu)} = \frac{c_\mu^2 w_\mu}{c_\lambda^2 w_\lambda}.$$

Since we may normalize dx on X according to G-orbits, we normalize as

$$v(K \cdot x_0) = 1, \quad v(K \cdot x_1) = q^{2n-1}\frac{(1-(-q^{-1})^n)^2}{1+q^{-1}},$$

where $x_0 = 1_{2n}$ and $x_1 = Diag(\pi, 1, \ldots, 1, \pi^{-1})$ (cf. Theorem 1.4). Then we obtain

Theorem 4.5 *(Plancherel formula on $\mathcal{S}(K\backslash X)$) By the normalization of G-invariant measure dx such that*

$$v(K \cdot x_\lambda) = c_\lambda^{-2} w_\lambda^{-1}, \qquad \lambda \in \Lambda_n^+,$$

one has, for any $\varphi, \psi \in \mathcal{S}(K\backslash X)$,

$$\int_X \varphi(x)\overline{\psi(x)}dx = \int_{\mathfrak{a}^*} F(\varphi)(z)\overline{F(\psi)(z)}d\mu(z).$$

Corollary 4.6 *(Inversion formula) For any $\varphi \in \mathcal{S}(K\backslash X)$,*

$$\varphi(x) = \int_{\mathfrak{a}^*} F(\varphi)(z)\Psi(x; z)d\mu(z), \quad x \in X.$$

Appendix

In [H5], we have considered, for each $T \in \mathcal{H}_n(k')$

$$X_T = \mathfrak{X}_T/U(T), \qquad \mathfrak{X}_T = \{x \in M_{2n,n}(k') \mid H_n[x] = T\}, \quad H_n = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix},$$

where $U(H_n)$ acts homogeneously on X_T by the left multiplication, and the stabilizer at a point in X_T is isomorphic to $U(T) \times U(T)$ (cf. [H5] Lemma 1,1).

The explicit formula of $\omega(x_\lambda; z)$ in Theorem 3.1 is the same as the explicit formula of $\omega_T(y_\lambda; z)$ on X_T at $y_\lambda \in X_T$ parametrized by $\lambda \in \Lambda_n^+$ in Theorem 3.3 in [H5]. We explain the relation between the spaces X and X_T's.

We assume T is diagonal and realize X_T as a set of k-rational points in an algebraic set defined over k. We consider the space

$$\mathfrak{X}_T(\overline{k}) \quad = \quad \left\{(x, y) \in M_{2n,n}(\overline{k}) \oplus M_{2n,n}(\overline{k}) \mid {}^t y H_n x = T\right\},$$

with the action of $\Gamma = Gal(\overline{k}/k)$ given by

$$\sigma(x, y) = \begin{cases} (x^\sigma, y^\sigma) & \text{if } \sigma|_{k'} = id \\ (y^\sigma, x^\sigma) & \text{if } \sigma|_{k'} = \tau, \end{cases}$$

where $x^\sigma = (x_{ij}{}^\sigma)$ for $x = (x_{ij})$ and $\sigma \in \Gamma$, and $\langle \tau \rangle = Gal(k'/k)$. We set

$$\begin{aligned}
&\mathbb{U}(H_n) = U(H_n)(\overline{k}) = \left\{ (g_1, g_2) \in GL_{2n}(\overline{k}) \times GL_{2n}(\overline{k}) \;\middle|\; {}^t g_2 H_n g_1 = H_n \right\}, \\
&\mathbb{U}(T) = U(T)(\overline{k}) = \left\{ (h_1, h_2) \in GL_n(\overline{k}) \times GL_n(\overline{k}) \;\middle|\; {}^t h_2 T h_1 = T \right\}, \\
&\mathbb{X}_T(\overline{k}) = \mathfrak{X}_T(\overline{k})/\mathbb{U}(T) \supset \mathbb{X}_T(k) = \mathbb{X}_T{}^\Gamma,
\end{aligned}$$

where we may consider the similar Γ-action on $\mathbb{U}(H_n)$ and $\mathbb{U}(T)$, since H_n and T are Γ-invariant. We identify

$$\begin{aligned}
&\mathbb{U}(H_n)^\Gamma = \{ (g, \overline{g}) \in GL_{2n}(k') \times GL_{2n}(k') \mid {}^t\overline{g} H_n g = H_n \} \quad \text{with } U(H_n), \\
&\mathbb{U}(T)^\Gamma = \left\{ (h, \overline{h}) \in GL_n(k') \times GL_n(k') \;\middle|\; {}^t\overline{h} T h = T \right\} \qquad \text{with } U(T),
\end{aligned}$$

where and henceforth we write $\overline{g}$ instead of g^τ for a matrix g with entries in k'. By the injective map

$$\varphi_T : X_T \longrightarrow \mathbb{X}_T(k), \; xU(T) \longmapsto (x, \overline{x})\mathbb{U}(T),$$

we understand X_T as a subspace of $\mathbb{X}_T(k)$. Set

$$T_1 = \begin{pmatrix} \pi & 0 \\ 0 & 1_{n-1} \end{pmatrix}, \quad \widetilde{\eta_\pi} = (\eta_\pi, \eta_\pi), \quad \eta_\pi = \begin{pmatrix} \sqrt{\pi}^{-1} & 0 \\ 0 & 1_{n-1} \end{pmatrix}.$$

Lemma A.1 *The map*

$$f : \mathbb{X}_{T_1}(\overline{k}) \longrightarrow \mathbb{X}_{1_n}(\overline{k}), \; (x, y)\mathbb{U}(T_1) \longmapsto (x\eta_\pi, y\eta_\pi)\mathbb{U}(1_n)$$

is well defined, it sends $\mathbb{X}_{T_1}(k)$ *into* $\mathbb{X}_{1_n}(k)$ *and* $f(X_{T_1}) \neq X_{1_n}$.

We set

$$\mathbb{N} = \left\{ \left(\begin{pmatrix} h_1 & 0 \\ 0 & k_1 \end{pmatrix}, \begin{pmatrix} h_2 & 0 \\ 0 & k_2 \end{pmatrix} \right) \;\middle|\; (h_1, h_2), \; (k_1, k_2) \in \mathbb{U}(1_n) \right\}$$

with the similar Γ-action on $\mathbb{N}$ as before, and identify $\mathbb{N}^\Gamma$ with

$$N = \left\{ \begin{pmatrix} h & 0 \\ 0 & k \end{pmatrix} \;\middle|\; h, k \in U(1_n) \right\}.$$

The stabilizer at $y_0 U(1_n) \in X_{1n}$ in $U(H_n)$, where

$$y_0 = \begin{pmatrix} \xi \\ 1_n \end{pmatrix}, \quad \xi = \frac{1 + \sqrt{\varepsilon}}{2},$$

is given by

$$\nu N \nu^{-1}, \qquad \nu = \begin{pmatrix} \xi 1_n & \overline{\xi} 1_n \\ 1_n & -1_n \end{pmatrix} \in GL_{2n}(\mathcal{O}_{k'}).$$

On the other hand, the stabilizer at 1_{2n} in $G = U(j_{2n})$ is given by

$$\mu N \mu^{-1}, \qquad \mu = \begin{pmatrix} 1_n & 1_n \\ j_n & -j_n \end{pmatrix} \in GL_{2n}(k).$$

Further we see

$$\mu\nu^{-1}U(H_n)\nu\mu^{-1} = G, \quad \mu\nu^{-1}\mathbb{U}(H_n)\nu\mu^{-1} = G(\overline{k}). \tag{A.1}$$

Thus we have a commutative diagram

$$\begin{array}{ccccccc} \mathbb{X}_{1_n}(\overline{k}) & \cong & \mathbb{U}(H_n)/\nu\mathbb{N}\nu^{-1} & \overset{\varphi}{\underset{\sim}{\longrightarrow}} & G(\overline{k})/\mu\mathbb{N}\mu^{-1} & \cong & X_n(\overline{k}) \\ \cup & & & & & & \cup \\ \mathbb{X}_{1_n}(k) \supset X_{1_n} & \cong & U(H_n)/\nu N\nu^{-1} & \overset{\varphi}{\underset{\sim}{\longrightarrow}} & G/\mu N\mu^{-1} \cong G\cdot 1_{2n} & \subset & X_n, \end{array} \tag{A.2}$$

where φ is the conjugation determined by (A.1). Then we have the following by (A.2), Lemma A.1, and Theorem 1.4.

Proposition A.2 *The above φ gives an isomorphism between the sets of k-rational points*

$$U(H_n)\backslash\mathbb{X}_{1_n}(k) \cong G\backslash X_n,$$

and $U(H_n)$-orbit decomposition

$$\mathbb{X}_{1_n}(k) = X_{1_n} \sqcup f(X_{T_1}); \quad X_{1_n} \cong G\cdot x_0, \quad f(X_{T_1}) \cong G\cdot x_1,$$

where x_0 and x_1 are the representatives of G-orbits in X_n given in Theorem 1.4.

Under the assumption q is odd, we have $\mu \in GL_{2n}(\mathcal{O}_{k'})$. Hence we see

$$K' := U(H_n) \cap GL_{2n}(\mathcal{O}_{k'}) = \nu\mu^{-1}K\mu\nu^{-1},$$

and the space X_T inherits the Cartan decomposition of X_n, and we have

Theorem A.3 *Assume k has odd residual characteristic and take any $T \in \mathcal{H}_n(k')$. Then*

$$\mathfrak{X}_T = \bigsqcup_{\substack{\lambda\in\Lambda_n^+ \\ \lambda\sim T}} K'y_\lambda h_\lambda U(T),$$

where

$$y_\lambda = \begin{pmatrix} \xi\pi^\lambda \\ 1_n \end{pmatrix} \in \mathfrak{X}_{\pi^\lambda},$$

$\lambda \sim T$ means that $|\lambda| \equiv v_\pi(\det(T)) \pmod 2$ and guarantees the existence of $h_\lambda \in GL_n(k')$ satisfying $T = \pi^\lambda[h_\lambda]$.

The above decomposition has been expected in [H5] Remark 4.2. In [H5], we have known the disjointness of orbits in the right hand side by explicit formulas of spherical functions $\omega_T(y;z)$, but we didn't know they are enough. By Theorem A.3, we see the spherical Fourier transform F_T is isomorphic in [H5, Theorem 4.1.].

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