数理解析研究所講究録
第 1826 巻 2013 年 129-137

MODULAR FORMS AND THE COHOMOLOGY OF MODULI SPACES

CAREL FABER

0. Introduction

In the conference, I talked about the ongoing joint work with Jonas Bergström and Gerard van der Geer. See [5], §§2–3 for a survey and [1] for a detailed presentation. Some of the results mentioned in the talk and below have not yet been written up.

It is a pleasure to thank Bergström and van der Geer for the continuing collaboration and Professors Ibukiyama and Moriyama for the kind invitation and hospitality.

1. Preliminaries

The subject of interest here is the cohomology of certain moduli spaces. The main characters are the moduli spaces M_g of smooth curves of genus g and A_g of principally polarized abelian varieties of dimension g, but the moduli spaces $\overline{M}_g$ of stable curves of genus g, and $M_{g,n}$ resp. $\overline{M}_{g,n}$ of smooth resp. stable n-pointed curves of genus g play a role as well. The natural action of the symmetric group Σ_n permuting the n ordered points will be important. All moduli spaces above are smooth over $\mathbb{Z}$; as a result, the modular forms that we will encounter will always be of level one.

For $g \geq 2$, the space $M_{g,n}$ is open in C_g^n, the n-fold fibre product of the universal curve $C_g = M_{g,1}$ over M_g; this is the moduli space of smooth curves of genus g with n ordered points, which may coincide. The fiber over $[C] \in M_g$ is C^n. Now the interesting cohomology of a curve is its first cohomology group $H^1(C)$. So, instead of studying $H^*(M_{g,n})$, it makes sense to focus on the cohomology of $\mathbb{V}^{\otimes n}$ on M_g, where $\mathbb{V}$ is the system of H^1's of curves of genus g (i.e., $\mathbb{V} = R^1\pi_*\mathbb{Q}$ or $R^1\pi_*\mathbb{Q}_\ell$ for $\pi\colon C_g \to M_g$). For $g = 1$, we study the corresponding local system on $M_{1,1}$.

Taking into account the Σ_n-action, we should also study the cohomology of $\mathrm{Sym}^j\mathbb{V}$, $\wedge^k\mathbb{V}$, and, more generally, of $\mathbb{V}_\lambda$, where $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \lambda_g \geq 0)$ corresponds to an irreducible representation of GSp_{2g}

(the one of highest weight in

$$\mathrm{Sym}^{\lambda_1-\lambda_2}V \otimes \mathrm{Sym}^{\lambda_2-\lambda_3}\left(\wedge^2 V\right) \otimes \ldots \mathrm{Sym}^{\lambda_g}\left(\wedge^g V\right),$$

where V, corresponding to $\mathbb{V}$, is the contragredient of the standard representation of GSp_{2g}). Note that $\mathbb{V} = \mathbb{V}_1$ comes with a symplectic pairing onto the Tate twisted trivial local system $\mathbb{V}_0(-1)$.

We will study $e_c(M_g, \mathbb{V}_\lambda)$, the Euler characteristic of the compactly supported cohomology, as this is good enough for much of what we want. The cohomology groups have a lot of structure (as, e.g., ℓ-adic Galois representations or mixed Hodge structures) and we remember this structure (so that e_c takes values in an appropriate Grothendieck group).

The $\mathbb{V}_\lambda$ are pulled back from A_g, via the Torelli morphism $t\colon M_g \to A_g$, sending $[C]$ to the class $[\mathrm{Jac}(C)]$ of its Jacobian. So we will also study $e_c(A_g, \mathbb{V}_\lambda)$.

2. Results

2.1. **Outline.** We have found a formula for $e_c(A_g, \mathbb{V}_\lambda)$ for $g \leq 3$. This was known before for $g = 1$, is conjectural, but known in many cases for $g = 2$, and is conjectural (with much evidence) for $g = 3$. From the formula for $g = 2$, we obtain one for $e_c(M_2, \mathbb{V}_\lambda)$, which by work of Getzler leads to a formula for the Σ_n-equivariant Euler characteristic $e_c^{\Sigma_n}(M_{2,n})$, and then, by Getzler-Kapranov [8] and the known results in genus 0 and 1, to a formula for $e_c^{\Sigma_n}(\overline{M}_{2,n})$, for all n. For $g = 3$, however, new phenomena appear; it doesn't suffice to know $e_c(A_3, \mathbb{V}_\lambda)$.

Summarily, our method is to count curves over finite fields and to interpret the data, in accordance with known results (as well as certain widely believed conjectures).

2.2. **Genus one.** Write $\lambda = a \in \mathbb{Z}_{\geq 0}$. For a odd, all cohomology vanishes due to the action of the elliptic involution, so assume a even. For $a > 0$, we have

$$e_c(M_{1,1}, \mathbb{V}_a) = e_c(A_1, \mathbb{V}_a) = -S[a+2] - 1.$$

Scholl [16] has constructed the $S[k]$ as motives. Considered as a Hodge structure, $S[k]$ satisfies

$$S[k] \otimes \mathbb{C} \cong S_k \oplus \overline{S_k},$$

where S_k is the vector space of holomorphic cusp forms for $\mathrm{SL}(2, \mathbb{Z})$ of weight k. So $\dim S[k] = 2 \dim S_k =: 2s_k$. The Hodge types are $(k-1, 0)$ and $(0, k-1)$. The trace of Frobenius at a prime p on $S[k]$

considered as an ℓ-adic Galois representation equals the trace of the Hecke operator $T(p)$ on S_k:

$$\mathrm{Tr}_{F_p} S[k] = \mathrm{Tr}_{T(p)} S_k .$$

For $a = 0$, we have of course $e_c(A_1, \mathbb{V}_0) = L$, the Lefschetz motive, the second cohomology group of a curve: $\dim L = 1$, the Hodge type is $(1,1)$, and $\mathrm{Tr}_{F_q} L = q$. To get a universal formula, we simply put $S[2] = -L - 1$, so that $s_2 := -1$ (sic).

It is clear that $\#M_{1,n}(\mathbb{F}_q)$ can be computed by counting elliptic curves over $\mathbb{F}_q$ and how many points they have; one need only keep in mind that each curve should be counted with the reciprocal of the order of its automorphism group. So, from counting elliptic curves over finite fields, one can compute traces of Hecke operators on S_k (there are other, more straightforward ways of doing this).

2.3. **Genus two.** Just as above, we may assume that the weight $a+b$ of $\lambda = (a,b)$ is even. Our conjecture (cf. [4]) reads as follows:

Conjecture 1. *For $a \geq b \geq 0$ and $a+b$ even,*

$$e_c(A_2, \mathbb{V}_{a,b}) = -S[a-b, b+3] - s_{a+b+4} S[a-b+2] L^{b+1} \\ + s_{a-b+2} - s_{a+b+4} L^{b+1} - S[a+3] + S[b+2] + \tfrac{1}{2}(1 + (-1)^a).$$

Here, $S[j,k]$ is the conjectural motive (constructed as a Galois representation by Weissauer (cf. [19]) for $j > 0$ and $k > 3$) associated to the space $S_{j,k}$ of vector valued Siegel cusp forms of type $\mathrm{Sym}^j \det^k$. In algebro-geometric terms,

$$S_{j,k} = H^0(A_2' \otimes \mathbb{C}, \mathrm{Sym}^j(\mathbb{E}) \otimes \det{}^k(\mathbb{E})(-D_\infty)),$$

where $A_2' = \overline{M}_2$ is the canonical toroidal compactification of A_2 with boundary divisor D_∞ and $\mathbb{E}$ is the Hodge bundle [6, p. 195]. The dimension of $S[j,k]$ equals $4 \dim S_{j,k} =: 4 s_{j,k}$. The trace of F_p on $S[j,k]$ equals the trace of the Hecke operator $T(p)$ on $S_{j,k}$. Again, special care is required in the case of a singular weight λ (i.e., $a = b$ or $b = 0$). First, $S[0,3]$ is defined as $-L^3 - L^2 - L - 1$, so that $s_{0,3} := -1$. Second, the submotive $SK[0, a+3]$ of $S[0, a+3]$ corresponding to the Saito-Kurokawa lifts must be defined as

$$S[2a+4] + s_{2a+4}(L^{a+1} + L^{a+2})$$

for a odd.

The conjecture is proved in the regular case $a > b > 0$, in the context of Galois representations, by combining the work of Weissauer (loc. cit.) and van der Geer [7] (see also the recent paper of Harder [9]).

In the regular case, the various terms in the conjecture have an interpretation that in general is not available in the singular case. The natural map $H^i_c \to H^i$ has kernel the Eisenstein cohomology H^i_{Eis} and image the inner cohomology $H^i_!$. Faltings has proved that $H^i_!(A_g, \mathbb{V}_\lambda) = 0$ for $i \neq g(g+1)/2$ and λ regular ($\lambda_1 > \lambda_2 > \ldots \lambda_g > 0$). The terms in the first line of the display are contributed by $H^3_!(A_2, \mathbb{V}_{a,b})$. The first term is a direct sum of 4-dimensional Galois representations corresponding to Hecke eigenforms (over a field containing the eigenvalues). The second term is the endoscopic contribution. The terms in the second line of the display form the contribution of the Eisenstein cohomology, cf. [9]. We write

$$e_c(A_2, \mathbb{V}_{a,b}) = -S[a-b, b+3] + e_{2,\mathrm{extra}}(a,b)$$

for future reference.

As is well-known, M_2 may be considered as an open substack of A_2. The difference, $A_{1,1} = \mathrm{Sym}^2 A_1$ presents no difficulties (see [1] for details). Let us note that the result of [6] on the possible degrees of the nonzero steps of the Hodge filtration on $H^i_c(A_g, \mathbb{V}_\lambda)$, i.e., that they belong to the set of 2^g partial sums of the g numbers $\lambda_1 + g$, $\lambda_2 + g - 1$, $\ldots$, $\lambda_g + 1$, doesn't hold for $e_c(M_2, \mathbb{V}_{a,b})$.

The conjecture was obtained by determining the trace of F_q on $e_c(M_2, \mathbb{V}_{a,b})$ for $q \leq 37$. Equivalently, we determined the Σ_n-equivariant trace of F_q on $e_c(M_{2,n})$; for this, it suffices to count smooth curves of genus 2 over $\mathbb{F}_q$ together with their numbers of points over $\mathbb{F}_q$ and $\mathbb{F}_{q^2}$. Of great help was Tsushima's formula for $\dim S_{j,k}$ for $k > 4$ (see [17]).

Using an optimized version of the Getzler-Kapranov formula, we have verified that the ensuing conjectural formula for $e_c^{\Sigma_n}(\overline{M}_{2,n})$ satisfies Poincaré Duality for $n \leq 22$. This appears to be a very non-trivial check. Besides the motives mentioned above, we find here also terms of the following types:

$$\wedge^2 S[k], \qquad \mathrm{Sym}^2 S[k], \qquad S[k] \otimes S[l].$$

See [5], §3.6, for some interesting consequences of the occurrence of such terms.

2.4. **Genus three.** As above, we assume that the weight $a+b+c$ of $\lambda = (a,b,c)$ is even. In [1], we formulate the following conjecture:

Conjecture 2. *For $a \geq b \geq c \geq 0$ and $a+b+c$ even,*

$$\begin{aligned} e_c(A_3, \mathbb{V}_{a,b,c}) = S[a-b, b-c, c+4] \\ - e_c(A_2, \mathbb{V}_{a+1,b+1}) + e_c(A_2, \mathbb{V}_{a+1,c}) - e_c(A_2, \mathbb{V}_{b,c}) \\ - e_{2,\mathrm{extra}}(a+1, b+1) \otimes S[c+2] + e_{2,\mathrm{extra}}(a+1, c) \otimes S[b+3] \\ - e_{2,\mathrm{extra}}(b, c) \otimes S[a+4]. \end{aligned}$$

The conjecture was obtained by determining the Frobenius traces on $e_c(M_3, \mathbb{V}_{a,b,c})$ for $q \leq 17$, or equivalently, the Σ_n-equivariant traces of F_q on $e_c(M_{3,n})$; to do this, we counted smooth curves of genus 3 over $\mathbb{F}_q$ together with their numbers of points over $\mathbb{F}_q$, $\mathbb{F}_{q^2}$, and $\mathbb{F}_{q^3}$.

Note that M_3 cannot be considered as an open substack of A_3, even though the Torelli map is an open immersion of the corresponding coarse moduli spaces. The stack M_3 is a stacky double cover of its image in A_3, the locus of Jacobians of smooth curves. The automorphism group of a non-hyperelliptic curve is an index-two subgroup of the automorphism group of its Jacobian, whereas equality holds for a hyperelliptic curve. The double cover is thus ramified along the hyperelliptic locus.

The curve count determines the corresponding count of Jacobians. The non-Jacobians can be dealt with inductively; naturally, this is more involved than in genus 2 (see [1], §8.3).

Conjecture 2 displays a striking recursive structure. In the case of a regular weight, the terms in the second line are explained by the structure of the rank one Eisenstein cohomology, see [7]. The remaining terms are not as well understood, although we can identify the endoscopic and Eisenstein contributions (see [1], §7.4).

Denote by $E_c(A_3, \mathbb{V}_{a,b,c})$ the integer-valued Euler characteristic, computed by my co-authors [2]. Conjecture 2 leads to a dimension prediction for the spaces of (in general vector valued) Siegel cusp forms of genus 3: $s_{a-b,b-c,c+4}$ should equal

$$\tfrac{1}{8}\left(E_c(A_3, \mathbb{V}_{a,b,c}) - E_{3,\mathrm{extra}}(a,b,c)\right).$$

The latter number is a nonnegative integer for all $\lambda = (a,b,c)$ with $a+b+c \leq 60$. In 317 cases it equals zero; then the Frobenius traces on $e_c(A_3, \mathbb{V}_{a,b,c})$ and $e_{3,\mathrm{extra}}(a,b,c)$ are equal for $q \leq 17$. When $a=b=c$, the prediction agrees with Tsuyumine's results [18] on scalar valued cusp forms of genus 3. In their recent work [3], Chenevier and Renard obtain partly conjectural dimension formulas for $s_{a-b,b-c,c+4}$ by very different means. In all explicitly computed cases, including 623 nonzero ones, their results agree with ours!

When the dimension prediction equals 1, we can compute the Hecke eigenvalues for primes $p \leq 17$ of a generator (137 cases with $a+b+c \leq 60$). This has enabled us to conjecture the existence of 3 types of lifts, one of which may occur for regular λ: for $a \geq b \geq c$ and Hecke eigenforms $f \in S_{b+3}$, $g \in S_{a+c+5}$, and $h \in S_{a-c+3}$, we conjecture the existence of a Hecke eigenform $F \in S_{a-b,b-c,c+4}$ with spinor L-function

$$L(F,s) = L(f \otimes g, s)\, L(f \otimes h, s-c-1).$$

See [1], Conj. 7.7; this extends work of Miyawaki [15] and Ikeda [13] in the scalar valued case.

We also expect the existence of lifts from G_2, following work of Gross and Savin; see [1], §9.1.

Finally, §10 of [1] discusses certain conjectural congruences for Hecke eigenvalues of various types of cusp forms. Below, I just give an overview; for more details and precise references, I refer to [1].

The base case is Harder's conjecture, tying an elliptic cusp form to a Siegel cusp form of degree 2, in general vector valued. The congruences are in this case modulo powers of an ordinary prime dividing a suitable 'critical value' of the elliptic cusp form (the actual critical value of the completed L-function divided by the appropriate period). They originate from denominators of certain Eisenstein classes in the Betti cohomology. Harder's original conjecture is trivially true when Saito-Kurokawa lifts are present. But one can refine it by considering only Siegel cusp eigenforms that aren't of this type. This refined statement has been proved in certain cases by Dummigan, Ibukiyama, and Katsurada.

Along analogous lines, we formulate a generalization to the vector-valued case of the Kurokawa-Mizumoto congruence, proved by Katsurada and Mizumoto. It originates from a different type of Eisenstein classes and relies on work of Satoh, Dummigan, and Harder.

In genus 2, there is also a Yoshida-type congruence, originating from the endoscopic contribution. The required critical values were here computed by Dummigan.

As to congruences in genus 3, we formulate two conjectural congruences of Eisenstein type connected to the two types of lifts mentioned above that can only occur for singular λ. The first one generalizes work of Katsurada on Miyawaki-Ikeda lifts and uses work of Mellit and Katsurada; in one case, the original conjecture was proved by Poor and Yuen. Furthermore, we have found examples for two other congruences of Eisenstein type; we also see possibilities for two additional such congruences, but haven't found examples yet.

We conclude [1] with a congruence connected to the type of lift that can occur for regular λ and a congruence connected to one of the two endoscopic contributions. This relies on work of Dummigan and Mellit. In both cases, we have one example. We have no examples for a congruence connected to the other endoscopic contribution.

2.5. **Curves of genus three.** As mentioned above, when considered as a map of stacks, the Torelli map $M_3 \to A_3$ is $2:1$ onto its image. The answers for a local system of even weight $a+b+c$ for A_3 and the loci of products will yield the answer for M_3. But the local systems of odd weight will in general have cohomology on M_3, whereas they have no cohomology on A_3. A priori, there doesn't seem to be a reason why this cohomology should be 'explainable' in terms of Siegel modular forms.

In fact, I have been able to prove that new types of motives do appear in the cohomology of local systems of odd weight on M_3 (the first examples are provided by two systems of weight 17). My method is based on three ingredients. The first is an explicit formula for the weight-zero term in the Euler characteristic of compactly supported cohomology of a symplectic local system on M_g. The formula is proved for all $g \leq 9$. The theoretical basis for this work is provided by the work of Getzler and Kapranov [8] on modular operads. The concrete formula was found by Zagier based on data obtained for $g \leq 8$ and then verified for $g = 9$. The second ingredient is provided by the data obtained with Bergström and van der Geer by counting curves of genus at most 3 over finite fields. The third ingredient is the realization that non-Tate twisted terms in the Euler characteristic are detected if the trace of Frobenius at a prime p and the weight-zero term differ modulo p, coupled with the fact that motives corresponding to modular forms of genus 1 and 2 are *not* detected at certain primes. Thus elliptic motives are not detected at $p \leq 7$, but $S[12]$ is detected at $p = 11$; certain genus 2 motives are detected at $p = 7$, but all relevant genus 2 motives are not detected at $p \leq 5$; finally, for two local systems of weight 17 on M_3, cohomology is detected at $p = 5$, which cannot possibly come from Siegel modular forms of genus ≤ 3.

Subsequently, in the style of the initial work done for A_2 and A_3, I was able to guess formulas for the motivic Euler characteristic of all local systems of weight ≤ 17 on M_3, the two local systems mentioned above being the *only* exceptions. For two new local systems, the answer was particularly interesting, since it pointed directly to the existence of some kind of modular form — if one assumes that a variant of the Faltings-Chai-Eichler-Shimura theory still holds. In particular, the existence of a classical modular form of weight 9 on M_3 is predicted this way. Luckily, such a form is known to exist; it is the square root of the classical Siegel modular form of weight 18 on A_3 which vanishes on the divisor of hyperelliptic Jacobians. Modular forms on M_g of this type were studied in detail by Ichikawa in the 1990's [10, 11, 12] and baptized *Teichmüller modular forms*. The form of weight 9 was already known to Klein and it has been studied recently in connection with the problem of distinguishing a three-dimensional Jacobian from a non-Jacobian [14].

Recently, van der Geer and I found a method for constructing vector valued Teichmüller modular forms, which apparently haven't been studied earlier. We still need to check certain details, but the first results are very promising. In particular, we construct Teichmüller modular forms corresponding to each of the special local systems mentioned above. Our method works just as well in genus 2 and provides us with a new way to study the modular forms occurring there.

We also want to study in detail the Galois representations associated to Teichmüller modular forms. In two cases, we know that representations associated to Siegel modular forms of genus 2 are involved, which suggests that the modular forms themselves are lifts in a suitable (new) sense.

An exciting development here is the recent appearance of the preprint [3] of Chenevier and Renard. They specifically study level one, which is very relevant to our work. At this point, it seems likely that two of the seven 6-dimensional symplectic Galois representations of motivic weight 23 associated to cusp forms for SO(7) identified by them appear in the cohomology of $M_{3,17}$.

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Department of Mathematics, KTH Royal Institute of Technology, 100 44 Stockholm, Sweden

E-mail address: `faber@kth.se`