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# A Simplified Characterisation of Provably Computable Functions of the System $\mathrm{ID}_1$ of Inductive Definitions (Extended Abstract)

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**Abstract**

We present a simplified and streamlined characterisation of provably total computable functions of the system $\mathrm{ID}_1$ of non-iterated inductive definitions. The idea of the simplification is to employ the method of operator-controlled derivations that was originally introduced by Wilfried Buchholz and afterwards applied by the second author to a streamlined characterisation of provably total computable functions of Peano arithmetic PA.

## 1 Introduction

As stated by Gödel's first incompleteness theorem, any reasonable consistent formal system has an unprovable $\Pi^0_2$-sentence that is true in the standard model of arithmetic. This means that the total (computable) functions whose totality is provable in a consistent system, which are known as *provably (total) computable functions*, form a proper subclass of total computable functions. Hence it is natural to ask how we can describe the provably computable functions of a given system. Not surprisingly provably computable functions are closely related to provable well-ordering, i.e., *ordinal analysis*. Several successful applications of techniques from ordinal analysis to provably computable functions have been provided by B. Blankertz and A. Weiermann

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[1], W. Buchholz [5], Buchholz, E. A. Cichon and Weiermann [6], or M. Michelbrink [9].

Modern ordinal analysis is based on the method of *local predicativity*, that was first introduced by W. Pohlers, cf. [10, 11]. Successful applications of local predicativity to provably computable functions contain works by Blankertz and Weiermann [12] and by Weiermann [2]. However, to the authors' knowledge, the most successful way in ordinal analysis is based on the method of *operator-controlled derivations*, an essential simplification of local predicativity, that was introduced by Buchholz [3]. In [13] the second author successfully applied the method of operator-controlled derivations to a streamlined characterisation of provably computable functions of PA. (See also [11, Section 2.1.5].) Technically this work aims to lift up the characterisation obtained in [13] to an impredicative system $\mathrm{ID}_1$ of non-iterated inductive definitions. We introduce an ordinal notation system $\mathcal{O}(\Omega)$ and define a computable function $f^\alpha$ for a starting numerical function $f : \mathbb{N} \to \mathbb{N}$ by transfinite recursion on $\alpha \in \mathcal{O}(\Omega)$. The transfinite definition of $f^\alpha$ stems from [13]. We show that a function is provably computable in $\mathrm{ID}_1$ if and only if it is a Kalmar elementary function in $\{\mathsf{s}^\alpha \mid \alpha \in \mathcal{O}(\Omega) \text{ and } \alpha < \Omega\}$, where $\mathsf{s}$ denotes the numerical successor function $m \mapsto m + 1$ and $\Omega$ denotes the least non-computable ordinal (Corollary 6.4).

This paper consists of two materials, a technical report [8] by the authors and a draft [14] by the second author. Section 3–6 consist of [8] and Section 7 consists of [14]. We mention in particular that the ordinal notation system $\mathcal{OT}(\mathcal{F})$ stems from [14]. Most of proofs are omitted due to the page limitation. We note however that there is a non-trivial error in the technical report [8, p. 8, Lemma 15.5]. We restate Lemma 4.4.5, provide its proof and discuss in detail about embedding (Section 5) affected by this correction. The full details of missing proofs will appear in [7].

## 2 Preliminaries

In order to make our contribution precise, in this preliminary section we collect the central notions. We write $\mathcal{L}_{\mathrm{PA}}$ to denote the standard language of first order theories of arithmetic. In particular we suppose that the constant 0 and the successor function symbol $S$ are included in $\mathcal{L}_{\mathrm{PA}}$. For each natural $m$ we use the notation $\underline{m}$ to denote the corresponding numeral built from 0 and $S$. Let a set variable $X$ denote a subset of $\mathbb{N}$. We write $X(t)$ instead of $t \in X$ and $\mathcal{L}_{\mathrm{PA}}(X)$ for $\mathcal{L}_{\mathrm{PA}} \cup \{X\}$. Let $\mathsf{FV}_1(A)$ denote the set of free number variables appearing in a formula $A$ and $\mathsf{FV}_2(A)$ the set of free set variables in $A$. And then let $\mathsf{FV}(A) := \mathsf{FV}_1(A) \cup \mathsf{FV}_2(A)$. For a fresh set variable $X$ we call an $\mathcal{L}_{\mathrm{PA}}(X)$-formula $\mathcal{A}(x)$ a *positive operator form* if $\mathsf{FV}_1(\mathcal{A}(x)) \subseteq \{x\}$, $\mathsf{FV}_2(\mathcal{A}(x)) = \{X\}$, and $X$ occurs only positively in $\mathcal{A}$.

Let $\mathsf{FV}_1(\mathcal{A}(x)) = \{x\}$. For a formula $F(x)$ such that $x \in \mathsf{FV}_1(F(x))$ we write $\mathcal{A}(F, t)$ to denote the result of replacing in $\mathcal{A}(t)$ every subformula $X(s)$ by $F(s)$. The language $\mathcal{L}_{\mathrm{ID}_1}$ of the *system* $\mathrm{ID}_1$ *of non-iterated inductive definitions* is defined by $\mathcal{L}_{\mathrm{ID}_1} := \mathcal{L}_{\mathrm{PA}} \cup \{P_{\mathcal{A}} \mid \mathcal{A} \text{ is a positive operator form}\}$ where for each positive operator form $\mathcal{A}$, $P_{\mathcal{A}}$ denotes a new unary predicate symbol. We write $\mathcal{T}(\mathcal{L}_{\mathrm{ID}_1}, \mathcal{V})$ to denote the set of $\mathcal{L}_{\mathrm{ID}_1}$-terms and $\mathcal{T}(\mathcal{L}_{\mathrm{ID}_1})$ to denote the set of closed $\mathcal{L}_{\mathrm{ID}_1}$-terms. The axioms of $\mathrm{ID}_1$ consist of the axioms of Peano arithmetic PA in the language $\mathcal{L}_{\mathrm{ID}_1}$ and the following new axiom schemata $(\mathsf{ID}_1)$ and $(\mathsf{ID}_2)$:

(ID1) $\forall x(\mathcal{A}(P_{\mathcal{A}}, x) \rightarrow P_{\mathcal{A}}(x))$.

(ID2) (The universal closure of) $\forall x(\mathcal{A}(F, x) \rightarrow F(x)) \rightarrow \forall x(P_{\mathcal{A}}(x) \rightarrow F(x))$, where $F$ is an $\mathcal{L}_{\mathrm{ID}_1}$-formula.

For each $n \in \mathbb{N}$ we write $\mathrm{I}\Sigma_n$ to denote the fragment of Peano arithmetic PA with induction restricted to $\Sigma^0_n$-formulas. Let $k$ be a natural number and $f : \mathbb{N}^k \rightarrow \mathbb{N}$ a numerical function and $T$ be a system of arithmetic containing $\mathrm{I}\Sigma_1$. Then we say that $f$ is *provably total computable in* $T$ or *provably computable in* $T$ for short if there exists a $\Sigma^0_1$-formula $A_f(x_1, \ldots, x_k, y)$ such that (i) $\mathsf{FV}(A_f) = \mathsf{FV}_1(A_f) = \{x_1, \ldots, x_k, y\}$, (ii) for all $\vec{m}, n \in \mathbb{N}$, $f(\vec{m}) = n$ holds if and only if $A_f(\underline{\vec{m}}, \underline{n})$ is true in the standard model $\mathbb{N}$ of PA, and (iii) $\forall \vec{x} \exists ! y A_f(\vec{x}, y)$ is a theorem in $T$.

# 3 A non-computable ordinal notation system $\mathcal{OT}(\mathcal{F})$

In this section we introduce a *non-computable* ordinal notation system $\mathcal{OT}(\mathcal{F}) = \langle \mathcal{OT}(\mathcal{F}), < \rangle$. This new ordinal notation system is employed in the next section. For an element $\alpha \in \mathcal{OT}(\mathcal{F})$ let $\mathcal{OT}(\mathcal{F}) \upharpoonright \alpha$ denote the set $\{\beta \in \mathcal{OT}(\mathcal{F}) \mid \beta < \alpha\}$.

**Definition 3.1** *We define three sets* $\mathsf{SC} \subseteq \mathbb{H} \subseteq \mathcal{OT}(\mathcal{F})$ *of ordinal terms and a set* $\mathcal{F}$ *of unary function symbols simultaneously. Let* $0$, $\varphi$, $\Omega$, $\mathsf{S}$, $\mathsf{E}$ *and* $+$ *be distinct symbols.*

1. $0 \in \mathcal{OT}(\mathcal{F})$ *and* $\Omega \in \mathsf{SC}$.
2. $\{\mathsf{S}, \mathsf{E}\} \subseteq \mathcal{F}$.
3. *If* $\alpha \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, *then* $\mathsf{S}(\alpha) \in \mathcal{OT}(\mathcal{F})$ *and* $\mathsf{E}(\alpha) \in \mathbb{H}$.
4. *If* $\{\alpha_1, \ldots, \alpha_l\} \subseteq \mathbb{H}$ *and* $\alpha_1 \geq \cdots \geq \alpha_l$, *then* $\alpha_1 + \cdots + \alpha_l \in \mathcal{OT}(\mathcal{F})$.
5. *If* $\{\alpha, \beta\} \subseteq \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, *then* $\varphi\alpha\beta \in \mathbb{H}$.
6. *If* $\alpha \in \mathcal{OT}(\mathcal{F})$ *and* $\xi \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, *then* $\Omega^\alpha \cdot \xi \in \mathbb{H}$.
7. *If* $F \in \mathcal{F}$, $\alpha \in \mathcal{OT}(\mathcal{F})$ *and* $\xi \in \mathcal{OT}(\mathcal{F}) \upharpoonright \Omega$, *then* $F^\alpha(\xi) \in \mathsf{SC}$.
8. *If* $F \in \mathcal{F}$ *and* $\alpha \in \mathcal{OT}(\mathcal{F})$, *then* $F^\alpha \in \mathcal{F}$.

We write $\omega^\alpha$ to denote $\varphi 0\alpha$ and $m$ to denote $\omega^0 \cdot m = \underbrace{\omega^0 + \cdots + \omega^0}_{m \text{ many}}$.

Let $\mathsf{Ord}$ denote the class of ordinals and $\mathsf{Lim}$ the class of limit ones. We define a semantic $[\cdot]$ for $\mathcal{OT}(\mathcal{F})$, i.e., $[\cdot] : \mathcal{OT}(\mathcal{F}) \to \mathsf{Ord}$. The well ordering $<$ on $\mathcal{OT}(\mathcal{F})$ is defined by $\alpha < \beta \Leftrightarrow [\alpha] < [\beta]$. Let $\Omega_1$ denote the least non-computable ordinal $\omega_1^{\mathrm{CK}}$. For an ordinal $\alpha$ we write $\alpha =_{NF} \Omega_1^{\alpha_1} \cdot \beta_1 + \cdots + \Omega_1^{\alpha_l} \cdot \beta_l$ if $\alpha > \alpha_1 > \cdots > \alpha_l$, $\{\beta_1, \ldots, \beta_l\} \subseteq \Omega_1$, and $\alpha = \Omega_1^{\alpha_1} \cdot \beta_1 + \cdots + \Omega_1^{\alpha_l} \cdot \beta_l$. Let $\varepsilon_\alpha$ denote the $\alpha$th epsilon number. One can observe that for each ordinal $\alpha < \varepsilon_{\Omega_1+1}$ there uniquely exists a set $\{\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l\}$ of ordinals such that $\alpha =_{NF} \Omega_1^{\alpha_1} \cdot \beta_1 + \cdots + \Omega_1^{\alpha_l} \cdot \beta_l$. For a set $K \subseteq \mathsf{Ord}$ and for an ordinal $\alpha$ we will write $K < \alpha$ to abbreviate $(\forall \xi \in K)\xi < \alpha$, and dually $\alpha \leq K$ to abbreviate $(\exists \xi \in K)\alpha \leq \xi$.

**Definition 3.2 (Collapsing operators)** *1. Let $\alpha$ be an ordinal such that $\alpha =_{NF} \Omega_1^{\alpha_1} \cdot \beta_1 + \cdots + \Omega_1^{\alpha_l} \cdot \beta_l < \varepsilon_{\Omega_1+1}$. The set $K_\Omega \alpha$ of* coefficients *of $\alpha$ is defined by*

$$K_\Omega \alpha = \{\beta_1, \ldots, \beta_l\} \cup K_\Omega \alpha_1 \cup \cdots \cup K_\Omega \alpha_l.$$

*2. Let $F : \mathsf{Ord} \to \mathsf{Ord}$ be an ordinal function. Then a function $F^\alpha : \mathsf{Ord} \to \mathsf{Ord}$ is defined by transfinite recursion on $\alpha \in \mathsf{Ord}$ by*

$$\begin{cases} F^0(\xi) & = & F(\xi), \\ F^\alpha(\xi) & = & \min\{\gamma \in \mathsf{Ord} \mid \omega^\gamma = \gamma,\ K_\Omega \alpha \cup \{\xi\} < \gamma \text{ and} \\ & & (\forall \eta < \gamma)(\forall \beta < \alpha)(K_\Omega \beta < \gamma \Rightarrow F^\beta(\eta) < \gamma)\}. \end{cases}$$

**Corollary 3.3** *Let $F : \mathsf{Ord} \to \mathsf{Ord}$ be an ordinal function. Then $F^\beta(\eta) < F^\alpha(\xi)$ holds if $(\beta < \alpha \wedge K_\Omega \beta \cup \{\eta\} < F^\alpha(\xi))$ or $(\alpha \leq \beta \wedge F^\beta(\eta) \leq K_\Omega \alpha)$.*

**Proposition 3.4** *Suppose that $\alpha < \varepsilon_{\Omega_1+1}$, a function $F : \mathsf{Ord} \to \mathsf{Ord}$ has a $\Sigma_1$-definition in the $\Omega_1$th stage $L_{\Omega_1}$ of the constructible hierarchy $(L_\alpha)_{\alpha \in \mathsf{Ord}}$ and that $F(\xi) < \Omega_1$ for all $\xi < \Omega_1$. Then $F^\alpha$ also has a $\Sigma_1$-definition in $L_{\Omega_1}$ and $F^\alpha(\xi) < \Omega_1$ holds for all $\xi < \Omega_1$.*

**Proposition 3.5** *For any $\alpha \in \mathsf{Ord}$, for any $\eta, \xi < \Omega_1$ and for any ordinal function $F : \Omega_1 \to \Omega_1$, if $\eta < F^\alpha(\xi)$, then $F^\alpha(\eta) \leq F^\alpha(\xi)$.*

**Definition 3.6** *We define the* value *$[\alpha] \in \mathsf{Ord}$ of an ordinal term $\alpha \in \mathcal{OT}(\mathcal{F})$ by recursion on the length of $\alpha$.*

1. $[0] = 0$ *and* $[\Omega] = \Omega_1$.

2. $[\alpha + \beta] = [\alpha] + [\beta]$.

3. $[\varphi\alpha\beta] = [\varphi]\,[\alpha][\beta]$, *where $[\varphi]$ is the standard Veblen function, i.e.,*

$$\begin{cases} [\varphi]0\beta & = & \omega^\beta, & \\ [\varphi]\,(\alpha+1)0 & = & \sup\{([\varphi]\,\alpha)^n 0 \mid n \in \omega\}, & \\ [\varphi]\,\gamma 0 & = & \sup\{[\varphi]\,\alpha 0 \mid \alpha < \gamma\} & \text{if } \gamma \in \mathsf{Lim}, \\ [\varphi]\,(\alpha+1)(\beta+1) & = & \sup\{([\varphi]\,\alpha)^n([\varphi](\alpha+1)\beta + 1 \mid n \in \omega\}, & \\ [\varphi]\,\gamma(\beta+1) & = & \sup\{[\varphi]\,\alpha([\varphi]\,\gamma\beta + 1) \mid \alpha < \gamma\} & \text{if } \gamma \in \mathsf{Lim}, \\ [\varphi]\,\alpha\gamma & = & \sup\{[\varphi]\,\alpha\beta \mid \beta < \gamma\} & \text{if } \gamma \in \mathsf{Lim}. \end{cases}$$

4. $[\Omega^\alpha \cdot \xi] = \Omega_1^{[\alpha]} \cdot [\xi]$.

5. $[\mathsf{S}(\alpha)] = [\mathsf{S}]([\alpha])$, *where* $[\mathsf{S}]$ *denotes the ordinal successor* $\alpha \mapsto \alpha + 1$. *Clearly* $\{[\mathsf{S}](\xi) \mid \xi < \Omega_1\} \subseteq \Omega_1$.

6. $[\mathsf{E}(\alpha)] = [\mathsf{E}]([\alpha])$, *where the function* $[\mathsf{E}] : \mathsf{Ord} \to \mathsf{Ord}$ *is defined by* $[\mathsf{E}](\alpha) = \min\{\xi \in \mathsf{Ord} \mid \omega^\xi = \xi$ *and* $\alpha < \xi\}$. *It is also clear that* $\{[\mathsf{E}](\xi) \mid \xi < \Omega_1\} \subseteq \Omega_1$ *holds.*

7. $[F^\alpha(\xi)] = [F]^{[\alpha]}([\xi])$.

**Definition 3.7** *For all* $\alpha, \beta \in \mathcal{OT}(\mathcal{F})$, $\alpha < \beta$ *if* $[\alpha] < [\beta]$, *and* $\alpha = \beta$ *if* $[\alpha] = [\beta]$.

We will identify each element $\alpha \in \mathcal{OT}(\mathcal{F})$ with its value $[\alpha] \in \mathsf{Ord}$. Accordingly we will write $K_\Omega \alpha$ instead of $K_\Omega[\alpha]$ for $\alpha \in \mathcal{OT}(\mathcal{F})$. Further for a finite set $K \subseteq \mathsf{Ord}$ we write $K_\Omega K$ to denote the finite set $\bigcup_{\xi \in K} K_\Omega \xi$. By this identification, $\mathbb{H}$ is the set of *additively indecomposable* ordinals and $\mathsf{SC}$ is the set of *strongly critical* ordinals, i.e, $\mathsf{SC} \subseteq \mathbb{H} \subseteq \mathsf{Lim} \cup \{1\} \subseteq \mathsf{Ord}$.

**Corollary 3.8** $F^\alpha(\xi) < \Omega$ *for any* $F \in \mathcal{F}$ *and* $\xi < \Omega$.

**Proof.** Proof by induction over the build-up of $F \in \mathcal{F}$. □

**Corollary 3.9** 1. $K_\Omega 0 = K_\Omega \Omega = \emptyset$.

2. *If* $K_\Omega \alpha < \xi$ *and* $\xi \in \mathsf{SC}$, *then* $K_\Omega \mathsf{S}(\alpha) < \xi$.

3. $K_\Omega \mathsf{E}(\alpha) = \{\mathsf{E}(\alpha)\}$ *(since* $\alpha < \Omega$*).*

4. *If* $K_\Omega \alpha \cup K_\Omega \beta < \xi$ *and* $\xi \in \mathsf{SC}$, *then* $K_\Omega(\alpha + \beta) < \xi$.

5. $K_\Omega \varphi\alpha\beta = \{\varphi\alpha\beta\}$ *(since* $\alpha, \beta < \Omega$*). Further, if* $\alpha, \beta < \xi$ *and* $\xi \in \mathsf{SC}$, *then* $\varphi\alpha\beta < \xi$.

6. $K_\Omega F^\alpha(\xi) = \{F^\alpha(\xi)\}$ *(since* $\xi < \Omega$*).*

By Corollary 3.8 each function symbol in $\mathcal{F}$ defines a weakly increasing function $F : \Omega \to \Omega$ such that $\xi < F(\xi)$ holds for all $\xi \in \Omega$. In the rest of this section let $F$ denote such a function. For a finite set $K \subseteq \mathsf{Ord}$ we will use the notation $F[K](\xi)$ to abbreviate $F(\max(K \cup \{\xi\}))$.

**Lemma 3.10** *Let* $K \subseteq \mathsf{Ord}$ *be a finite set such that* $K < \Omega$. *Then* $(F[K])^\alpha(\xi) \leq F^\alpha[K](\xi)$ *for all* $\xi < \Omega$.

**Lemma 3.11** $(F^\alpha)^\beta(\xi) \leq F^{\alpha+\beta}(\xi)$ *for all* $\xi < \Omega$.

# 4 An infinitary proof system $\mathrm{ID}_1^\infty$

In this section we introduce the main definition of this paper, a new infinitary proof system $\mathrm{ID}_1^\infty$, to which the new ordinal notation system $\mathcal{OT}(\mathcal{F})$ is connected, and into which every (finite) proof in $\mathrm{ID}_1$ can be embedded in good order. For each positive operator form $\mathcal{A}$ and for each ordinal term $\alpha \in (\mathcal{OT}(\mathcal{F}) \restriction \Omega) \cup \{\Omega\}$ let $P_{\mathcal{A}}^{<\alpha}$ be a new unary predicate symbol. Let us define an infinitary language $\mathcal{L}^*$ of $\mathrm{ID}_1^\infty$ by $\mathcal{L}^* = \mathcal{L}_{\mathrm{PA}} \cup \{\neq, \not\leq\} \cup \{P_{\mathcal{A}}^{<\alpha}, \neg P_{\mathcal{A}}^{<\alpha} \mid \alpha \in (\mathcal{OT}(\mathcal{F}) \restriction \Omega) \cup \{\Omega\}$ and $\mathcal{A}$ is a positive operator form$\}$. Let us write $P_{\mathcal{A}}^{<\Omega}$ to denote $P_{\mathcal{A}}$ to have the inclusion $\mathcal{L}_{\mathrm{ID}_1} \subseteq \mathcal{L}^*$. We write $\mathcal{T}(\mathcal{L}^*)$ to denote the set of closed $\mathcal{L}^*$-terms. Specifically, the language $\mathcal{L}^*$ contains complementary predicate symbol $\neg P$ for each predicate symbol $P \in \mathcal{L}^*$. We note that the negation $\neg$ nor the implication $\rightarrow$ is not included as a logical symbol. The negation $\neg A$ is defined via de Morgan's law by $\neg(\neg P(\vec{t})) :\equiv P(\vec{t})$ for an atomic formula $P(\vec{t})$, $\neg(A \wedge B) :\equiv \neg A \vee \neg B$, $\neg(A \vee B) :\equiv \neg A \wedge \neg B$, $\neg \forall x A :\equiv \exists x \neg A$ and $\neg \exists x A :\equiv \forall x \neg A$. The implication $A \rightarrow B$ is defined by $\neg A \vee B$. We start with technical definitions.

**Definition 4.1 (Complexity measures $\mathsf{lh}$, $\mathsf{rk}$, $\mathsf{k}^\Pi$ $\mathsf{k}^\Sigma$, $\mathsf{k}$ of $\mathcal{L}^*$-formulas)**

1. *The* length $\mathsf{lh}(A)$ *of an $\mathcal{L}^*$-formula $A$ is the number of the symbols $P_{\mathcal{A}}^{<\alpha}$, $\neg P_{\mathcal{A}}^{<\alpha}$, $\vee$, $\wedge$, $\exists$ and $\forall$ occurring in $A$.*

2. *The* rank $\mathsf{rk}(A)$ *of an $\mathcal{L}^*$-formula $A$.*

   (a) $\mathsf{rk}(P_{\mathcal{A}}^{<\alpha}(t)) := \mathsf{rk}(\neg P_{\mathcal{A}}^{<\alpha}(t)) := \omega \cdot \alpha$.

   (b) $\mathsf{rk}(A) := 0$ *if $A$ is an $\mathcal{L}_{\mathrm{ID}_1}$-literal.*

   (c) $\mathsf{rk}(A \wedge B) := \mathsf{rk}(A \vee B) := \max\{\mathsf{rk}(A), \mathsf{rk}(B)\} + 1$.

   (d) $\mathsf{rk}(\forall x A) := \mathsf{rk}(\exists x A) := \mathsf{rk}(A) + 1$.

3. *The set* $\mathsf{k}^\Pi(A)$ *of* $\Pi$-coefficients *of an $\mathcal{L}^*$-formula $A$.*

   (a) $\mathsf{k}^\Pi(P_{\mathcal{A}}^{<\alpha}(t)) := \{0\}$, $\mathsf{k}^\Pi(\neg P_{\mathcal{A}}^{<\alpha}(t)) := \{0, \alpha\}$.

   (b) $\mathsf{k}^\Pi(A) := \{0\}$ *if $A$ is an $\mathcal{L}_{\mathrm{ID}_1}$-literal.*

   (c) $\mathsf{k}^\Pi(A \wedge B) := \mathsf{k}^\Pi(A \vee B) := \mathsf{k}^\Pi(A) \cup \mathsf{k}^\Pi(B)$.

   (d) $\mathsf{k}^\Pi(\forall x A) := \mathsf{k}^\Pi(\exists x A) := \mathsf{k}^\Pi(A)$.

4. *The set* $\mathsf{k}^\Sigma(A)$ *of* $\Sigma$-coefficients *of an $\mathcal{L}^*$-formula $A$.*
   $\mathsf{k}^\Sigma(A) := \mathsf{k}^\Pi(\neg A)$.

5. *The set* $\mathsf{k}(A)$ *of all the* coefficients *of an $\mathcal{L}^*$-formula $A$.*
   $\mathsf{k}(A) := \mathsf{k}^\Pi(A) \cup \mathsf{k}^\Sigma(A)$.

6. *The set* $\mathsf{k}_\Omega^\Pi(A)$ *of* $\Pi$-coefficients *of an $\mathcal{L}^*$-formula $A$ less than $\Omega$.*
   $\mathsf{k}_\Omega^\Pi(A) := \mathsf{k}^\Pi(A) \restriction \Omega$.
   *The set $\mathsf{k}_\Omega^\Sigma(A)$ and $\mathsf{k}_\Omega(A)$ are defined accordingly.*

By definition $\mathsf{rk}(A) = \mathsf{rk}(\neg A)$, $\mathsf{k}(A) = \mathsf{k}(\neg A)$ and $\mathsf{k}_\Omega(A) = \mathsf{k}_\Omega(\neg A)$.

**Definition 4.2 (Complexity measures val, ord, $N$ of $\mathcal{L}^*$-terms)**

1. *The* value $\mathsf{val}(t)$ *of a term* $t \in \mathcal{T}(\mathcal{L}_{\mathrm{ID}_1}) = \mathcal{T}(\mathcal{L}_{\mathrm{PA}})$ *is the value of the closed term* $t$ *in the standard model* $\mathbb{N}$ *of the Peano arithmetic* PA.

2. *A complexity measure* $\mathsf{ord} : \mathcal{T}(\mathcal{L}^*) \to (\mathcal{OT}(\mathcal{F}) \restriction \Omega) \cup \{\Omega\}$ *is defined by*

$$\begin{cases} \mathsf{ord}(t) := 0 & \text{if } t \in \mathcal{T}(\mathcal{L}_{\mathrm{ID}_1}), \\ \mathsf{ord}(\alpha) := \alpha & \text{if } \alpha \in \mathcal{OT}(\mathcal{F}). \end{cases}$$

3. *The* norm $N(\alpha)$ *of* $\alpha \in \mathcal{OT}(\mathcal{F})$.

   (a) $N(0) = 0$ *and* $N(\Omega) = 1$.

   (b) $N(\mathsf{S}(\alpha)) = N(\alpha) + 1$.

   (c) $N(\mathsf{E}(\alpha)) = N(\alpha) + 1$.

   (d) $N(\alpha + \beta) = N(\alpha) + N(\beta)$.

   (e) $N(\varphi\alpha\beta) = N(\alpha) + N(\beta) + 1$,

   (f) $N(\Omega^\alpha \cdot \xi) = N(\alpha) + N(\xi) + 1$.

   (g) $N(F^\alpha(\xi)) = N(F(\xi)) + N(\alpha)$. *(Note that* $F(\xi) \in \mathcal{OT}(\mathcal{F})$ *if* $F^\alpha(\xi) \in \mathcal{OT}(\mathcal{F})$.*)*

   *The norm is extended to a complexity measure* $N : \mathcal{T}(\mathcal{L}^*) \to \mathbb{N}$ *by*

$$\begin{cases} N(t) := \mathsf{val}(t) & \text{if } t \in \mathcal{T}(\mathcal{L}_{\mathrm{ID}_1}), \\ N(\alpha) := N(\alpha) & \text{if } \alpha \in \mathcal{OT}(\mathcal{F}). \end{cases}$$

By definition $N(\omega^\alpha) = N(\varphi 0 \alpha) = N(\alpha) + 1$ and $N(m) = N(\omega^0 \cdot m) = m$ for any $m < \omega$. This seems to be a good point to explain why we contain the constant $\Omega$ in $\mathcal{OT}(\mathcal{F})$. Having that $N(\Omega) = 1$ removes some technicalities.

**Definition 4.3** *We define a relation* $\simeq$ *between* $\mathcal{L}^*$*-sentences and (infinitary) propositional* $\mathcal{L}^*$*-sentences.*

1. $\neg P_{\mathcal{A}}^{<\alpha}(t) :\simeq \bigwedge_{\xi \in \mathcal{OT}(\mathcal{F})\restriction\alpha} \neg \mathcal{A}(P_{\mathcal{A}}^{<\xi}, t)$ *and* $P_{\mathcal{A}}^{<\alpha}(t) :\simeq \bigvee_{\xi \in \mathcal{OT}(\mathcal{F})\restriction\alpha} \mathcal{A}(P_{\mathcal{A}}^{<\xi}, t)$.

2. $A \wedge B :\simeq \bigwedge_{\iota \in \{\underline{0},\underline{1}\}} A_\iota$ *and* $A \vee B :\simeq \bigvee_{\iota \in \{\underline{0},\underline{1}\}} A_\iota$ *where* $A_{\underline{0}} \equiv A$ *and* $A_{\underline{1}} \equiv B$.

3. $\forall x A(x) :\simeq \bigwedge_{t \in \mathcal{T}(\mathcal{L}_{\mathrm{ID}_1})} A(t)$ *and* $\exists x A(x) :\simeq \bigvee_{t \in \mathcal{T}(\mathcal{L}_{\mathrm{ID}_1})} A(t)$.

We call an $\mathcal{L}^*$-sentence $A$ a $\bigwedge$-*type* (conjunctive type) if $A \simeq \bigwedge_{\iota \in J} A_\iota$ for some $A_\iota$, and a $\bigvee$-*type* (disjunctive type) if $A \simeq \bigvee_{\iota \in J} A_\iota$ for some $A_\iota$. For the sake of simplicity we will write $\bigwedge_{\xi<\alpha} A_\xi$ instead of $\bigwedge_{\xi \in \mathcal{OT}(\mathcal{F})\restriction\alpha} A_\xi$ and write $\bigvee_{\xi<\alpha} A_\xi$ accordingly.

**Lemma 4.4** *1. If either* $A \simeq \bigwedge_{\iota\in J} A_\iota$ *or* $A \simeq \bigvee_{\iota\in J} A_\iota$*, then for all* $\iota \in J$, $\mathsf{k}^\Pi(A_\iota) \subseteq \{\mathsf{ord}(\iota)\} \cup \mathsf{k}^\Pi(A)$ *and* $\mathsf{k}^\Sigma(A_\iota) \subseteq \{\mathsf{ord}(\iota)\} \cup \mathsf{k}^\Sigma(A)$.

*2. For any* $\alpha \in \mathcal{OT}(\mathcal{F})$*, if* $A \simeq \bigwedge_{\xi<\alpha} A_\xi$*, then* $(\exists\sigma \in \mathsf{k}^\Pi(A))(\forall\xi < \alpha)[\xi \le \sigma]$.

*3. For any* $\mathcal{L}^*$*-sentence* $A$, $\mathsf{rk}(A) = \omega \cdot \max \mathsf{k}(A) + n$ *for some* $n \le \mathsf{lh}(A)$.

*4. If* $\mathsf{rk}(A) = \Omega$*, then either* $A \equiv P_{\mathcal{A}}^{<\Omega}(t)$ *or* $A \equiv \neg P_{\mathcal{A}}^{<\Omega}(t)$.

*5. If either* $A \simeq \bigwedge_{\iota\in J} A_\iota$ *or* $A \simeq \bigvee_{\iota\in J} A_\iota$*, then* $N(\mathsf{rk}(A_\iota)) \le \max(\{N(\mathsf{rk}(A))\} \cup \{2 \cdot N(\iota) + \mathsf{lh}(\mathcal{A}(\cdot,*)) \mid P_{\mathcal{A}}^{<\xi}$ *or* $\neg P_{\mathcal{A}}^{<\xi}$ *occurs in* $A\})$ *for all* $\iota \in J$.

**Proof.** We only show the non-trivial property, Property 5. By Property 3, $\mathsf{rk}(A) = \omega \cdot \max \mathsf{k}(A) + n$ for some $n \le \mathsf{lh}(A)$.

CASE. $n > 0$: In this case $\mathsf{rk}(A_\iota) = \omega \cdot \max \mathsf{k}(A) + n_0$ for some $n_0 < n \le \mathsf{lh}(A)$. Hence clearly $N(\mathsf{rk}(A_\iota)) \le N(\mathsf{rk}(A))$.

CASE. $n = 0$: In this case without loss of generality let us assume $A$ is of the form $P_{\mathcal{A}}^{<\alpha}(t) \simeq \bigvee_{\xi<\alpha} \mathcal{A}(P_{\mathcal{A}}^{<\xi}, t)$ and hence $A_\xi \simeq \mathcal{A}(P_{\mathcal{A}}^{<\xi}, t)$. Let $\iota := \xi < \alpha$. Then $\mathsf{rk}(A_\iota) = \omega \cdot \xi + n_\iota$ for some $n_\iota \le \mathsf{lh}(\mathcal{A}(\cdot, t))$. Hence $N(\mathsf{rk}(A_\iota)) \le 2 \cdot N(\xi) + \mathsf{lh}(\mathcal{A}(\cdot, *))$.
□

Throughout this section we use the symbol $F$ to denote a weakly increasing ordinal function $F : \Omega \to \Omega$ and the symbol $f$ to denote a numerical function $f : \mathbb{N} \to \mathbb{N}$ that enjoys the following conditions.

($f$.1) $f$ is a strictly increasing function such that $2m + 1 \le f(m)$ for all $m$. Hence, in particular, $n + f(m) \le f(n + m)$ for all $m$ and $n$.

($f$.2) $2 \cdot f(m) \le f(f(m))$ for all $m$.

We will use the notation $f[n](m)$ to abbreviate $f(n + m)$. It is easy to see that if the conditions ($f$.1) and ($f$.2) hold, then for a fixed $n$ the conditions ($f[n]$.1) and ($f[n]$.2) also hold.

**Definition 4.5** *Let* $f : \mathbb{N} \to \mathbb{N}$ *be a numerical function. Then a function* $f^\alpha : \mathbb{N} \to \mathbb{N}$ *is defined by transfinite recursion on* $\alpha \in \mathcal{OT}(\mathcal{F})$ *by*

$$\begin{aligned} f^0(m) &= f(m), \\ f^\alpha(m) &= \max\{f^\beta(f^\beta(m)) \mid \beta < \alpha \text{ and } N(\beta) \le f[N(\alpha)](m)\} \quad \text{if } 0 < \alpha. \end{aligned}$$

**Corollary 4.6** *1. If* $f$ *is strictly increasing, then so is* $f^\alpha$ *for any* $\alpha \in \mathcal{OT}(\mathcal{F})$.

*2. If* $\beta < \alpha$ *and* $N(\beta) \le f[N(\alpha)](m)$*, then* $f^\beta(m) < f^\alpha(m)$.

*3.* $f^\alpha(f^\alpha(m)) \le f^{\alpha+1}(m)$.

We note that the function $f^\alpha$ is not a computable function in general even if $f$ is computable since the ordinal notation system $\langle \mathcal{OT}(\mathcal{F}), < \rangle$ is not a computable system.

**Lemma 4.7** *Let* $\alpha \in \mathcal{OT}(\mathcal{F})$ *and* $F \in \mathcal{F}$. *Then* $N(\alpha) \leq f^{F^{\alpha}(0)}(0)$.

**Lemma 4.8** *Let* $\{\alpha, \beta\} \subseteq \mathcal{OT}(\mathcal{F}) \restriction \Omega$ *and* $F \in \mathcal{F}$. *Then* $(f^{\alpha})^{\beta}(m) \leq f^{F^{\Omega\cdot\alpha+\beta}(0)}(m)$ *for all* $m$.

**Lemma 4.9** *1.* $f^{\alpha}[n](m) \leq (f[n])^{\alpha}(m)$.

*2. If* $n \leq m$, *then* $(f[n])^{\alpha}(m) \leq f^{\alpha}[f^{\alpha}(f(m))](f(m))$.

We write $f[n][m]$ to abbreviate $(f[n])(m)$ and $f[n]^{\alpha}$ to abbreviate $(f[n])^{\alpha}$.

**Corollary 4.10** *If* $n \leq m$, *then* $(f[n])^{\alpha}(m) \leq f^{\alpha+2}(m)$.

We define a relation $f, F \vdash^{\alpha}_{\rho} \Gamma$ for a quintuple $(f, F, \alpha, \rho, \Gamma)$ where $\alpha < \varepsilon_{\Omega+1}$, $\rho < \Omega+\omega$ and $\Gamma$ is a sequent of $\mathcal{L}^*$-sentences. In this paper a "sequent" means a finite set of formulas. We write $\Gamma, A$ or $A, \Gamma$ to denote $\Gamma \cup \{A\}$. Let us recall that for a finite set $K \subseteq \mathsf{Ord}$, $F[K](\xi)$ denotes $F(\max(K \cup \{\xi\}))$. We will write $F[\mu](\xi)$ to denote $F[\{\mu\}](\xi)$. We write $\mathsf{TRUE}_0$ to denote the set $\{A \mid A$ is an $\mathcal{L}_{\mathrm{PA}}$-literal true in the standard model $\mathbb{N}$ of PA$\}$.

**Definition 4.11** $f, F \vdash^{\alpha}_{\rho} \Gamma$ *if*

$$\max\{N(F(0)), N(\alpha)\} \leq f(0), \quad K_{\Omega}\alpha < F(0), \qquad (\mathsf{HYP}(f; F; \alpha))$$

*and one of the following holds.*

(Ax1) $\exists A(x)$*: an* $\mathcal{L}_{\mathrm{ID}_1}$*-literal,* $\exists s, t \in \mathcal{T}(\mathcal{L}_{\mathrm{ID}_1})$ *s.t.* $\mathsf{FV}(A) = \{x\}$, $\mathsf{val}(s) = \mathsf{val}(t)$ *and* $\{\neg A(s), A(t)\} \subseteq \Gamma$.

(Ax2) $\Gamma \cap \mathsf{TRUE}_0 \neq \emptyset$.

($\bigvee$) $\exists A \simeq \bigvee_{\iota \in J} A_{\mu} \in \Gamma$, $\exists \alpha_0 < \alpha$, $\exists \iota_0 \in J$ *s.t.* $N(\iota_0) \leq f(0)$, $\mathsf{ord}(\iota_0) < \min\{\alpha, F(0)\}$ *and* $f, F \vdash^{\alpha_0}_{\rho} \Gamma, A_{\iota_0}$.

($\bigwedge$) $\exists A \simeq \bigwedge_{\iota \in J} A_{\iota} \in \Gamma$ *s.t.* $\max\{N(\sigma) \mid \sigma \in \mathsf{k}^{\Pi}_{\Omega}(A)\} \leq f(0)$, $\mathsf{k}^{\Pi}_{\Omega}(A) < F(0)$ *and* $(\forall \iota \in J)$ $(\exists \alpha_{\iota} < \alpha)$ $[f[N(\iota)], F[\mathsf{ord}(\iota)] \vdash^{\alpha_{\iota}}_{\rho} \Gamma, A_{\iota}]$.

($\mathsf{Cl}_{\Omega}$) $\exists t \in \mathcal{T}(\mathcal{L}_{\mathrm{ID}_1})$, $\exists \alpha_0 < \alpha$ *s.t.* $P^{<\Omega}_{\mathcal{A}}(t) \in \Gamma$, $\Omega < \alpha$ *and* $f, F \vdash^{\alpha_0}_{\rho} \Gamma, \mathcal{A}(P^{<\Omega}_{\mathcal{A}}, t)$.

(Cut) $\exists C$*: an* $\mathcal{L}^*$*-sentence of* $\bigvee$*-type,* $\exists \alpha_0 < \alpha$ *s.t.* $\max(\{N(\sigma) \mid \sigma \in \mathsf{k}_{\Omega}(C)\} \cup \{\mathsf{lh}(C)\}) \leq f(0)$, $\mathsf{k}_{\Omega}(C) < F(0)$, $\mathsf{rk}(C) < \rho$, $f, F \vdash^{\alpha_0}_{\rho} \Gamma, C$, *and* $f, F \vdash^{\alpha_0}_{\rho} \Gamma, \neg C$.

*We will call the pair* $(f, F)$ operators *controlling the derivation that forms* $f, F \vdash^{\alpha}_{\rho} \Gamma$.

In the sequel we always assume that the operator $F$ enjoys the following condition $\mathsf{HYP}(F)$:

$$\eta < F(\xi) \Rightarrow F(\eta) \leq F(\xi) \quad \text{for any ordinals } \xi, \eta < \Omega. \qquad (\mathsf{HYP}(F))$$

We note that the hypothesis $\mathsf{HYP}(F)$ reflects the fact stated in Proposition 3.5. It is not difficult to see that if the condition $\mathsf{HYP}(F)$ holds, then the condition $\mathsf{HYP}(F[K])$ also holds for any finite set $K < \Omega$.

**Lemma 4.12 (Inversion)** *Assume that $A \simeq \bigwedge_{\iota \in J} A_\iota$. If $f, F \vdash^\alpha_\rho \Gamma, A$, then for all $\iota \in J$, $f[N(\iota)], F[\mathsf{ord}(\iota)] \vdash^\alpha_\rho \Gamma, A_\iota$.*

We write $f \circ g$ to denote the result of composing $f$ and $g$: $m \mapsto f(g(m))$.

**Lemma 4.13 (Cut-reduction)** *Assume $C \simeq \bigvee_{\iota \in J} C_\iota$, $\mathsf{rk}(C) = \rho \neq \Omega$, $\max(\{N(\sigma) \mid \sigma \in \mathsf{k}_\Omega(C)\} \cup \{\mathsf{lh}(C)\}) \leq f(g(0))$, and $\mathsf{k}_\Omega(C) < F(0)$. If $f, F \vdash^\alpha_\rho \Gamma, \neg C$ and $g, F \vdash^\beta_\rho \Gamma, C$, then $f \circ g, F \vdash^{\alpha+\beta}_\rho \Gamma$.*

For a sequent $\Gamma$ we write $\mathsf{k}^\Pi_\Omega(\Gamma)$ to denote the set $\bigcup_{B \in \Gamma} \mathsf{k}^\Pi_\Omega(B)$.

**Lemma 4.14 (First Cut-elimination)** *Let $k < \omega$. If $f, F \vdash^\alpha_{\Omega+k+2} \Gamma$, then $f^{F^\alpha(0)+1}$, $F \vdash^{\Omega^\alpha}_{\Omega+k+1} \Gamma$.*

**Lemma 4.15 (Predicative Cut-elimination)** *Assume that $\{\alpha, \beta, \gamma\} < \Omega$, $N(\alpha) \leq f^\gamma(0)$ and $K_\Omega \alpha < F(0)$. If $f^\gamma, F \vdash^\beta_{\rho+\omega^\alpha} \Gamma$, then $f^{F^{\Omega \cdot \alpha + \gamma + \beta}(0)+1}, F \vdash^{\varphi\alpha\beta}_\rho \Gamma$.*

**Definition 4.16** *For each $\mathcal{L}^*$-formula $B$ let $B^\alpha$ be the result of replacing in $B$ every occurrence of $P^{<\Omega}_A$ by $P^{<\alpha}_A$.*

**Lemma 4.17 (Boundedness)** *Assume that $f, F \vdash^\alpha_\rho \Gamma, A$. Then for all $\xi$ if $\alpha \leq \xi \leq F(0)$, $N(\xi) \leq f(0)$ and $K_\Omega \xi < F(0)$, then $f, F \vdash^\alpha_\rho \Gamma, A^\xi$.*

We will write $f, F \vdash^\alpha_{\boldsymbol{\cdot}} \Gamma$ instead of $f, F \vdash^\alpha_\alpha \Gamma$.

**Lemma 4.18 (Impredicative Cut-elimination)**
*If $f, F \vdash^\alpha_{\Omega+1} \Gamma$, then $f^{F^\alpha(0)+1}, F^{\alpha+1} \vdash^{F^\alpha(0)}_{\boldsymbol{\cdot}} \Gamma$.*

**Lemma 4.19 (Witnessing)** *For each $j < l$ let $B_j(x)$ be a $\Delta^0_0$-$\mathcal{L}_{\mathrm{PA}}$-formula such that $\mathsf{FV}(B_j(x)) = \{x\}$. Let $\Gamma \equiv \exists x_0 B_0(x_0), \dots, \exists x_{l-1} B_{l-1}(x_{l-1})$. If $f, F \vdash^\alpha_0 \Gamma$ for some $\alpha \in \mathcal{OT}(\mathcal{F})$, then there exists a sequence $m_0, \dots, m_{l-1}$ of naturals such that $\max\{m_j \mid j < l\} \leq f(0)$ and $B_0(\underline{m_0}) \vee \cdots \vee B_{l-1}(\underline{m_{l-1}})$ is true in the standard model $\mathbb{N}$ of PA.*

# 5 Embedding $\mathrm{ID}_1$ into $\mathrm{ID}_1^\infty$

In this section we embed the system $\mathrm{ID}_1$ into the infinitary system $\mathrm{ID}_1^\infty$. Following conventions in the previous section we use the symbol $f$ to denote a strict increasing function $f : \mathbb{N} \to \mathbb{N}$ that enjoys the conditions $(f.1)$ and $(f.2)$ (p. 8). Let us recall that the function symbol $\mathsf{E} \in \mathcal{F}$ denotes the function $\mathsf{E} : \Omega \to \Omega$ such that $\mathsf{E}(\alpha) = \min\{\xi < \Omega \mid \omega^\xi = \xi \text{ and } \alpha < \xi\}$. It is easy to see that the condition $\mathsf{HYP}(\mathsf{E})$ holds since $\mathsf{E}(\xi) = \varepsilon_0 \leq \mathsf{E}(0)$ for all $\xi < \mathsf{E}(0) = \varepsilon_0$.

**Lemma 5.1 (Tautology lemma)** *Let $s, t \in \mathcal{T}(\mathcal{L}_{\mathrm{ID}_1})$, $\Gamma$ be a sequent of $\mathcal{L}^*$-sentences, and $A(x)$ be an $\mathcal{L}^*$-formula such that $\mathsf{FV}(A) = \{x\}$. If $\mathsf{val}(s) = \mathsf{val}(t)$, then*

$$f[n], \mathsf{E}[\mathsf{k}_\Omega(A)] \vdash_0^{\mathsf{rk}(A)\cdot 2} \Gamma, \neg A(s), A(t), \tag{1}$$

*where $n := \max(\{N(\mathsf{rk}(A))\} \cup \{2 \cdot N(\sigma) + \mathsf{lh}(\mathcal{A}(\cdot, *)) \mid \sigma \in \mathsf{k}_\Omega(A) \text{ and } P_{\mathcal{A}}^{<\xi} \text{ or } \neg P_{\mathcal{A}}^{<\xi}$ occurs in $A\})$.*

**Proof.** By induction on $\mathsf{rk}(A)$. Let $n := \max(\{N(\mathsf{rk}(A))\} \cup \{2 \cdot N(\sigma) + \mathsf{lh}(\mathcal{A}(\cdot, *)) \mid \sigma \in \mathsf{k}_\Omega(A) \text{ and } P_{\mathcal{A}}^{<\xi} \text{ or } \neg P_{\mathcal{A}}^{<\xi} \text{ occurs in } A\})$. From Lemma 4.4.3 one can check that the condition $\mathsf{HYP}(f[n]; \mathsf{E}(\mathsf{k}_\Omega(A)); \mathsf{rk}(A) \cdot 2)$ holds. If $\mathsf{rk}(A) = 0$, then $A$ is an $\mathcal{L}_{\mathrm{ID}_1}$-literal, and hence (1) is an instance of (Ax1). Suppose that $\mathsf{rk}(A) > 0$. Without loss of generality we can assume that $A \simeq \bigvee_{\iota \in J} A_\iota$. Let $\iota \in J$. By Lemma 4.4.5 we observe that $N(\mathsf{rk}(A_\iota) \leq f(n) = f[n][N(\iota)](0)$ since $2m + 1 \leq f(m)$ for all $m$ by the condition $(f.1)$. Further by Lemma 4.4.1 $K_\Omega(\mathsf{rk}(A_\iota) \cdot 2) \subseteq \mathsf{k}_\Omega(A) \cup \{\mathsf{ord}(\iota)\} \leq \mathsf{E}[\mathsf{k}_\Omega(A)][\mathsf{ord}(\iota)]$. Summing up, we have the condition

$$\mathsf{HYP}(f[n][N(\iota)]; \mathsf{E}[\mathsf{k}_\Omega(A)][\mathsf{ord}(\iota)]; \mathsf{rk}(A_\iota) \cdot 2).$$

Hence by IH we can obtain the sequent

$$f[n][N(\iota)], \mathsf{E}[\mathsf{k}_\Omega(A)][\mathsf{ord}(\iota)] \vdash_0^{\mathsf{rk}(A_\iota)\cdot 2} \Gamma, \neg A_\iota(s), A_\iota(t). \tag{2}$$

It is not difficult to see $\mathsf{ord}(\iota) \leq \mathsf{rk}(A_\iota) < \mathsf{rk}(A_\iota) \cdot 2 + 1$ and $N(\mathsf{rk}(A_\iota) \cdot 2 + 1) = N(\mathsf{rk}(A_\iota) \cdot 2) + 1 \leq f[n][N(\iota)](0)$. This allows us to apply $(\bigvee)$ to the sequent (2) yielding

$$f[n][N(\iota)], \mathsf{E}[\mathsf{k}_\Omega(A)][\mathsf{ord}(\iota)] \vdash_0^{\mathsf{rk}(A_\iota)\cdot 2+1} \Gamma, \neg A_\iota(s), A(t).$$

We can see that $\mathsf{rk}(A_\iota) \cdot 2 + 1 < \mathsf{rk}(A) \cdot 2$, $\max\{N(\sigma) \mid \sigma \in \mathsf{k}_\Omega^\Pi(A)\} \leq f[n](0)$ and $\mathsf{k}_\Omega^\Pi(A) < \mathsf{E}[\mathsf{k}_\Omega(A)]$. Hence we can apply $(\bigwedge)$ concluding (1). □

**Lemma 5.2** *Let $B_j$ be an $\mathcal{L}_{\mathrm{ID}_1}$-sentence for each $j = 0, \ldots, l-1$. Suppose that $B_0 \vee \cdots \vee B_{l-1}$ is a logical consequence in the first order predicate logic with equality. Then there exists a natural $k < \omega$ such that $f[m+k], \mathsf{E} \vdash_0^{\Omega\cdot 2+k} \Gamma, B_0, \ldots, B_{l-1}$, where $m = \max(\{N(\mathsf{rk}(B_j)) \mid 0 \leq j \leq l-1\} \cup \{\mathsf{lh}(\mathcal{A}(\cdot, *)) \mid P_{\mathcal{A}}^{<\xi} \text{ or } \neg P_{\mathcal{A}}^{<\xi}$ occurs in $B_j$ for some $j\})$.*

**Proof.** Let $B_j$ be an $\mathcal{L}_{\mathrm{ID}_1}$-sentence for each $j = 0, \ldots, l-1$ and suppose that $B_0 \vee \cdots \vee B_{l-1}$ is a logical consequence in the first order predicate logic with equality. Then we can find a cut-free proof of the sequent $\Gamma, B_0, \ldots, B_{l-1}$ in an LK-style sequent calculus. More precisely we can find a cut-free proof $P$ of $\Gamma, B_0, \ldots, B_{l-1}$ in the sequent calculus that is known as $\mathrm{G3_m}$. Let $h$ denote the tree height of the cut-free proof $P$. Then by induction on $h$ one can find a witnessing natural $k$ such that $f[m+k], \mathsf{E} \vdash_0^\alpha \Gamma, B_0, \ldots, B_{l-1}$ for all $\alpha \geq \Omega + k$. In case $h = 0$ Tautology lemma (Lemma 5.1) can be applied since for any $\mathcal{L}_{\mathrm{ID}_1}$-sentence $A$, $\mathsf{rk}(A) \in \omega \cup \{\Omega + k \mid k < \omega\}$ and $\mathsf{k}(A) \subseteq \{0, \Omega\}$, and hence $\mathsf{k}_\Omega(A) = \{0\}$ and $\max\{N(\sigma) \mid \sigma \in \mathsf{k}_\Omega(A)\} = 0$. □

**Lemma 5.3** *Let $m \in \mathbb{N}$ and $A(x)$ be an $\mathcal{L}_{\mathrm{ID}_1}$-formula such that $\mathsf{FV}(A(x)) = \{x\}$. Then for any $t \in \mathcal{T}(\mathcal{L}_{\mathrm{ID}_1})$ and for any sequent $\Gamma$ of $\mathcal{L}_{\mathrm{ID}_1}$-sentences, if $\mathsf{val}(t) = m$, then*

$$f[n+m], \mathsf{E} \vdash_0^{(\mathsf{rk}(A)+m)\cdot 2} \Gamma, \neg A(0), \neg\forall x(A(x) \to A(S(x))), A(t), \tag{3}$$

*where $n := \max(\{N(\mathsf{rk}(A))\} \cup \{\mathsf{lh}(\mathcal{A}(\cdot,*)) \mid P_{\mathcal{A}}^{<\xi} \text{ or } \neg P_{\mathcal{A}}^{<\xi} \text{ occurs in } A\})$*

**Proof.** By induction on $m$. The base case $\mathsf{val}(t) = m = 0$ follows from Tautology lemma (Lemma 5.1). For the induction step suppose $\mathsf{val}(t) = m+1$. Fix a sequent $\Gamma$ of $\mathcal{L}_{\mathrm{ID}_1}$-sentences. Then (3) holds by IH. On the other hand again by Tautology lemma,

$$f[n], \mathsf{E} \vdash_0^{\mathsf{rk}(A)\cdot 2} \Gamma, \neg A(0), \exists x(A(x) \wedge \neg A(S(x))), A(\underline{m}), \neg A(\underline{m}). \tag{4}$$

An application of $(\bigwedge)$ to the two sequents (3) and (4) yields

$$f[n+m], \mathsf{E} \vdash_0^{\alpha_m \cdot 2+1} \Gamma, \neg A(0), \exists x(A(x) \wedge \neg A(S(x))), A(t), A(\underline{m}) \wedge \neg A(\underline{m}),$$

The final application of $(\bigvee)$ yields

$$f[n+m+1], F \vdash_0^{(\mathsf{rk}(A)+m+1)\cdot 2} \Gamma, \neg A(0), \exists x(A(x) \wedge \neg A(S(x))), A(t).$$

□

**Lemma 5.4** *Let $\xi \leq \Omega$, $F(x)$ be an $\mathcal{L}_{\mathrm{ID}_1}$-formula such that $\mathsf{FV}(F(x)) = \{x\}$ and $B(X)$ be an $X$-positive $\mathcal{L}_{\mathrm{PA}}(X)$-formula such that $\mathsf{FV}(B) = \emptyset$. Then*

$$f[n], \mathsf{E}[K_\Omega \xi] \vdash_0^{(\sigma+\alpha+1)\cdot 2} \Gamma, \neg\forall x(\mathcal{A}(F,x) \to F(x)), \neg B(P_{\mathcal{A}}^{<\xi}), B(F),$$

*where $\sigma := \mathsf{rk}(F)$, $\alpha := \mathsf{rk}(B(P_{\mathcal{A}}^{<\xi}))$ and $n := \max(\{N(\sigma + \alpha + 1)\} \cup \{\mathsf{lh}(\mathcal{B}) \mid P_{\mathcal{B}}^{<\gamma} \text{ or } \neg P_{\mathcal{B}}^{<\gamma} \text{ occurs in } F\})$.*

**Proof.** By main induction on $\xi$ and side induction on $\mathsf{rk}(B(P_{\mathcal{A}}^{<\xi}))$. Let $\mathsf{Cl}_{\mathcal{A}}(F)$ denote $\neg\forall x(\mathcal{A}(F,x) \to F(x))$. Then $\neg\mathsf{Cl}_{\mathcal{A}}(F) \equiv \exists x(\mathcal{A}(F,x) \wedge \neg F(x))$. The argument splits into several cases depending on the shape of the formula $B(X)$.

CASE. $B(X)$ is an $\mathcal{L}_{\text{PA}}$-literal: In this case $B$ does not contain the set free variable $X$, and hence Tautology lemma (Lemma 5.1) can be applied. Note that the operator form $\mathcal{B}$ does not occur in $B$.

CASE. $B \equiv X(t)$ for some $t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$: In this case $\neg B(P_{\mathcal{A}}^{<\xi}) \equiv \neg P_{\mathcal{A}}^{<\xi}(t) \equiv \bigwedge_{\eta<\xi} \neg \mathcal{A}(P_{\mathcal{A}}^{<\eta}, t)$. Let $\eta < \xi$. Then by MIH

$$f[n_\eta], \mathsf{E}[K_\Omega \eta] \vdash_0^{(\sigma+\alpha_\eta+1)\cdot 2} \Gamma, \neg \mathsf{Cl}_{\mathcal{A}}(F), \neg \mathcal{A}(P_{\mathcal{A}}^{<\eta}, t), \mathcal{A}(F,t), F(t)$$

where $\alpha_\eta := \mathsf{rk}(\mathcal{A}(P_{\mathcal{A}}^{<\eta}, t))$ and $n_\eta := \max(\{N(\sigma + \alpha_\eta + 1)\} \cup \{\mathsf{lh}(\mathcal{B}) \mid P_{\mathcal{B}}^{<\gamma} \text{ or } \neg P_{\mathcal{B}}^{<\gamma}$ occurs in $F\})$. We note that $\eta < \xi \leq \Omega$ and hence $K_\Omega \eta = \{\eta\} = \{\mathsf{ord}(\eta)\}$. Hence this yields the sequent

$$f[n][N(\eta)], \mathsf{E}[\mathsf{ord}(\eta)] \vdash_0^{(\sigma+\alpha_\eta+1)\cdot 2} \Gamma, \neg \mathsf{Cl}_{\mathcal{A}}(F), \neg \mathcal{A}(P_{\mathcal{A}}^{<\eta}, t), \mathcal{A}(F,t), F(t).$$

An application of $(\bigwedge)$ yields the sequent

$$f[n], \mathsf{E}[K_\Omega \xi] \vdash_0^{(\sigma+\alpha)\cdot 2} \Gamma, \neg \mathsf{Cl}_{\mathcal{A}}(F), \neg P_{\mathcal{A}}^{<\xi} t, \mathcal{A}(F,t), F(t). \tag{5}$$

On the other hand by Tautology lemma (Lemma 5.1),

$$f[n], \mathsf{E}[K_\Omega \xi] \vdash_0^{\mathsf{rk}(F)\cdot 2} \Gamma, \neg \mathsf{Cl}_{\mathcal{A}}(F), \neg P_{\mathcal{A}}^{<\xi} t, \neg F(t), F(t). \tag{6}$$

Another application of $(\bigwedge)$ to the two sequents (5) and (5) yields the sequent

$$f[n], \mathsf{E}[K_\Omega \xi] \vdash_0^{(\sigma+\alpha)\cdot 2+1} \Gamma, \neg \mathsf{Cl}_{\mathcal{A}}(F), \neg P_{\mathcal{A}}^{<\xi} t, \mathcal{A}(F,t) \wedge \neg F(t), F(t).$$

An application of $(\bigvee)$ allows us to conclude

$$f[n], \mathsf{E}[K_\Omega \xi] \vdash_0^{(\sigma+\alpha+1)\cdot 2} \Gamma, \neg \mathsf{Cl}_{\mathcal{A}}(F), \neg P_{\mathcal{A}}^{<\xi} t, F(t).$$

CASE. $B(X) \equiv \forall y B_0(X,y)$ for some $\mathcal{L}_{\text{PA}}$-formula $B_0(X,y)$: Let $\alpha_0$ denote the ordinal $\mathsf{rk}(B_0(P_{\mathcal{A}}^{<\xi}, \underline{0}))$. Then $\alpha = \alpha_0 + 1$. By the definition of the rank function $\mathsf{rk}$, $\alpha_0 = \mathsf{rk}(B_0(P_{\mathcal{A}}^{<\xi}, t))$ for all $t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$. Fix a closed term $t \in \mathcal{T}(\mathcal{L}_{\text{ID}_1})$. Then from SIH we have the sequent

$$f[n], \mathsf{E}[K_\Omega \xi] \vdash_0^{(\sigma+\alpha)\cdot 2} \Gamma, \neg \mathsf{Cl}_{\mathcal{A}}(F), \neg B_0(P_{\mathcal{A}}^{<\xi}, t), B_0(P_{\mathcal{A}}^{<\xi}, t).$$

An application of $(\bigvee)$ yields the sequent

$$f[n], \mathsf{E}[K_\Omega \xi] \vdash_0^{(\sigma+\alpha)\cdot 2+1} \Gamma, \neg \mathsf{Cl}_{\mathcal{A}}(F), \neg \forall y B_0(P_{\mathcal{A}}^{<\xi}, y), B_0(P_{\mathcal{A}}^{<\xi}, t).$$

And an application of $(\bigwedge)$ allows us to conclude.

The other cases can be treated in similar ways. □

**Lemma 5.5** *1.* $f[n], \mathsf{E} \vdash_0^{\Omega\cdot 2+\omega} \Gamma, \forall x(\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, x) \to P_{\mathcal{A}}^{<\Omega}(x))$,
*where* $n := \max\{N(\mathsf{rk}(\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, \underline{0})), \mathsf{lh}(\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, \underline{0})\}$

*2.* $f[3+l], \mathsf{E} \vdash_0^{\Omega\cdot 2+\omega} \Gamma, \forall \vec{y}[\forall x\{\mathcal{A}(F(\cdot,\vec{y}),x) \to F(x,\vec{y})\} \to \forall x\{P_{\mathcal{A}}^{<\Omega}(x) \to F(x,\vec{y})\}]$, *where* $\vec{y} = y_0, \ldots, y_{l-1}$.

**Proof.** 1. Let $\alpha = \mathsf{rk}(\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, \underline{0})$ and $t \in \mathcal{T}(\mathcal{L}_{\mathrm{ID}_1})$. By the definition of $\mathsf{rk}$ we can find a natural $k \leq \mathsf{lh}(\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, \underline{0})$ such that $\alpha = \mathsf{rk}(\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, t) = \Omega + k$. This implies $\mathsf{k}(\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, t)) = \{0, \Omega\}$ and hence $\mathsf{k}_\Omega(\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, t)) = \{0\} < \mathsf{E}(0)$. By Tautology lemma (Lemma 5.1),

$$f[n], \mathsf{E} \vdash_0^{\Omega\cdot 2+k} \Gamma, P_{\mathcal{A}}^{<\Omega}(t), \neg\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, t), \mathcal{A}(P_{\mathcal{A}}^{<\Omega}, t).$$

Since $\Omega < \Omega \cdot 2 + k + 1$, we can apply the closure rule $(\mathsf{Cl}_\Omega)$ obtaining the sequent

$$f[n], \mathsf{E} \vdash_0^{\Omega\cdot 2+k+1} \Gamma, \neg\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, t), P_{\mathcal{A}}^{<\Omega}(t).$$

An application of $(\bigwedge)$ followed by an application of $(\bigvee)$ enables us to conclude

$$f[n], \mathsf{E} \vdash_0^{\Omega\cdot 2+\omega} \Gamma, \forall x(\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, x) \to P_{\mathcal{A}}^{<\Omega} x).$$

2. By definition $\mathsf{rk}(P_{\mathcal{A}}^{<\Omega}) = \omega \cdot \Omega = \Omega$, On the other hand $\mathsf{rk}(F) < \omega$ and hence $(\mathsf{rk}(F) + \mathsf{rk}(P_{\mathcal{A}}^{<\Omega}) + 1) \cdot 2 = \Omega \cdot 2 + 2$. Let $s, \vec{t} = s, t_0, \ldots t_{l-1} \in \mathcal{T}(\mathcal{L}_{\mathrm{ID}_1})$. Then by the previous lemma (Lemma 5.4)

$$f[2], \mathsf{E} \vdash_0^{\Omega\cdot 2+1} \neg\forall x(\mathcal{A}(F(\cdot,\vec{t}),x) \to F(x,\vec{t})), \neg P_{\mathcal{A}}^{<\Omega}(t), F(s,\vec{t})$$

since $N(\Omega + 1) = 2$. It is not difficult to see that applications of $(\bigvee)$, $(\bigwedge)$ and $(\bigvee)$ in this order yield the sequent

$$f[3], \mathsf{E} \vdash_0^{\Omega\cdot 2+5} \forall x(\mathcal{A}(F(\cdot,\vec{t}),x) \to F(x,\vec{t})) \to \forall x(P_{\mathcal{A}}^{<\Omega}(x) \to F(x,\vec{t}))$$

Finally, $l$-fold application of $(\bigwedge)$ allows us to conclude. □

Let us recall that $\mathsf{s}$ denotes the numerical successor $m \mapsto m+1$.

**Theorem 5.6** *Let $A \equiv \forall\vec{x}\exists y B(\vec{x},y)$ be a $\Pi^0_2$-sentence for a $\Delta^0_0$-formula $B(\vec{x},y)$ such that $\mathsf{FV}(B(\vec{x},y)) = \{\vec{x},y\}$. If $\mathrm{ID}_1 \vdash A$, then we can find an ordinal term $\alpha \in \mathcal{OT}(\mathcal{F}) \restriction \Omega$ built up without the Veblen function symbol $\varphi$ such that for all $\vec{m} = m_0, \ldots, m_{l-1} \in \mathbb{N}$ there exists $n \leq \mathsf{s}^\alpha(m_0 + \cdots + m_{l-1})$ such that $B(\vec{m},n)$ is true in the standard model $\mathbb{N}$ of* PA.

**Proof.** Assume $\mathrm{ID}_1 \vdash A$. Then there exist $\mathrm{ID}_1$-axioms $A_0, \ldots, A_{k-1}$ such that $(\neg A_0) \vee \cdots \vee (\neg A_{k-1}) \vee A$ is a logical consequence in the first order predicate logic with equality. Hence by Lemma 5.2,

$$f[c_0], \mathsf{E} \vdash_0^{\Omega\cdot 3} \neg A_0, \ldots, \neg A_{k-1}, A$$

for some constant $c_0 < \omega$ depending on $N(\mathsf{rk}(A_0)), \ldots, N(\mathsf{rk}(A_{k-1}))$, $N(\mathsf{rk}(A))$ and $\max\{\mathsf{lh}(\mathcal{A}(\cdot,*)) \mid P_{\mathcal{A}}^{<\xi}$ or $\neg P_{\mathcal{A}}^{<\xi}$ occurs in $A_j$ or $A\}$, and depending also on the tree height of a cut-free LK-derivation of the sequent $\neg A_0, \ldots, \neg A_{k-1}, A$. By Lemma 5.3 and 5.5, for each $j \leq k-1$, there exists a constant $c_j$ depending on $\mathsf{rk}(A_j)$ such that $f[c_j], \mathsf{E} \vdash_0^{\Omega\cdot 2+\omega} A_j$. Hence $k$-fold application of $(\mathsf{Cut})$ yields $f[c], \mathsf{E} \vdash_{\Omega+d+1}^{\Omega\cdot 3}$

$A$, where $c := \max(\{k\} \cup \{c_j \mid j \le k-1\} \cup \{\mathsf{lh}(A_j) \mid j \le k-1\})$ and $d := \max(\{\Omega, \mathsf{rk}(A_0), \dots, \mathsf{rk}(A_{k-1})\})$.

For each $n \in \mathbb{N}$ and $\alpha \in \mathcal{OT}(\mathcal{F})$ let us define ordinal $\Omega_n(\alpha)$ and $\gamma_n$ by

$$\begin{array}{rclcrcl} \Omega_0(\alpha) & = & \alpha, & & \gamma_0 & = & \Omega \cdot 3, \\ \Omega_{n+1}(\alpha) & = & \Omega^{\Omega_n(\alpha)}, & & \gamma_{n+1} & = & \mathsf{E}^{\gamma_n}(0) + 1. \end{array}$$

Then $d$-fold iteration of Cut-reduction lemma (Lemma 4.13) yields the sequent $f[c]^{\gamma_d}, \mathsf{E} \vdash^{\Omega_d(\Omega\cdot 3)}_{\Omega+1} A$. Hence Impredicative cut-elimination lemma (Lemma 4.18) yields

$$(f[c]^{\gamma_d})^{\mathsf{E}^{\Omega_d(\Omega\cdot 3)}(0)}, \mathsf{E}^{\Omega_d(\Omega\cdot 3)+1} \vdash^{\mathsf{E}^{\Omega_d(\Omega\cdot 3)}(0)}_{\cdot} A.$$

Let $F := \mathsf{E}^{\Omega_d(\Omega\cdot 3)+1}$ and $\beta := \mathsf{E}^{\Omega_d(\Omega\cdot 3)}(0)$. Then $(f[c]^{\gamma_d})^\beta, F \vdash^{\beta}_{\omega^\beta} A$ holds. It is not difficult to check that $\beta < \Omega$, $N(\beta) \le (f[c]^{\gamma_d})^\beta$ and $K_\Omega \beta < F(0)$. Hence Predicative cut-elimination lemma (Lemma 4.15) yields the sequent

$$(f[c]^{\gamma_d})^{F^{\Omega\cdot\beta+\beta\cdot 2}(0)+1} F \vdash^{\varphi\beta\beta}_{0} A.$$

Now let $f$ denote $\mathsf{s}^\omega$. One can check that the conditions ($\mathsf{s}^\omega$.1) and ($\mathsf{s}^\omega$.2) hold. One will also see that $\mathsf{s}^\omega[c](m) \le \mathsf{s}^\omega(\mathsf{s}^c(m)) \le \mathsf{s}^{\omega+c+1}(m)$ for all $m$. By these we have the inequality

$$(\mathsf{s}[c]^{\gamma_d})^{F^{\Omega\cdot\beta+\beta\cdot 2}(0)+1}(0) \le ((\mathsf{s}^{\omega+c+1})^{\gamma_d})^{F^{\Omega\cdot\beta+\beta\cdot 2}(0)+1}(0).$$

Thanks to Lemma 4.8 we can find an ordinal $\alpha \in \mathcal{OT}(\mathcal{F}) \restriction \Omega$ built up without the Veblen function symbol $\varphi$ such that

$$((\mathsf{s}^{\omega+c+1})^{\gamma_d})^{F^{\Omega\cdot\beta+\beta\cdot 2}(0)+1}(0) \le \mathsf{s}^\alpha(0).$$

This together with ($l$-fold application of) Inversion lemma (Lemma 4.12) yields the sequent

$$\mathsf{s}^\alpha[m_0]\cdots[m_{l-1}], F \vdash^{\varphi\beta\beta}_{0} \exists y B(\vec{\underline{m}}, y),$$

where $\vec{m} = m_0, \dots, m_{l-1}$. By Witnessing lemma (Lemma 4.19) we can find a natural $n \le \mathsf{s}^\alpha[m_0]\cdots[m_{l-1}](0) = \mathsf{s}^\alpha(m_0 + \cdots + m_{l-1})$ such that $B(\vec{m}, n)$ is true in the standard model $\mathbb{N}$ of PA. □

We say a function $f$ is *elementary* (in another function $g$) if $f$ is definable explicitly from the successor $\mathsf{s}$, projection, zero 0, addition $+$, multiplication $\cdot$, cut-off subtraction $\dot{-}$ (and $g$), using composition, bounded sums and bounded products.

**Corollary 5.7** *Every function provably computable in* $\mathrm{ID}_1$ *is elementary in* $\{\mathsf{s}^\alpha \mid \alpha \in \mathcal{OT}(\mathcal{F}) \restriction \Omega\}$.

# 6 A computable ordinal notation system $\mathcal{O}(\Omega)$

In order to obtain a precise characterisation of the provably computable functions of $\mathrm{ID}_1$, we introduce a *computable* ordinal notation system $\langle \mathcal{O}(\Omega), < \rangle$. Essentially $\mathcal{O}(\Omega)$ is a subsystem of $\mathcal{OT}(\mathcal{F})$.

**Definition 6.1** *We define three sets* $\mathsf{SC} \subseteq \mathbb{H} \subseteq \mathcal{O}(\Omega)$ *of ordinal terms simultaneously. Let* $0$, $\Omega$, $\mathsf{S}$, *and* $+$ *be distinct symbols.*

1. $0 \in \mathcal{O}(\Omega)$ *and* $\Omega \in \mathsf{SC}$.
2. *If* $\alpha \in \mathcal{OT}(\mathcal{F}) \restriction \Omega$, *then* $\mathsf{S}(\alpha) \in \mathcal{O}(\Omega)$.
3. *If* $\{\alpha_1, \dots, \alpha_l\} \subseteq \mathbb{H}$ *and* $\alpha_1 \geq \cdots \geq \alpha_l$, *then* $\alpha_1 + \cdots + \alpha_l \in \mathcal{O}(\Omega)$.
4. *If* $\alpha \in \mathcal{O}(\Omega)$, *then* $\omega^\alpha \in \mathbb{H}$.
5. *If* $\alpha \in \mathcal{O}(\Omega)$ *and* $\xi \in \mathcal{O}(\Omega) \restriction \Omega$, *then* $\Omega^\alpha \cdot \xi \in \mathbb{H}$.
6. *If* $\alpha \in \mathcal{O}(\Omega)$ *and* $\xi \in \mathcal{O}(\Omega) \restriction \Omega$, *then* $\mathsf{S}^\alpha(\xi) \in \mathsf{SC}$.

The relation $<$ on $\mathcal{O}(\Omega)$ is defined in the obvious way. One will see that $\mathcal{O}(\Omega)$ is indeed a computable ordinal notation system. Let us define the norm $N(\omega^\alpha)$ of $\omega^\alpha$ in the most natural way, i.e., $N(\omega^\alpha) = N(\alpha) + 1$.

**Lemma 6.2** *Let* $\alpha$ *denote an ordinal term built up in* $\mathcal{OT}(\mathcal{F})$ *without the Veblen function symbol* $\varphi$. *Then there exists an ordinal term* $\alpha' \in \mathcal{O}(\Omega)$ *such that* $\alpha \leq \alpha'$ *and* $N(\alpha) \leq N(\alpha')$.

**Proof.** By induction over the term construction of $\alpha \in \mathcal{OT}(\mathcal{F})$. In the base case let us observe that $\mathsf{E}(\alpha) \leq \mathsf{S}^1(\alpha)$ for all $\alpha < \Omega$ and that $N(\mathsf{E}(\alpha)) = N(\alpha)+1 < N(\mathsf{S}(\alpha))+1 = N(\mathsf{S}^1(\alpha))$. In the induction case we employ Lemma 3.11. □

**Lemma 6.3** *For any ordinal term* $\alpha \in \mathcal{OT}(\mathcal{F})$ *built up without the Veblen function symbol* $\varphi$ *there exists an ordinal term* $\alpha' \in \mathcal{O}(\Omega)$ *such that* $\mathsf{s}^\alpha(m) \leq \mathsf{s}^{\alpha'}(m)$ *for all* $m$.

**Corollary 6.4** *A function is provably computable in* $\mathrm{ID}_1$ *if and only if it is elementary in* $\{\mathsf{s}^\alpha \mid \alpha \in \mathcal{O}(\Omega) \restriction \Omega\}$.

The "only if" direction follows from Corollary 5.7 and Lemma 6.3. The "if" direction can be seen as follows. One can show that for each $\alpha \in \mathcal{O}(\Omega) \restriction \Omega$ the system $\mathrm{ID}_1$ proves that the initial segment $\langle \mathcal{O}(\Omega) \restriction \alpha, < \rangle$ of $\langle \mathcal{O}(\Omega), < \rangle$ is a well-ordering. For the full proof, we kindly refer the readers to, e.g., Pohlers [11, §29]. From this one can show that for each $\alpha \in \mathcal{O}(\Omega) \restriction \Omega$ the function $\mathsf{s}^\alpha$ is provably computable in $\mathrm{ID}_1$, and hence the assertion.

# 7 A quick proof-theoretic analysis of $\mathrm{ID}_1$

In the final section we show that the collapsing function $F : \Omega_1 \times \varepsilon_{\Omega_1} \to \Omega_1$; $(\xi, \alpha) \mapsto F^\alpha(\xi)$ can be used for a smooth proof-theoretic analysis of $\mathrm{ID}_1$. Suppose a positive operator form $\mathcal{A}$. Let $\Phi_\mathcal{A} : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ denote the operator induced by the operator form $\mathcal{A}$. Namely $\Phi_\mathcal{A}(X) = \{n \in \mathbb{N} \mid \mathbb{N} \models \mathcal{A}(X, n)\}$ if $X \subseteq \mathbb{N}$. By positiveness of $\mathcal{A}$ the operator $\Phi_\mathcal{A}$ is monotone, i.e., $X \subseteq Y \Rightarrow \Phi_\mathcal{A}(X) \subseteq \Phi_\mathcal{A}(Y)$, and hence $\Phi_\mathcal{A}$ has the least fixed point $I_{\Phi_\mathcal{A}}$ that corresponds to the predicate $P_\mathcal{A}$. Further, for an ordinal $\alpha$, let $I^\alpha_{\Phi_\mathcal{A}}$ denote the $\alpha$-th stage of iterating $\Phi_\mathcal{A}$. More precisely, corresponding to the predicate $P^{<\alpha}_\mathcal{A}$, $I^\alpha_{\Phi_\mathcal{A}}$ is defined by $I^0_{\Phi_\mathcal{A}} = \emptyset$ and $I^\alpha_{\Phi_\mathcal{A}} = \Phi_\mathcal{A}(\bigcup_{\xi<\alpha} I^\xi_{\Phi_\mathcal{A}})$ $(0 < \alpha)$. Recall that $\Omega_1$ denotes the least non-computable ordinal $\omega_1^{\mathrm{CK}}$. From an elementary fact in generalised recursion theory, it is known that $I^\alpha_{\Phi_\mathcal{A}}$ is consumed at $\alpha = \Omega_1$, i.e., $I^{\Omega_1}_{\Phi_\mathcal{A}} = I_{\Phi_\mathcal{A}}$. The *norm* $|n|_{\Phi_\mathcal{A}}$ of a natural number $n$ is defined by $|n|_{\Phi_\mathcal{A}} = \min\{\alpha \in \mathsf{Ord} \mid n \in I^\alpha_{\Phi_\mathcal{A}}\}$. It is natural to ask what can be said about the norm $|n|_{\Phi_\mathcal{A}}$ in case that $\mathrm{ID}_1 \vdash P_\mathcal{A}(\underline{n})$ holds. An elegant proof-theoretic way to answer this question can be found in lecture notes [4] by W. Buchholz. (See [4, Theorem 9.19].) By slightly modifying the exposition in [4] we present an alternative simplified way to answer this question.

In contrast to the infinitary system $\mathrm{ID}_1^\infty$ we investigate the associated semiformal system $\mathrm{ID}_1^*$ which is modelled following the lecture notes [4]. As until the previous section we will identify each element $\alpha \in \mathcal{OT}(\mathcal{F})$ with its value $[\alpha] \in \mathsf{Ord}$, e.g., $\Omega \in \mathcal{OT}(\mathcal{F})$ with $\Omega_1 \in \mathsf{Ord}$. We also follow a convention that $F : \Omega \to \Omega$ denotes a weakly increasing function such that $\xi < F(\xi)$ for all $\xi < \Omega$. Further in this section we use an additional convention that $\omega^{F(\xi)} = F(\xi)$, and hence $\mathsf{E}(\xi) \leq F(\xi)$ for all $\xi < \Omega$. (Recall $\mathsf{E}(\alpha) = \min\{\xi \in \mathsf{Ord} \mid \omega^\xi = \xi \text{ and } \alpha < \xi\}$.) Let us recall that for a sequent $\Gamma$, $\mathsf{k}^\Pi_\Omega(\Gamma)$ denotes the set $\bigcup_{B\in\Gamma} \mathsf{k}^\Pi_\Omega(B)$.

**Definition 7.1** *$F \vdash^\alpha_\rho \Gamma$ if $\mathsf{k}^\Pi_\Omega(\Gamma) \cup K_\Omega\alpha < F(0)$ and one of the following holds.*

(Ax1) *$\exists A(x)$: an $\mathcal{L}_{\mathrm{ID}_1}$-literal, $\exists s, t \in \mathcal{T}(\mathcal{L}_{\mathrm{ID}_1})$ s.t. $\mathsf{FV}(A) = \{x\}$, $\mathsf{val}(s) = \mathsf{val}(t)$ and $\{\neg A(s), A(t)\} \subseteq \Gamma$.*

(Ax2) *$\Gamma \cap \mathsf{TRUE}_0 \neq \emptyset$.*

($\bigvee$) *$\exists A \simeq \bigvee_{\iota\in J} A_\mu \in \Gamma$, $\exists \alpha_0 < \alpha$, $\exists \iota_0 \in J$ s.t. $\mathsf{ord}(\iota_0) < F(0)$, and $F \vdash^{\alpha_0}_\rho \Gamma, A_{\iota_0}$.*

($\bigwedge$) *$\exists A \simeq \bigwedge_{\iota\in J} A_\iota \in \Gamma$ s.t. $(\forall \iota \in J)$ $(\exists \alpha_\iota < \alpha)$ $F[\mathsf{ord}(\iota)] \vdash^{\alpha_\iota}_\rho \Gamma, A_\iota]$.*

($\mathsf{Cl}_\Omega$) *$\exists t \in \mathcal{T}(\mathcal{L}_{\mathrm{ID}_1})$, $\exists \alpha_0 < \alpha$ s.t. $P^{<\Omega}_\mathcal{A}(t) \in \Gamma$ and $F \vdash^{\alpha_0}_\rho \Gamma, \mathcal{A}(P^{<\Omega}_\mathcal{A}, t)$.*

($\mathsf{Cut}$) *$\exists C$: an $\mathcal{L}^*$-sentence of $\bigvee$-type, $\exists \alpha_0 < \alpha$ s.t. $\mathsf{rk}(C) < \rho$, $F \vdash^{\alpha_0}_\rho \Gamma, C$, and $F \vdash^{\alpha_0}_\rho \Gamma, \neg C$.*

**Lemma 7.2 (Inversion)** *Assume that $A \simeq \bigwedge_{\iota\in J} A_\iota$. If $F \vdash^\alpha_\rho \Gamma, A$, then $F[\mathsf{ord}(\iota)] \vdash^\alpha_\rho \Gamma, A_\iota$ for all $\iota \in J$.*

**Proof.** By induction on $\alpha$. □

**Lemma 7.3 (Cut-reduction)** *Assume that* $C \simeq \bigvee_{\iota \in J} C_\iota$ *and* $\mathsf{rk}(C) = \Omega + k + 1$. *If* $F \vdash^{\alpha}_{\Omega+k+1} \Gamma, \neg C$ *and* $F \vdash^{\beta}_{\Omega+k+1} \Gamma, C$, *then* $F \vdash^{\alpha+\beta}_{\Omega+k+1} \Gamma$.

**Proof.** By induction on $\beta$. □

**Lemma 7.4 (Cut-elimination)** *Let* $k < \omega$. *If* $F \vdash^{\alpha}_{\Omega+k+2} \Gamma$, *then* $F \vdash^{\Omega^\alpha}_{\Omega+k+1} \Gamma$.

**Lemma 7.5** $F[\xi]^\alpha(\xi) \leq F^\alpha(\xi)$.

**Proof.** By induction on $\alpha$. □

**Lemma 7.6** *If* $\eta < \xi$ *and* $\alpha_\eta < \alpha$ *and* $K\alpha_\eta < F[\eta](0)$ *then* $F[\eta]^{\alpha_\eta}(\xi) \leq F^\alpha(\xi)$.

**Lemma 7.7** *If* $\eta < F(0)$ *and* $\alpha_\eta < \alpha$ *and* $K\alpha_\eta < F[\eta](0)$ *then* $F[\eta]^{\alpha_\eta}(\xi) \leq F^\alpha(\xi)$.

**Definition 7.8** *For each* $\mathcal{L}^*$*-formula* $B$ *let* $B^{\alpha,\beta}$ *denote the result of replacing in* $B$ *every negative occurrence of* $P^{<\Omega}_{\mathcal{A}}$ *by* $P^{<\alpha}_{\mathcal{A}}$ *and every positive occurrence of* $P^{<\Omega}_{\mathcal{A}}$ *by* $P^{<\beta}_{\mathcal{A}}$. *For each sequent* $\Gamma$ *consisting of* $\mathcal{L}^*$*-formulas let* $\Gamma^{\alpha,\beta} := \{B^{\alpha,\beta} \mid B \in \Gamma\}$. *It is known that, viewing* $\mathrm{ID}_1$ *as a subsystem of set theory in a standard way,* $L_\Omega \models \mathrm{ID}_1$ *holds for the* $\Omega$*th stage* $L_\Omega$ *of the constructible hierarchy* $(L_\alpha)_{\alpha \in \mathsf{Ord}}$. *We will just write* $\models B$ *(*$B$ *is an* $\mathcal{L}^*$ *sentence) or* $\models \Gamma$ *(*$\Gamma$ *is an* $\mathcal{L}^*$ *sequent) to refer to this relation if no confusion arises.*

**Theorem 7.9 (Witnessing)** *If* $F \vdash^{\alpha}_{\Omega+1} \Gamma$, *then* $\models \Gamma^{\xi, F^\alpha(\xi)}$ *for all* $\xi < \Omega$.

**Proof.** By induction on $\alpha$. □

In embedding $\mathrm{ID}_1$ into $\mathrm{ID}_1^*$, we follow (very closely) the exposition in the lecture notes [4] and indicate how the operators can be adapted accordingly. As in case of embedding $\mathrm{ID}_1$ into $\mathrm{ID}_1^\infty$, the condition $\mathsf{HYP(E)}$ on page 10 holds.

**Lemma 7.10 (Tautology lemma)** *Let* $s, t \in \mathcal{T}(\mathcal{L}_{\mathrm{ID}_1})$, $\Gamma$ *a sequent of* $\mathcal{L}^*$*-sentences, and* $A(x)$ *be an* $\mathcal{L}^*$*-formula such that* $\mathsf{FV}(A) = \{x\}$. *If* $\mathsf{val}(s) = \mathsf{val}(t)$, *then* $F \vdash^{\mathsf{rk}(A)\cdot 2}_0 \Gamma, \neg A(s), A(t)$, *provided* $\mathsf{k}^\Pi_\Omega(\Gamma) \cup \mathsf{k}^\Pi_\Omega(A) < F(0)$.

**Proof.** By induction on $\mathsf{rk}(A)$. □

**Lemma 7.11** *Let* $B_j$ *be an* $\mathcal{L}_{\mathrm{ID}_1}$*-sentence for each* $j = 0, \ldots, l-1$. *Suppose that* $B_0 \vee \cdots \vee B_{l-1}$ *is a logical consequence in the first order predicate logic with equality. Then there exists a natural* $k < \omega$ *such that* $F \vdash^{\Omega\cdot 2+k}_0 \Gamma, B_0, \ldots, B_{l-1}$, *provided* $\mathsf{k}^\Pi_\Omega(\Gamma) < F(0)$.

This can be shown like Lemma 5.2.

**Lemma 7.12** *Let $m \in \mathbb{N}$ and $A(x)$ be an $\mathcal{L}_{\mathrm{ID}_1}$-formula such that* $\mathsf{FV}(A(x)) = \{x\}$. *Then for any $t \in \mathcal{T}(\mathcal{L}_{\mathrm{ID}_1})$ and for any sequent $\Gamma$ of $\mathcal{L}_{\mathrm{ID}_1}$-sentences*

$$F \vdash_0^{(\mathsf{rk}(A)+\mathsf{val}(t))\cdot 2} \Gamma, \neg A(0), \neg\forall x(A(x) \rightarrow A(S(x))), A(t),$$

*provided* $\mathsf{k}_\Omega^\Pi(\Gamma) \cup \mathsf{k}_\Omega^\Pi(A) < F(0)$.

**Proof.** By induction on $\mathsf{val}(t)$. □

**Lemma 7.13** *Let $\xi \leq \Omega$, $A(x)$ be an $\mathcal{L}_{\mathrm{ID}_1}$-formula such that* $\mathsf{FV}(A(x)) = \{x\}$ *and $B(X)$ be an $X$-positive $\mathcal{L}_{\mathrm{PA}}(X)$-formula such that* $\mathsf{FV}(A) = \emptyset$. *Then*

$$F \vdash_0^{(\mathsf{rk}(A)+\alpha+1)\cdot 2} \Gamma, \neg\forall x(\mathcal{A}(A,x) \rightarrow A(x)), \neg B(P_{\mathcal{A}}^{<\xi}), B(A),$$

*provided* $\mathsf{k}_\Omega^\Pi(\Gamma) \cup \mathsf{k}_\Omega^\Pi(A) \cup \{\mathsf{ord}(\xi)\} < F(0)$ *where* $\alpha := \mathsf{rk}(B(P_{\mathcal{A}}^{<\xi}))$.

**Proof.** By induction on $\mathsf{rk}(B(P_{\mathcal{A}}^{<\xi}))$. □

**Lemma 7.14** *1. $F \vdash_0^{\Omega+\omega} \Gamma, \forall x(\mathcal{A}(P_{\mathcal{A}}^{<\Omega}, x) \rightarrow P_{\mathcal{A}}^{<\Omega}(x))$, provided* $\mathsf{k}_\Omega^\Pi(\Gamma) < F(0)$.

*2. $F \vdash_0^{\Omega\cdot 2+\omega} \Gamma, \forall\vec{y}[\forall x(\mathcal{A}(B(\cdot,\vec{y}),x) \rightarrow B(x,\vec{y})] \rightarrow \forall x(P_{\mathcal{A}}^{<\Omega}(x) \rightarrow B(x,\vec{y}))]$, provided* $\mathsf{k}_\Omega^\Pi(\Gamma) \cup \mathsf{k}_\Omega^\Pi(B) < F(0)$.

Let us recall that $\mathsf{S}$ denotes the ordinal successor.

**Theorem 7.15** *Let $n \in \mathbb{N}$. If $\mathrm{ID}_1 \vdash P_{\mathcal{A}}(\underline{n})$, then there exists an ordinal $\alpha < \varepsilon_{\Omega+1}$ such that $|n|_{\mathcal{A}} < \mathsf{S}^\alpha(0)$.*

Note that the latter bound is sharp in the sense that for each $\alpha < \mathsf{S}^{\varepsilon_{\Omega+1}}(0) := \sup\{\mathsf{S}^{\Omega_m(\Omega+1)}(0) \mid m < \omega\}$ there exists an operator form $\mathcal{A}$ and a natural number $n$ such that $\mathrm{ID}_1 \vdash P_{\mathcal{A}}(\underline{n})$ and $\alpha \leq |n|_{\mathcal{A}}$.

# 8 Conclusion

In [13] the second author has started a new approach to provably total computable functions, providing a streamlined characterisation of those functions provably computable in PA. In this work we extend this approach to those functions provably computable in the system $\mathrm{ID}_1$ of non-iterated inductive definitions. The approach introduced in this work should be extended to stronger impredicative systems. The obvious next step is to extension to the system $\mathrm{ID}_2$ of an iterated inductive definitions. This extension seems to be made possible by employing an additional ordinal operator, i.e., $f, F_0, F_1 \vdash_\rho^\alpha \Gamma$ where $F_0$ is an ordinal function $F_0 : \Omega_1 \rightarrow \Omega_1$, $F_1$ is another ordinal function $F_1 : \Omega_2 \rightarrow \Omega_2$, and $\Omega_2$ denotes the least recursively regular ordinal above $\Omega_1$.

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