

Notes on the first-order part of Ramsey's theorem for pairs

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Abstract

We give the Π_2^0 -part, the Π_3^0 -part and the Π_4^0 -part of RT_2^2 and related combinatorial principles.

1 Introduction

Determinating the first-order part of $\text{WKL}_0 + \text{RT}_2^2$ and other important combinatorial principles is a one of the crucial topics in the study of Reverse Mathematics (see, e.g., [2, 4]). The usual approach for these questions is using forcing arguments to construct a second-order part for the target combinatorial principle. On the other hand, there is a traditional way to study the strength of combinatorial principles by using indicator functions. (For the details of indicator functions, see [6].) In [1], Bovykin and Weiermann gave the Π_2^0 -part of $\text{WKL}_0 + \text{RT}_2^2$ by means of an indicator function defined by a density notion, using the idea of Paris [7] and Paris/Kirby [8]. Using similar arguments, we can show that the Π_2^0 -part of $\text{WKL}_0^* + \text{RT}_2^2$ is equivalent to Elementary Function Arithmetic (see [9]). In this paper, we give the Π_3^0 -part and the Π_4^0 -part of $\text{WKL}_0 + \text{RT}_2^2$ based on [1]. We will also give several density notions to characterize the Π_2^0 -part, the Π_3^0 -part and the Π_4^0 -part of $\text{RT}_{<\infty}^2$, SRT_2^2 , $\text{SRT}_{<\infty}^2$ and EM.

2 The Π_2^0 -part of $\text{WKL}_0 + \text{RT}_2^2$

This section is essentially due to Bovykin/Weiermann[1].

Definition 2.1 (within IS_1). For a finite set X , we define the notion of n -density as follows.

- A finite set X is said to be 0 -dense if $|X| > \min X$.
- A finite set X is said to be $n + 1$ -dense if for any (coloring) function $P : [X]^2 \rightarrow 2$, there exists a subset $Y \subseteq X$ such that Y is n -dense and Y is P -homogeneous, i.e., P is constant on $[Y]^2$.

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Note that “ X is m -dense” can be expressed by a Σ_0 -formula.

Definition 2.2. $n\text{PH}_2^2$ asserts that for any a there exists an n -dense set X such that $\min X > a$.

Define $T_0 := \{k\text{PH}_2^2 \mid k \in \omega\} \cup \text{IS}_1$.

Lemma 2.1. • $\text{WKL}_0 + \text{RT}_2^2 \vdash n\text{PH}_2^2$ for any $n \in \omega$.

• $\text{IS}_1 \vdash m\text{PH}_2^2 \rightarrow \text{PH}_{m+1}^2$.

Proof. Easy. □

Lemma 2.2 (Bovykin/Weiermann[1]). *Let M be a countable model of IS_1 , and let $X \subseteq M$ is a (M -)finite set which is k -dense for any $k \in \omega$. Then, there exists a cut $I \subseteq M$ such that $\min X \in I < \max X$, $X \cap I$ is unbounded in I and $(I, \text{Cod}(I/M) \models \text{WKL}_0 + \text{RT}_2^2$.*

Proof. See [1]. □

Theorem 2.3 (Bovykin/Weiermann[1]). *A Π_2^0 sentence ψ is provable in $\text{WKL}_0 + \text{RT}_2^2$ if and only if it is provable in T_0 .*

Proof. See [1]. □

In fact, we can generalize this theorem as follows.

Theorem 2.4. *A Π_2^0 formula ψ (ψ may contain set parameters) is provable in $\text{WKL}_0 + \text{RT}_2^2$ if and only if it is provable in $\text{IS}_1^0 \cup \{k\text{PH}_2^2 \mid k \in \omega\}$. (Here, IS_1^0 is a system of second-order arithmetic which contains basic axioms and induction axioms for Σ_1^0 -formulas with set parameters.)*

3 The Π_3^0 -part of $\text{WKL}_0 + \text{RT}_2^2$

Definition 3.1. Let $\theta(a, x, y)$ be a Σ_0 -formula. We say that a finite set $X = \{a_i \mid i \leq l\}$ dominates $\theta(a, \cdot, \cdot)$ if $\forall i < l \forall x \leq a_i \exists y \leq a_{i+1} \theta(a, x, y)$ holds. We define several variations of PH_2^2 as follows:

- $\theta\text{-}n\text{PH}_2^2 := \forall a (\forall x \exists y \theta(a, x, y) \rightarrow \exists X (X \text{ is finite, } n\text{-dense, and dominates } \theta(a, \cdot, \cdot)))$,
- $n\widetilde{\text{PH}}_2^2 := \forall X (\forall x \exists y \geq x \exists y \in X \rightarrow \exists Y (Y \text{ is finite, } n\text{-dense, and } Y \subseteq X))$.

Define $T_1 := \{\theta\text{-}k\text{PH}_2^2 \mid k \in \omega, \theta \in \Sigma_0\} \cup \text{IS}_1$ and $\widetilde{T}_1 := \{k\widetilde{\text{PH}}_2^2 \mid k \in \omega\} \cup \text{RCA}_0$. Note that T_1 is a Π_3^0 -theory, i.e., T_1 is a set of Π_3^0 -sentences.

Lemma 3.1. *Let $\theta(a, x, y)$ be a Σ_0 -formula, and let $n \in \omega$. Then, $\text{WKL}_0 + \text{RT}_2^2 \vdash \theta\text{-}n\text{PH}_2^2$, and $\text{WKL}_0 + \text{RT}_2^2 \vdash n\text{PH}_2^2$.*

Proof. Easy. □

Theorem 3.2. A Π_3^0 sentence ψ is provable in $\text{WKL}_0 + \text{RT}_2^2$ if and only if it is provable in T_1 . Thus, T_1 is the Π_3^0 -part of $\text{WKL}_0 + \text{RT}_2^2$.

Proof. We show that $T_1 \not\vdash \psi$ implies $\text{WKL}_0 + \text{RT}_2^2 \not\vdash \psi$ for any Π_3^0 -sentence ψ . Assume that $\psi \equiv \forall a \exists x \forall y \theta(a, x, y)$ is not provable from T_1 . Then, there exists a nonstandard countable model $M \models T_1$ such that $M \models \forall x \exists y \neg \theta(a, x, y)$ for some $a \in M$. By $(\neg\theta)$ - $k\text{PH}_2^2$ and overspill, there exists an m -dense set X which dominates $\neg\theta(a, \cdot, \cdot)$ for some $m \in M \setminus \omega$. By Lemma 2.2, there exists an initial segment $I \subseteq_e M$ such that $(I, \text{Cod}(I/M)) \models \text{WKL}_0 + \text{RT}_2^2$ and $I \cap X$ is unbounded in I . Since X dominates $\neg\theta$, for any $x \in I$ there exists $y \in I$ such that $I \models \neg\theta(a, x, y)$. Thus, we have $(I, \text{Cod}(I/M)) \models \neg\psi$, which means that $\text{WKL}_0 + \text{RT}_2^2 \not\vdash \psi$. \square

Theorem 3.3. A Π_3^0 formula ψ is provable in $\text{WKL}_0 + \text{RT}_2^2$ if and only if it is provable in \widetilde{T}_1 .

Proof. Similar to the proof of Theorem 3.2. \square

Note that \widetilde{T}_1 is equivalent to $\text{I}\Sigma_1^0 \cup \{\forall A \forall a (\forall x \exists y \theta(A, a, x, y) \rightarrow \exists X (X \text{ is finite, } n\text{-dense, and dominates } \theta(A, a, \cdot, \cdot))) \mid n \in \omega, \theta \in \Sigma_0^0\}$ with respect to Π_1^1 -sentences.

4 The Π_4^0 -part of $\text{WKL}_0 + \text{RT}_2^2$

Definition 4.1 (within $\text{I}\Sigma_1$). Let $\theta(a, x, y, z)$ be a Σ_0 -formula. Then, we define the notion of *weakly domination* as follows.

- A 0-dense set X *weakly dominates* $\theta(a, \cdot, \cdot, \cdot)$.
- An $n + 1$ -dense set X *weakly dominates* $\theta(a, \cdot, \cdot, \cdot)$ if for any coloring $P : [X]^2 \rightarrow 2$, there exists a P -homogeneous set $Y \subseteq X$ such that $\forall x < \min X \exists y < \min Y \forall z < \max Y \theta(a, x, y, z)$, Y is n -dense and weakly dominates $\theta(a, \cdot, \cdot, \cdot)$.

Note that “ X is m -dense and weakly dominates $\theta(a, \cdot, \cdot, \cdot)$ ” can be expressed by a Σ_0 formula.

Definition 4.2. Let $\theta(a, x, y, z)$ be a Σ_0 -formula. Then, the assertion $\theta^*\text{-}n\text{PH}_2^2$ is the following

$$\forall a \forall b (\forall x \exists y \forall z \theta(a, x, y, z) \rightarrow \exists X (X \text{ is } n\text{-dense, weakly dominates } \theta(a, \cdot, \cdot, \cdot) \text{ and } \min X > b)).$$

Define $T_2 := \{\theta^*\text{-}n\text{PH}_2^2 \mid n \in \omega, \theta(a, x, y, z) \in \Sigma_0\} \cup \text{I}\Sigma_1$. Note that T_2 is a Π_4^0 -theory.

Lemma 4.1. Let $\theta(a, x, y, z)$ be a Σ_0 -formula, and let $n \in \omega$. Then, $\text{WKL}_0 + \text{RT}_2^2 \vdash \theta^*\text{-}n\text{PH}_2^2$.

Proof. Easy. \square

Theorem 4.2. A Π_4^0 sentence ψ is provable in $\text{WKL}_0 + \text{RT}_2^2$ if and only if it is provable in T_2 . Thus, T_2 is the Π_4^0 -part of $\text{WKL}_0 + \text{RT}_2^2$.

Proof. We show that $T_2 \not\vdash \psi$ implies $\text{WKL}_0 + \text{RT}_2^2 \not\vdash \psi$ for any Π_4^0 -sentence ψ . Assume that $\psi \equiv \forall a \exists x \forall y \forall z \theta(a, x, y, z)$ is not provable from T_2 . Then, there exists a nonstandard countable model $M \models T_2$ such that $M \models \forall x \exists y \neg \theta(a, x, y, z)$ for some $a \in M$. By $(k, \neg\theta)\text{PH}_2^2$ and overspill, there exists an $(m, \theta(a, \cdot, \cdot, \cdot))$ -dense set X such that $\min X > a$ for some $m \in M \setminus \omega$. As the proof of Theorem 1 of [1], we can construct a descending sequence $X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$ which satisfies the following:

- $I = \sup\{\min X_i \mid i \in \omega\} \subseteq_e M$,
- $(I, \text{Cod}(I/M)) \models \text{WKL}_0 + \text{RT}_2^2$,
- $I \cap X$ is unbounded in I ,
- $\forall x \leq \min X_i \exists y \leq \min X_{i+1} \forall z \leq \max X_{i+1} \neg \theta(a, x, y, z)$ for any $i \in \omega$.

Since $\min X_i < \min X_{i+1} < I < \max X_{i+1}$ for any $i \in \omega$, we have $I \models \forall x \exists y \forall z \neg \theta(a, x, y, z)$, i.e., $(I, \text{Cod}(I/M)) \models \neg \psi$. This means that $\text{WKL}_0 + \text{RT}_2^2 \not\vdash \psi$. \square

Remark 4.3. Adding set parameters, we can easily show the following: a Π_4^0 formula ψ is provable in $\text{WKL}_0 + \text{RT}_2^2$ if and only if it is provable in

$$\text{ISigma}_1^0 \cup \{ \forall A \forall a \forall b (\forall x \exists y \forall z \theta(A, a, x, y, z) \rightarrow \exists X (X \text{ is } n\text{-dense, weakly dominates } \theta(A, a, \cdot, \cdot, \cdot) \text{ and } \min X > b) \mid n \in \omega, \theta \in \Sigma_0^0 \}.$$

5 PH_2^2 with stronger largeness notion

In this section, we compare $n\text{PH}_2^2$ with PH_2^2 plus “stronger largeness”.

Definition 5.1 (within ISigma_1). • A finite set X is said to be 0-large if $X \neq \emptyset$.

- A finite set X is said to be $r + 1$ -large if there is a partition $X = \bigsqcup_{i \leq \min X} Y_i$ such that $\max Y_i < \min Y_{i+1}$ for any $i < \min X$ and each Y_i is r -large.

Remark 5.1. 1. For any $r \in \omega$, ISigma_1 proves that for any a , there exists a finite set X such that $\min X > a$ and X is r -large.

2. $Q(a, b) := \max\{r \mid [a, b] \text{ is } r\text{-large}\}$ is an indicator function for WKL_0 .

3. More generally, if M is a model of ISigma_1 and $X \subseteq M$ is r -large for some $r \in M \setminus \omega$, then there exists a cut $I \subseteq_e M$ such that $(I, \text{Cod}(I/M)) \models \text{WKL}_0$ and $X \cap I$ is unbounded in I .

Definition 5.2. 1. $\text{PH}_{2,r}^2$ asserts that for any a , there exists a finite set X such that $\min X > a$ and for any coloring $P : [X]^2 \rightarrow 2$, there exists a P -homogeneous set $Y \subseteq X$ which is r -large.

2. $\widetilde{\text{PH}}_{2,r}^2$ asserts that for any infinite set A , there exists a finite set X such that $X \subseteq A$ and for any coloring $P : [X]^2 \rightarrow 2$, there exists a P -homogeneous set $Y \subseteq X$ which is r -large.
3. In general, $\widetilde{n\text{PH}}_{2,r}^2$ asserts that for any infinite set A , there exists a finite set X such that $X \subseteq A$ and X is (n, r) -dense, where the notion of (n, r) -density is defined as follows:
 - A finite set X is said to be $(0, r)$ -dense if X is r -large.
 - A finite set X is said to be $(n + 1, r)$ -dense if for any coloring $P : [X]^2 \rightarrow 2$, there exists a P -homogeneous set $Y \subseteq X$ which is (n, r) -dense.

Proposition 5.2. $\text{I}\Sigma_1 \vdash n\text{PH}_2^2 \rightarrow \text{PH}_{2,n}^2$.

Proof. Easy. □

The strength of $\text{PH}_{2,r}^2$ is related to the strength of $n\text{PH}_2^2$ in the following meaning.

Proposition 5.3. Assume that $\text{WKL}_0 \vdash \widetilde{\text{PH}}_{2,r}^2$ for all $r \in \omega$, then we have $\text{WKL}_0 \vdash \widetilde{n\text{PH}}_2^2$ for all $n \in \omega$.

Proof. Our assumption is $\text{WKL}_0 \vdash \widetilde{1\text{PH}}_{2,r}^2$ for any $r \in \omega$. We will show by induction on n that $\text{WKL}_0 \vdash \widetilde{n\text{PH}}_{2,r}^2$ for any $r \in \omega$ and for any $n \in \omega$. Let $\text{WKL}_0 \vdash \widetilde{n\text{PH}}_{2,r}^2$ for any $r \in \omega$. Assume for the sake of contradiction that $\text{WKL}_0 \not\vdash \widetilde{(n+1)\text{PH}}_{2,r}^2$ for some $r \in \omega$. Then, there exists a model $(M, S) \models \text{WKL}_0$ and $A \in S$ such that $M \not\cong \omega$, A is unbounded in M and any (M) -finite subset of A is not $(n+1, r)$ -dense. By the assumption, there exists an (n, s) -dense subset of A for any $s \in \omega$. Thus, by overspill, for some $m \in M \setminus \omega$, we can take an (n, m) -dense subset $X \subseteq A$. We will show that this X is in fact $(n+1, r)$ -dense, which leads to a contradiction. By the definition of (n, m) -density, for any coloring $P : [X]^2 \rightarrow 2$, there exists a P -homogeneous set $Y_1 \subseteq X$ which is $(n-1, m)$ -dense, and we can repeat this process n -times then the result set Y_n is m -large. By Remark 5.1.3, there exists a cut $I \subseteq_e M$ such that $(I, \text{Cod}(I/M)) \models \text{WKL}_0$ and $Y_n \cap I$ is unbounded in I . Thus, there exists a finite subset of $Y_n \cap I$ which is $(1, r)$ -dense. This means that Y_n is $(1, r)$ -dense, and hence X is $(n+1, r)$ -dense. □

Thus, if $\text{WKL}_0 \vdash \widetilde{\text{PH}}_{2,r}^2$, then $\text{WKL}_0 + \text{RT}_2^2$ is a Π_2^0 -conservative extension of WKL_0 . This may give a new approach to study the proof-theoretic strength of $\text{WKL}_0 + \text{RT}_2^2$.

Question 5.3. Is $\text{I}\Sigma_1 \cup \{n\text{PH}_2^2 \mid n \in \omega\}$ equivalent to $\text{I}\Sigma_1 \cup \{\text{PH}_{2,r}^2 \mid r \in \omega\}$?

6 Other combinatorial principles

In this section, we give several density notions for SRT_2^2 , $\text{RT}_{<\infty}^2$, $\text{SRT}_{<\infty}^2$, EM and ADS. (For the definitions of these combinatorial principles, see [2, 5, 1].) Using these notions, we can characterize Π_2^0 , Π_3^0 or Π_4^0 part of the target combinatorial principle as in Sections 2,3 and 4.

We reason within IS_1 .

Proposition 6.1. *The Π_2^0 -part, Π_3^0 -part and the Π_4^0 -part of $\text{WKL}_0 + \text{SRT}_2^2$ is characterized by the following density notion.*

A finite set X is said to be

- 0-dense if $|X| > \min X$, and
- $m + 1$ -dense if for any $P : [X]^2 \rightarrow 2$,
 - there exists a P -homogeneous subset $Y \subseteq X$ which is m -dense, or,
 - there exists $Y = \{y_0 < y_1 < \dots < y_l\} \subseteq X$ such that $P(y_0, y_i) \neq P(y_0, y_{i+1})$ for any $0 < i < l$ and Y is m -dense.

For the strength of SRT_2^2 , see also Chong/Slaman/Yang [3].

Proposition 6.2. *The Π_2^0 -part, Π_3^0 -part and the Π_4^0 -part of $\text{WKL}_0 + \text{RT}_{<\infty}^2$ is characterized by the following density notion.*

A finite set X is said to be

- 0-dense if $|X| > \min X$, and
- $m + 1$ -dense if for any coloring $P : [X]^2 \rightarrow k$ such that $k < \min X$, there exists a P -homogeneous subset $Y \subseteq X$ which is m -dense.

Proposition 6.3. *The Π_2^0 -part, Π_3^0 -part and the Π_4^0 -part of $\text{WKL}_0 + \text{SRT}_{<\infty}^2$ is characterized by the following density notion.*

A finite set X is said to be

- 0-dense if $|X| > \min X$, and
- $m + 1$ -dense if for any coloring $P : [X]^2 \rightarrow k$ such that $k < \min X$,
 - there exists a P -homogeneous subset $Y \subseteq X$ which is m -dense, or,
 - there exists $Y = \{y_0 < y_1 < \dots < y_l\} \subseteq X$ such that $P(y_0, y_i) \neq P(y_0, y_{i+1})$ for any $0 < i < l$ and Y is m -dense,

Proposition 6.4. *The Π_2^0 -part, Π_3^0 -part and the Π_4^0 -part of $\text{WKL}_0 + \text{EM}$ is characterized by the following density notion.*

A finite set X is said to be

- 0-dense if $|X| > \min X$, and
- $m + 1$ -dense if
 - for any coloring $P : [X]^2 \rightarrow 2$, there exists $Y \subseteq X$ such that P is transitive on Y and Y is m -dense, and,
 - there is a partition $X = \bigsqcup_{i \leq \min X} Y_i$ such that $\max Y_i < \min Y_{i+1}$ for any $i < \min X$ and each Y_i is m -dense.

Here, a coloring P is said to be transitive if $P(a, b) = P(b, c) \Rightarrow P(a, b) = P(a, c)$.

Proposition 6.5. *The Π_2^0 -part, Π_3^0 -part and the Π_4^0 -part of $\text{WKL}_0 + \text{ADS}$ is characterized by the following density notion.*

A finite set X is said to be

- 0-dense if $|X| > \min X$, and
- $m + 1$ -dense if for any transitive coloring $P : [X]^2 \rightarrow 2$, there exists a P -homogeneous subset $Y \subseteq X$ which is m -dense.

In fact, Slaman/Chong/Yang[4] showed that $\text{WKL}_0 + \text{ADS}$ is a Π_1^1 -conservative extension of $\text{B}\Sigma_2^0$. Thus, for any $n \in \omega$, WKL_0 actually proves for any a , there exists a finite set X such that $\min X > a$ and X is n -dense for ADS.

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References

- [1] Andrey Bovykin and Andreas Weiermann. The strength of infinitary ramseyan principles can be accessed by their densities. to appear.
- [2] Peter A. Cholak, Carl G. Jockusch, and Theodore A. Slaman. On the strength of Ramsey's theorem for pairs. *Journal of Symbolic Logic*, 66(1):1–55, 2001.
- [3] C. T. Chong, Theodore A. Slaman, and Yue Yang. The metamathematics of Stable Ramsey's Theorem for pairs. to appear.
- [4] C. T. Chong, Theodore A. Slaman, and Yue Yang. Π_1^1 -conservation of combinatorial principles weaker than Ramsey's theorem for pairs. *Advances in Mathematics*, 230(3):1060–1077, 2012.

- [5] Denis R. Hirschfeldt and Richard A. Shore. Combinatorial principles weaker than Ramsey's theorem for pairs. *Journal of Symbolic Logic*, 72(1):171–206, 2007.
- [6] Richard Kaye. *Models of Peano Arithmetic*. Oxford Logic Guides, 15. Oxford University Press, 1991. x+292 pages.
- [7] J. B. Paris. Some independence results for Peano Arithmetic. *Journal of Symbolic Logic*, 43(4):725–731, 1978.
- [8] J. B. Paris and L. A. S. Kirby. Σ_n -collection schemas in arithmetic. In *Logic Colloquium '77 (Proc. Conf., Wrocław, 1977)*, volume 96 of *Stud. Logic Foundations Math.*, 1978.
- [9] Keita Yokoyama. On the strength of Ramsey's theorem without Σ_1 -induction. to appear.