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TOWARD OBTAINING A TABLE OF LINK-HOMOTOPY CLASSES: THE SECOND HOMOLOGY OF A REDUCED KNOT QUANDLE

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1. INTRODUCTION

Link-homotopy, introduced by Milnor [11], gives rise to an equivalence relation on oriented and ordered links. More precisely, two links are said to be link-homotopic if they are related to each other by a finite sequence of ambient isotopies and self-crossing changes, keeping the orientation and ordering. Here, a self-crossing change is a homotopy for a single component of a link depicted in Figure 1, supported in a small ball whose intersection with the component consists of two segments. The classification problem of links up to link-homotopy is already solved by Habegger and Lin [5] completely. They gave an algorithm which determines whether given links are link-homotopic or not. On the other hand, a table consisting of all representatives of link-homotopy classes is still not known other than partial ones given by Milnor [11, 12] for links with 3 or fewer components and by Levine [9] for links with 4 components. The comparison algorithm never gives us a complete table. To obtain such a table, we should require link-homotopy invariants. Indeed, both of Milnor and Levine utilized numerical invariants to obtain the tables.

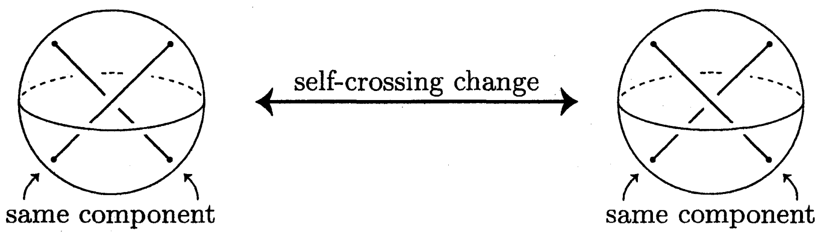

FIGURE 1

Although numerical link-homotopy invariants had not known other than the ones given by Milnor and Levine, the author [7] showed that we have a lot of numerical link-homotopy invariants utilizing quandle theory. A quandle, introduced by Joyce [8], is an algebraic system consisting of a set together with a binary operation whose definition is strongly motivated in knot theory. Hughes [6] defined the reduced knot quandle of a link, which is a certain quotient of the knot quandle given by Joyce [8], and showed that reduced knot quandles are isomorphic if associated links are link-homotopic to each other. The author [7] showed that we have the fundamental classes in the second quandle homology group of a reduced knot quandle, which are invariant under link-homotopy, derived from each components of an associated link if we modify the definition of quandle homology slightly.

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(It is shown by Carter et al. [1, 2] that we have the fundamental classes in the second quandle homology group of a knot quandle which are invariant under ambient isotopy.) The numerical invariants are given by evaluating the images of the fundamental classes by homomorphisms from the second (modified) quandle homology group of a reduced knot quandle to that of a quandle with a 2-cocycle.

The capability of the numerical invariants for classifying links up to link-homotopy essentially depends on the second (modified) quandle homology group of a reduced knot quandle. We are thus interested in the homology group. In this paper, we show that the second (modified) quandle homology group of a reduced knot quandle is completely generated by the fundamental classes derived from the components of an associated link being non-trivial up to link-homotopy (Theorem 4.1). It means, the numerical invariants detect that each component of a link is trivial or not up to link-homotopy. Theorem 4.1 is analogous to the work of Eisermann [3] showing that the second quandle homology group of a knot quandle is freely generated by the fundamental classes derived from the non-trivial components of an associated link.

Throughout this paper, links are assumed to be oriented, ordered and in S^3.

2. Quandle

In this section, we review a quandle, a knot quandle and a reduced knot quandle briefly. We refer the reader to [2, 8] for details about quandles and knot quandles, and to [6, 7] for details about reduced knot quandles.

We first review the definition of a quandle. A *quandle* is a non-empty set X equipped with a binary operation $* : X \times X \to X$ satisfying the following axioms:

(Q1) For each $x \in X$, $x * x = x$.
(Q2) For each $x \in X$, a map $* x : X \to X$ $(w \mapsto w * x)$ is bijective.
(Q3) For each $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

The notion of a homomorphism between quandles is appropriately defined. We will write the image $(* y)^\varepsilon(x)$ as $x *^\varepsilon y$ for any $x, y \in X$ and $\varepsilon \in \{\pm 1\}$.

Associated with a link L, we have a quandle as follows. Let N be a subspace of $\mathbb{C}$ which is the union of the closed unit disk D and a segment $\{z \in \mathbb{C} \mid 1 \leq z \leq 5\}$. Assume that D is oriented counterclockwise. A *noose* of L is a continuous map $\nu : N \to S^3$ satisfying the following conditions:

- The map ν sends $5 \in N$ to a fixed base point $p \in S^3 \setminus L$.
- The restriction map $\nu|_D : D \to S^3$ is an embedding.
- The link L intersects with $\operatorname{Im} \nu$ transversally only at $\nu(0)$.
- The intersection number between L and $\operatorname{Im} \nu|_D$ is $+1$.

The left-hand side of Figure 2 depicts an image of a noose ν. We define a product $*$ of two nooses μ and ν by

$$(\mu * \nu)(z) = \begin{cases} \mu(z) & \text{if } |z| \leq 1, \\ \mu(4z - 3) & \text{if } 1 \leq z \leq 2, \\ \nu(13 - 4z) & \text{if } 2 \leq z \leq 3, \\ \nu(\exp(2(z-3)\pi i)) & \text{if } 3 \leq z \leq 4, \\ \nu(4z - 15) & \text{if } 4 \leq z \leq 5. \end{cases}$$

The right-hand side of Figure 2 shows what happens if we take this product. Let $Q(L)$ be the set consisting of all homotopy classes of nooses of L. The product $*$ of nooses is obviously well-defined on $Q(L)$ and satisfies the axioms of a quandle. We call this quandle $Q(L)$ with $*$ the *knot quandle* of L. By definition, a knot quandle is obviously invariant under ambient isotopy. It is known by Joyce [8] and independently by Matveev [10] that knot quandles are isomorphic if and only if associated knots are week equivalent, i.e., there is a homeomorphism of S^3 sending an associated knot to the other.

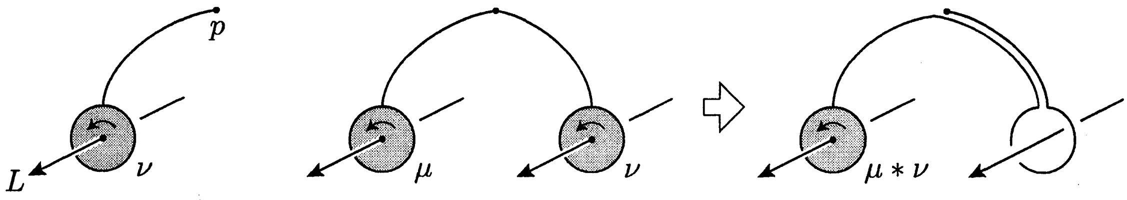

FIGURE 2

Although a knot quandle is not invariant under link-homotopy, we next see that its certain quotient is invariant under link-homotopy. For a link L, let $RQ(L)$ be the quotient of $Q(L)$ by the moves depicted in Figure 3. Then the product $*$ of nooses is well-defined on $RQ(L)$ and still satisfies the axioms of a quandle. We call this quandle $RQ(L)$ with $*$ the *reduced knot quandle* of L. It is known by Hughes [6] that reduced knot quandles are isomorphic if associated links are link-homotopic[1].

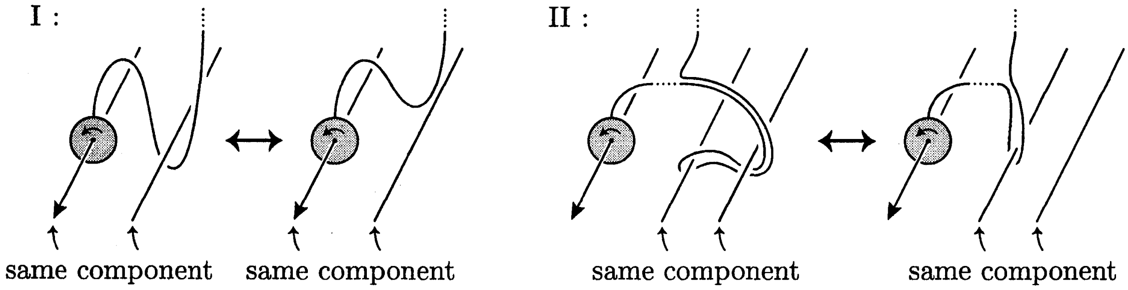

FIGURE 3

We finish up this section by discussing an algebraic property of a reduced knot quandle. We start with reviewing the following notions. An *automorphism group* $\mathrm{Aut}(X)$ of a quandle X is, as usual, the group consisting of all automorphisms of X. The axiom (Q3) of a quandle says that the bijection $*x : X \to X$ is an automorphism of X for each $x \in X$. An *inner automorphism group* $\mathrm{Inn}(X)$ of X is the subgroup of $\mathrm{Aut}(X)$ generated by the automorphisms $*\,x : X \to X$. We call an element of the inner automorphism group an *inner automorphism.*

Nooses μ and ν of a link L intersect with the same component of L if and only if there is an inner automorphism of the knot quandle $Q(L)$ sending the homotopy class of μ to that of ν. Thus I-moves depicted in Figure 3 are algebraically described as the following relation in $Q(L)$:

[1]This definition of a reduced knot quandle is given by the author. In his paper [6], Hughes defined a reduced knot quandle in an algebraic way and more complicated geometric way.

$(*)$ For each $a \in Q(L)$ and $\varphi \in \mathrm{Inn}(Q(L))$, $a * \varphi(a) = a$.
Further II-moves depicted in Figure 3 are described as the relation $a*(b*\varphi(b)) = a*b$ for each $a, b \in Q(L)$ and $\varphi \in \mathrm{Inn}(Q(L))$. Since this relation is an consequence of the relation $(*)$, the reduced knot quandle $RQ(L)$ is algebraically described as the quotient of $Q(L)$ by the relation $(*)$.

We call a quandle X to be *quasi-trivial* [7] if X satisfies the condition $x * \varphi(x) = x$ for each $x \in X$ and $\varphi \in \mathrm{Inn}(X)$. A reduced knot quandle is of course quasi-trivial.

Remark 2.1. For a quandle X, let $F(X)$ be the free group generated by all elements of X and $N(X)$ the subgroup of $F(X)$ normally generated by all elements in the form $y^{-1}xy(x*y)^{-1}$ with some $x, y \in X$. We call the quotient group $F(X)/N(X)$ the *associated group* of X and denote it by $\mathrm{As}(X)$. Since $w*(x*y) = ((w*^{-1}y)*x)*y$ for any $w, x, y \in X$, we have a homomorphism $\mathrm{As}(X) \to \mathrm{Inn}(X)$ sending x to $*\,x$ $(x \in X)$. Thus $\mathrm{As}(X)$ acts on X from the right through this homomorphism. We will write the image of $x \in X$ by the right action of $g \in \mathrm{As}(X)$ as $x \triangleleft g$.

For a link L, it is known that the associated group $\mathrm{As}(Q(L))$ of the knot quandle $Q(L)$ is isomorphic to the knot group $G(L)$ of L (see [4, 8] for example). An isomorphism $\mathrm{As}(Q(L)) \to G(L)$ is given by restricting each noose of L to the union of ∂D and the segment $\{z \in \mathbb{C} \mid 1 \le z \le 5\}$ (this is a positive meridian of L, by definition). Therefore, as Hughes mentioned in [6], the associated group $\mathrm{As}(RQ(L))$ of the reduced knot quandle $RQ(L)$ is isomorphic to the reduced knot group $RG(L)$, where $RG(L)$ is the quotient group of $G(L)$ by the subgroup normally generated by all elements in the form $[g, h^{-1}gh]$ with some $g, h \in G(L)$.

3. Quandle homology

This section is devoted to reviewing homology theory of quandles. We see that we have the fundamental classes in the second homology group of a knot quandle, which are invariant under ambient isotopy, derived from each components of an associated link. Although we do not have the fundamental classes in the second homology group of a reduced knot quandle which are invariant under link-homotopy, modifying the definition of quandle homology slightly, we define the fundamental classes being invariant under link-homotopy. We refer the reader to [1, 2] for details about quandle homology, and to [7] for details about modified quandle homology.

We first review the definition of quandle homology. Let X be a quandle. Consider the free abelian group $C_n^R(X)$ generated by all n-tuples $(x_1, x_2, \ldots, x_n) \in X^n$ for each $n \ge 1$. We let $C_0^R(X) = \mathbb{Z}$. Define a map $\partial_n : C_n^R(X) \to C_{n-1}^R(X)$ by

$$\partial_n(x_1, x_2, \ldots, x_n) = \sum_{i=2}^{n} (-1)^i \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \\ - (x_1 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n)\}$$

for $n \ge 2$, and $\partial_1 = 0$. Then we have $\partial_{n-1} \circ \partial_n = 0$. Thus $(C_n^R(X), \partial_n)$ is a chain complex. Let $C_n^D(X)$ be a subgroup of $C_n^R(X)$ generated by n-tuples $(x_1, x_2, \ldots, x_n) \in X^n$ with $x_i = x_{i+1}$ for some i if $n \ge 2$, and let $C_n^D(X) = 0$ otherwise. It is routine to check that $\partial_n(C_n^D(X)) \subset C_{n-1}^D(X)$. Therefore, putting $C_n^Q(X) = C_n^R(X)/C_n^D(X)$, we have a chain complex $(C_n^Q(X), \partial_n)$. Let G be an abelian group. The n-th *quandle homology group* $H_n^Q(X; G)$ with coefficients in G is the n-th homology group of the chain complex $(C_n^Q(X) \otimes G, \partial_n \otimes \mathrm{id})$. The n-th *quandle cohomology group* $H^n_Q(X; G)$ with coefficients in

G is the n-th cohomology group of the cochain complex $(\mathrm{Hom}(C_n^Q(X), G), \mathrm{Hom}(\partial_n, \mathrm{id}))$. We will use the symbol $[\cdot]$ to denote a class of quandle homology or cohomology.

Let L be an n-component link and D its diagram. To arcs $\alpha, \beta, \ldots$ of D, we assign elements $a, b, \ldots$ of the knot quandle $Q(L)$ respectively in the same manner as Wirtinger generators. For each i $(1 \leq i \leq n)$, consider an element $W_i = \sum \varepsilon \cdot (a, b) \in C_2^Q(Q(L))$, where the sum runs over the crossings of D which consist of under arcs α and γ belonging to the i-th component and an over arc β as depicted in Figure 4, and ε is 1 or -1 depending on whether the crossing is positive or negative respectively. Then, by construction, W_i is a 2-cycle. Suppose D' is a diagram of L obtained from D by a single Reidemeister move and $W_i' \in C_2^Q(Q(L))$ the 2-cycle derived from D'. The axioms (Q1), (Q2) and (Q3) of a quandle ensure that the difference $W_i' - W_i$ is in the second boundary group $B_2^Q(Q(L))$ (see [1, 2]). Thus the homology class $[W_i] \in H_2^Q(Q(L))$ does not depend on the choice of D, i.e., it is invariant under ambient isotopy. We call this homology class the *fundamental class* of the knot quandle $Q(L)$ derived from the i-th component, and denote it by $[K_i] \in H_2^Q(Q(L))$.

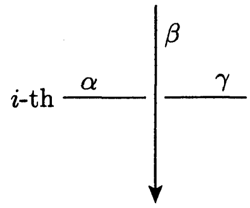

FIGURE 4

For the reduced knot quandle $RQ(L)$, we of course have a 2-cycle $W_i \in C_2^Q(RQ(L))$ derived from D in the same manner. However, if we let D'' be a diagram of L obtained from D by a self-crossing change at a crossing of the i-th component, then the difference $W_i'' - W_i$ is $\pm\,(a, \varphi(a)) \mp (\varphi(a), a)$ with some $a \in RQ(L)$ and $\varphi \in \mathrm{Inn}(RQ(L))$. This difference is not in the second boundary group $B_2^Q(RQ(L))$ in general. Therefore, we do not have fundamental classes in $H_2^Q(RQ(L))$ being invariant under link-homotopy. To solve this problem, we consider to modify the definition of quandle homology as follows.

Suppose X is a quasi-trivial quandle. Let $C_n^{D,qt}(X)$ be a subgroup of $C_n^R(X)$ which is generated by n-tuples $(x_1, x_2, \ldots, x_n) \in X^n$ with $x_i = x_{i+1}$ for some i and n-tuples $(x_1, \varphi(x_1), x_3, \ldots, x_n) \in X^n$ with some $\varphi \in \mathrm{Inn}(X)$ for $n \geq 2$, and $C_n^{D,qt}(X) = 0$ for $n = 0, 1$. By the assumption that X is quasi-trivial, $\partial_n(C_n^{D,qt}(X)) \subset C_{n-1}^{D,qt}(X)$. Thus, putting $C_n^{Q,qt}(X) = C_n^R(X)/C_n^{D,qt}(X)$, we have a chain complex $(C_n^{Q,qt}(X), \partial_n)$. For an abelian group G, let $H_n^{Q,qt}(X; G)$ denote the n-th homology group of the chain complex $(C_n^{Q,qt}(X) \otimes G, \partial_n \otimes \mathrm{id})$, and $H^n_{Q,qt}(X; G)$ the n-th cohomology group of the cochain complex $(\mathrm{Hom}(C_n^{Q,qt}(X), G), \mathrm{Hom}(\partial_n, \mathrm{id}))$. We will use the symbol $[\cdot]^{qt}$ to denote a class of these modified quandle homology or cohomology.

Let L, D and D'' be the same as above. Then we obviously have 2-cycles W_i and W_i'' in $C_2^{Q,qt}(RQ(L))$ derived from D and D'' respectively. Remark that the difference $W_i'' - W_i$ is equal to zero in $C_2^{Q,qt}(RQ(L))$ because $\pm\,(a, \varphi(a)) \mp (\varphi(a), a)$ is an element of $C_2^{D,qt}(RQ(L))$. Therefore, the homology class $[W_i]^{qt} \in H_2^{Q,qt}(RQ(L))$ is invariant under link-homotopy. We call this homology class the *fundamental class* of the reduce knot quandle $RQ(L)$ derived from the i-th component, and denote it by $[K_i]^{qt} \in H_2^{Q,qt}(RQ(L))$.

Remark 3.1. Let X be a quandle, G an abelian group and $\theta \in \operatorname{Hom}(C_2^Q(X), G)$ a 2-cocycle. For an n-component link L, consider the multi-set consisting of n-tuples

$$(\langle[\theta]\,|\,f\,|\,[K_1]\rangle, \langle[\theta]\,|\,f\,|\,[K_2]\rangle, \ldots, \langle[\theta]\,|\,f\,|\,[K_n]\rangle) \in G^n$$

derived from all homomorphisms $f : Q(L) \to X$, where $\langle[\theta]\,|\,f\,|\,[K_i]\rangle \in G$ denotes the value obtained by evaluating the image of $[K_i] \in H_2^Q(Q(L))$ by the homomorphism $H_2^Q(Q(L)) \to H_2^Q(X)$ induced from f with $[\theta] \in H_Q^2(X; G)$. This multi-set, introduced by Carter et al. [2], is obviously invariant under ambient isotopy and is called a *quandle cocycle invariant.*

Assume that X is quasi-trivial and θ a 2-cocycle in $\operatorname{Hom}(C_2^{Q,qt}(X), G)$. Then obviously the multi-set consisting of n-tuples

$$(\langle[\theta]^{qt}\,|\,f\,|\,[K_1]^{qt}\rangle, \langle[\theta]^{qt}\,|\,f\,|\,[K_2]^{qt}\rangle, \ldots, \langle[\theta]^{qt}\,|\,f\,|\,[K_n]^{qt}\rangle) \in G^n$$

derived from all homomorphisms $f : RQ(L) \to X$ is invariant under link-homotopy. A numerical link-homotopy invariant introduced by the author in [7] is exactly this multi-set, i.e., a type of quandle cocycle invariant.

4. The second homology of a reduced knot quandle

The aim of this section is to show the following theorem:

Theorem 4.1. *Let L be an n-component link. If the $i_1, i_2, \ldots, i_m$-th components of L are non-trivial and the other components are trivial, up to link-homotopy, then*

$$H_2^{Q,qt}(RQ(L)) = \langle[K_{i_1}]^{qt}\rangle \oplus \langle[K_{i_2}]^{qt}\rangle \oplus \cdots \oplus \langle[K_{i_m}]^{qt}\rangle.$$

Here, a component of a link is said to be trivial up to link-homotopy if the component bounds a disk which is disjoint from the other components of the link, after deforming the link by link-homotopy. The second homology group $H_2^{Q,qt}(RQ(L))$ is not always torsion-free (see Remark 4.8).

Theorem 4.1 is an analogue of the following theorem introduced by Eisermann [3]:

Theorem 4.2 (Eisermann [3]). *Let L be an n-component link. If the $i_1, i_2, \ldots, i_m$-th components of L are non-trivial and the other components are trivial, then $H_2^Q(Q(L))$ is freely generated by $[K_{i_1}], [K_{i_2}], \ldots, [K_{i_m}]$, i.e.,*

$$H_2^Q(Q(L)) = \langle[K_{i_1}]\rangle \oplus \langle[K_{i_2}]\rangle \oplus \cdots \oplus \langle[K_{i_m}]\rangle = \operatorname{span}_{\mathbb{Z}}\{[K_{i_1}], [K_{i_2}], \ldots, [K_{i_m}]\}.$$

We prove Theorem 4.1 in a similar way to the proof of Theorem 4.2 which Eisermann gave in [3]. We first review the notion of a quandle covering. Let X and $\widetilde{X}$ be quandles. An epimorphism $p : \widetilde{X} \to X$ is said to be a *covering* if $p(\widetilde{x}) = p(\widetilde{y})$ implies $\widetilde{w} * \widetilde{x} = \widetilde{w} * \widetilde{y}$ for any $\widetilde{w}, \widetilde{x}, \widetilde{y} \in \widetilde{X}$. In other words, the natural map $\widetilde{X} \to \operatorname{Inn}(\widetilde{X})$ sending $\widetilde{x}$ to $*\widetilde{x}$ factors through p. This property of a covering enables us to write an element $\widetilde{w} * \widetilde{x}$ as $\widetilde{w} * p(\widetilde{x})$.

A reduced knot quandle has the universal covering. To see it, we consider the following situation. Let L be an n-component link and $\mathscr{D}$ an embedded oriented disk in S^3 with which each component of L intersects only once transversally and positively. Choose a diagram D of L so that the image of $\mathscr{D}$ is a segment intersecting with each component of L in order (see the left-hand side of Figure 5). Furthermore, let T_i be a $(1,1)$-tangle obtained from (S^3, L) by removing a small regular neighborhood of the intersection point of the i-th component of L and $\mathscr{D}$. We remark that we have a diagram D_i of T_i, removing

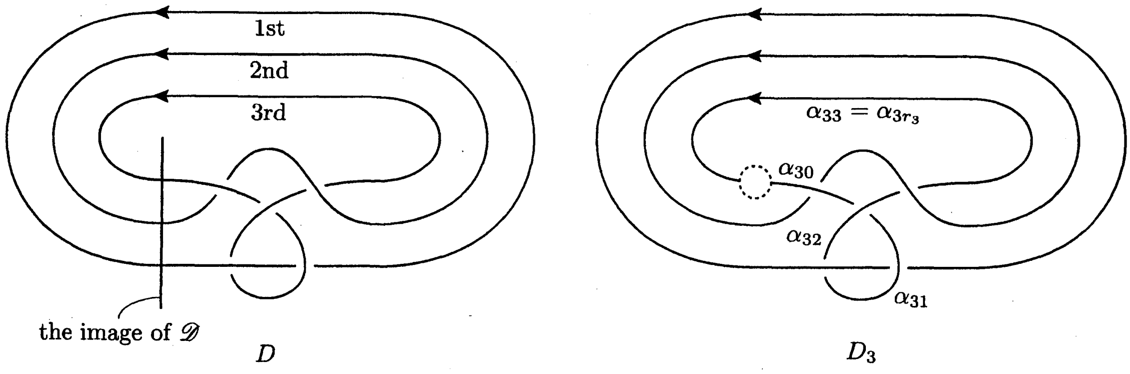

FIGURE 5

a small regular neighborhood of the intersection point of the i-th component and the image of $\mathscr{D}$ in (S^2, D) (see the right-hand side of Figure 5).

For each i $(1 \le i \le n)$, consider the set consisting of the homotopy classes of nooses of T_i which intersect with the i-th component. Let $\widetilde{RQ}(L)$ be the union of the quotients of these sets by I- and II-moves depicted in Figure 3, i.e.,

$$\widetilde{RQ}(L) = \bigcup_{i=1}^{n} ((\{\text{noose of } T_i \text{ i.w. } i\text{-th component}\} \,/\, \text{homotopy}) \,/\, \text{I- and II-moves}).$$

For each noose μ of T_i intersecting with the i-th component and noose ν of T_j intersecting with the j-th component, regarding ν as a noose of T_i in a natural way, we define the product $\mu * \nu$ in the same manner as in Section 2. This product $*$ is well-defined on $\widetilde{RQ}(L)$, and satisfies the axioms of a quandle and the condition for a quasi-trivial quandle. That is, $\widetilde{RQ}(L)$ with $*$ is a quasi-trivial quandle. Inclusion maps $(S^3 \setminus (\text{small ball}), T_i) \hookrightarrow (S^3, L)$ naturally induce a projection $\pi : \widetilde{RQ}(L) \to RQ(L)$. By definition, the natural map $\widetilde{RQ}(L) \to \mathrm{Inn}(\widetilde{RQ}(L))$ factors through π. Thus π is a covering.

To claim that π is universal, we further introduce the following notations. For each i $(1 \le i \le n)$, let α_{ij} denote an arc of D_i which is a part of the i-th component $(0 \le j \le r_i)$, in the way as depicted in the right-hand side of Figure 5. We assign $a_{ij} \in \widetilde{RQ}(L)$ to each α_{ij} in the same manner as a Wirtinger generator. Note that we have $\pi(a_{ir_i}) = \pi(a_{i0})$. Let β_{ij} be the arc separating $\alpha_{i,j-1}$ and α_{ij} $(1 \le j \le r_i)$, and $b_{ij} \in \widetilde{RQ}(L)$ the element assigned to β_{ij}. Then we have a relation $a_{ij} = a_{i,j-1} *^{\varepsilon_{ij}} b_{ij}$ in $\widetilde{RQ}(L)$, where ε_{ij} is 1 or -1 depending on whether the crossing consisting of $\alpha_{i,j-1}$, α_{ij} and β_{ij} is positive or negative respectively. We note that $\widetilde{RQ}(L)$ is generated by all elements of the set $\{a_{ij} \mid 1 \le i \le n,\ 0 \le j \le r_i\}$ and any relation in $\widetilde{RQ}(L)$ is a consequence of the relations $\{a_{ij} = a_{i,j-1} *^{\varepsilon_{ij}} b_{ij} \mid 1 \le i \le n,\ 1 \le j \le r_i\}$.

Proposition 4.3. *Let X and $\widetilde{X}$ be quasi-trivial quandles and $p : \widetilde{X} \to X$ a covering. Then, for each homomorphism $f : \widetilde{RQ}(L) \to X$ sending a_{i0} to x_i, we have a unique lift $\widetilde{f} : \widetilde{RQ}(L) \to \widetilde{X}$ of f sending a_{i0} to $\widetilde{x}_i \in p^{-1}(x_i)$ (i.e., $\widetilde{f}$ is a homomorphism satisfying $p \circ \widetilde{f} = f$). In particular, the natural projection $\pi : \widetilde{RQ}(L) \to RQ(L)$ is the universal covering.*

Proof. We inductively define a map $\widetilde{f} : \{a_{ij} \mid 1 \leq i \leq n,\ 0 \leq j \leq r_i\} \to \widetilde{X}$ as follows. To start with we let $\widetilde{f}(a_{i0}) = \widetilde{x}_i$ for all i $(1 \leq i \leq n)$ so that $p \circ \widetilde{f}(a_{i0}) = x_i$. At each crossing, we set $\widetilde{f}(a_{ij}) = \widetilde{f}(a_{i,j-1}) *^{\varepsilon_{ij}} f(b_{ij})$. Then, by induction, we have $p \circ \widetilde{f}(a_{ij}) = f(a_{ij})$ and so $\widetilde{f}(a_{ij}) = \widetilde{f}(a_{i,j-1}) *^{\varepsilon_{ij}} \widetilde{f}(b_{ij})$ for all i and j $(1 \leq i \leq n,\ 1 \leq j \leq r_i)$. It means that $\widetilde{f}$ uniquely extends to a homomorphism $\widetilde{f} : \widetilde{RQ}(L) \to \widetilde{X}$ satisfying $p \circ \widetilde{f} = f$. □

Remark 4.4. Remember that the reduced knot group $RG(L)$ acts on $RQ(L)$ from the right. Thus $RG(L)$ also acts on $\widetilde{RQ}(L)$ from the right, because $\pi : \widetilde{RQ}(L) \to RQ(L)$ is a covering. For each i $(1 \leq i \leq n)$, let $RG_i(L)$ denote the reduced knot group for the sublink of L obtained by removing the i-th component. Since $\widetilde{RQ}(L)$ is quasi-trivial, $RG_i(L)$ acts on each element of $\widetilde{RQ}(L)$ intersecting with the i-th component from the right through the quotient map $RG(L) \to RG_i(L)$. Therefore, each element of $\widetilde{RQ}(L)$ can be written as $a_{j0} \triangleleft u$ with some j $(1 \leq j \leq n)$ and $u \in RG_j(L)$. Identifying an element of $\widetilde{RQ}(L)$ with an element of $RG_i(L)$, consider the element

$$l_i = b_{i1}^{\varepsilon_{i1}} b_{i2}^{\varepsilon_{i2}} \cdots b_{ir_i}^{\varepsilon_{ir_i}} \in RG_i(L).$$

Then, by definition, we have $a_{ir_i} = a_{i0} \triangleleft l_i$.

Let X and $\widetilde{X}$ be (not necessary quasi-trivial) quandles. Assume that an abelian group G acts on $\widetilde{X}$ from the left. We call an epimorphism $E : G \curvearrowright \widetilde{X} \to X$ to be a *central extension* if the following conditions hold:

(E1) For each $g \in G$ and $\widetilde{x}, \widetilde{y} \in \widetilde{X}$, $(g \cdot \widetilde{x}) * \widetilde{y} = g \cdot (\widetilde{x} * \widetilde{y})$ and $\widetilde{x} * (g \cdot \widetilde{y}) = \widetilde{x} * \widetilde{y}$.
(E2) The abelian group G acts freely and transitively on each fiber $E^{-1}(x)$.

By definition, a central extension is a covering equipped with special properties. We next see that, for a quasi-trivial quandle X and its central extension $G \curvearrowright \widetilde{X} \to X$ with some quasi-trivial quandle $\widetilde{X}$, when a homomorphism $RQ(L) \to X$ lifts to $RQ(L) \to \widetilde{X}$.

Two central extensions $E_1 : G \curvearrowright \widetilde{X}_1 \to X$ and $E_2 : G \curvearrowright \widetilde{X}_2 \to X$ are said to be *equivalent* if there is a G-equivariant isomorphism $f : \widetilde{X}_1 \to \widetilde{X}_2$ satisfying $E_1 = E_2 \circ f$. For a quasi-trivial quandle X and an abelian group G, let $\mathscr{E}^{qt}(X, G)$ be the set consisting of all equivalence classes of central extensions $G \curvearrowright \widetilde{X} \to X$ with some quasi-trivial quandle $\widetilde{X}$. Then we have the following lemma:

Lemma 4.5. *There is a bijection between $\mathscr{E}^{qt}(X, G)$ and $H^2_{Q,qt}(X; G)$.*

Proof. For a central extension $E : G \curvearrowright \widetilde{X} \to X$, choose a section $s : X \to \widetilde{X}$ and define a map $\theta : X \times X \to G$ so that $s(x) * s(y) = \theta(x, y) \cdot s(x * y)$. We remark that θ is well-defined because G acts freely and transitively on each fiber and we have $s(x) * s(\varphi(x)) = s(x)$ for all $x \in X$ and $\varphi \in \mathrm{Inn}(X)$. It is easy to see that θ is a 2-cocycle in $\mathrm{Hom}(C_2^{Q,qt}(X), G)$. Suppose $\theta' \in \mathrm{Hom}(C_2^{Q,qt}(X), G)$ is a 2-cocycle derived from another section $s' : X \to \widetilde{X}$. Then the difference $\theta' - \theta$ is in the second coboundary group $B^2_{Q,qt}(X; G)$. Indeed, with a map $\eta : X \to G$ defined so that $s'(x) = \eta(x) \cdot s(x)$, we have $\theta'(x, y) - \theta(x, y) = \eta(\partial_1(x, y))$. We thus have a unique class $[\theta]^{qt} \in H^2_{Q,qt}(X; G)$ associated with E. Further, consider equivalent central extensions $G \curvearrowright \widetilde{X}_1 \to X$ and $G \curvearrowright \widetilde{X}_2 \to X$ with a G-equivariant isomorphism $f : \widetilde{X}_1 \to \widetilde{X}_2$, and a 2-cocycle θ derived from a section $s : X \to \widetilde{X}_1$. Then, $f \circ s : X \to \widetilde{X}_2$ is of course a section and we have $(f \circ s)(x) * (f \circ s)(y) = \theta(x, y) \cdot (f \circ s)(x * y)$ for all $x, y \in X$. Therefore, we have a map $\Phi : \mathscr{E}^{qt}(X, G) \to H^2_{Q,qt}(X; G)$.

On the other hand, for a 2-cocycle $\theta \in \mathrm{Hom}(C_2^{Q,qt}(X), G)$, define a binary operation $*$ on $G \times X$ by $(g,x) * (h,y) = (g + \theta(x,y), x * y)$. Then $G \times X$ with $*$ is in fact a quasi-trivial quandle. We let $G \times_\theta X$ denote this quasi-trivial quandle. The abelian group G acts on $G \times_\theta X$ from the left by $h \cdot (g,x) = (g+h, x)$. We thus have a central extension $G \curvearrowright G \times_\theta X \to X$ sending (g,x) to x. Consider a map $\eta : X \to G$ and a 2-cocycle $\theta' = \theta + \eta \circ \partial_1$ cohomologous to θ. Then the central extensions $G \curvearrowright G \times_\theta X \to X$ and $G \curvearrowright G \times_{\theta'} X \to X$ are equivalent with a G-equivariant isomorphism $G \times_\theta X \to G \times_{\theta'} X$ sending (g,x) to $(g - \eta(x), x)$. Therefore, we have a map $\Psi : H^2_{Q,qt}(X; G) \to \mathscr{E}^{qt}(X,G)$.

For a 2-cocycle $\theta \in \mathrm{Hom}(C_2^{Q,qt}(X), G)$, define a section $s : X \to G \times_\theta X$ by $s(x) = (0,x)$. Then we have $s(x) * s(y) = \theta(x,y) \cdot s(x*y)$ for all $x, y \in X$. It means that $\Phi \circ \Psi = \mathrm{id}$. Conversely, for a central extension $E : G \curvearrowright \widetilde{X} \to X$, suppose $\theta \in \mathrm{Hom}(C_2^{Q,qt}(X), G)$ is a 2-cocycle derived from a section $s : X \to \widetilde{X}$. Then the central extensions $G \times_\theta X \to X$ and E are equivalent with a G-equivariant isomorphism $f : G \times_\theta X \to \widetilde{X}$ sending (g,x) to $g \cdot s(x)$. It means that $\Psi \circ \Phi = \mathrm{id}$. □

Let X and $\widetilde{X}$ be quasi-trivial quandles, G an abelian group and $E : G \curvearrowright \widetilde{X} \to X$ a central extension. Consider a homomorphism $f : RQ(L) \to X$ sending $\pi(a_{i0})$ to x_i. Then we have a homomorphism $f \circ \pi : \widetilde{RQ}(L) \to X$ sending a_{i0} to x_i, and so its unique lift $\widetilde{f \circ \pi} : \widetilde{RQ}(L) \to \widetilde{X}$ sending a_{i0} to $\widetilde{x}_i \in E^{-1}(x_i)$ in the light of Proposition 4.3. Suppose $\theta \in \mathrm{Hom}(C_2^{Q,qt}(X), G)$ is a 2-cocycle derived from a section $s : X \to \widetilde{X}$. Then we have the following proposition, of which a necessary and sufficient condition for the existence of a lift $RQ(L) \to \widetilde{X}$ of a homomorphism $RQ(L) \to X$ is given as a corollary:

Proposition 4.6. *For each i $(1 \le i \le n)$ and $u \in RG_i(L)$, we have*

$$\widetilde{f \circ \pi}(a_{i0} \lhd l_i u) = \langle [\theta]^{qt} \,|\, f \,|\, [K_i]^{qt} \rangle \cdot \widetilde{f \circ \pi}(a_{i0} \lhd u).$$

Proof. By a straightforward calculus, we have

$$s(f \circ \pi(a_{ij})) = \begin{cases} -\theta(f \circ \pi(a_{i,j-1}), f \circ \pi(b_{ij})) \cdot (s(f \circ \pi(a_{i,j-1})) * s(f \circ \pi(b_{ij}))) & \text{if } \varepsilon_{ij} = 1, \\ \theta(f \circ \pi(a_{ij}), f \circ \pi(b_{ij})) \cdot (s(f \circ \pi(a_{i,j-1})) *^{-1} s(f \circ \pi(b_{ij}))) & \text{if } \varepsilon_{ij} = -1 \end{cases}$$

for all j $(1 \le j \le r_i)$. Therefore, by definition, we have

$$\widetilde{f \circ \pi}(a_{i0} \lhd l_i) = \widetilde{f \circ \pi}(a_{ir_i}) = \langle [\theta]^{qt} \,|\, f \,|\, [K_i]^{qt} \rangle \cdot \widetilde{f \circ \pi}(a_{i0}).$$

It is easy to see that we have the equation in the proposition from the above equation. □

Corollary 4.7. *A homomorphism $f : RQ(L) \to X$ sending $\pi(a_{i0})$ to x_i uniquely lifts to a homomorphism $\widetilde{f} : RQ(L) \to \widetilde{X}$ sending $\pi(a_{i0})$ to $\widetilde{x}_i \in E^{-1}(x_i)$ if and only if $\langle [\theta]^{qt} \,|\, f \,|\, [K_i]^{qt} \rangle = 0$ for all i $(1 \le i \le n)$.*

Proof. Since $\pi(a_{i0} \lhd l_i) = \pi(a_{i0})$ for all i, the lift $\widetilde{f \circ \pi}$ is decomposed as $\widetilde{f} \circ \pi$ if and only if $\langle [\theta]^{qt} \,|\, f \,|\, [K_i]^{qt} \rangle = 0$ for all i $(1 \le i \le n)$. □

We now prove Theorem 4.1.

Proof of Theorem 4.1. We first remark that $[K_i]^{qt} = 0$ if the i-th component of L is trivial up to link-homotopy. Indeed, we have a diagram of a link being link-homotopic to L in which the i-th component has no crossings.

Milnor [11] showed that $l_i \in RG_i(L)$ is trivial if and only if the i-th component of L is trivial up to link-homotopy. We thus have the cyclic subgroups $\langle l_{i_1} \rangle, \langle l_{i_2} \rangle, \ldots, \langle l_{i_m} \rangle$

of $RG_{i_1}(L), RG_{i_2}(L), \ldots, RG_{i_m}(L)$ respectively, which are not trivial. We note that the orders of these cyclic subgroups are not always infinite.

For each $(1,1)$-tangle T_i $(i = i_1, i_2, \ldots, i_m)$, consider its reduced knot quandle $RQ(T_i)$ in the same manner as in Section 2. We then have a natural projection $\pi_i : RQ(T_i) \to RQ(L)$, which is obviously a covering. Define a left action of $\langle l_i \rangle$ on $RQ(T_i)$ by

$$l_i \cdot (a_{j0} \triangleleft u) = \begin{cases} a_{i0} \triangleleft l_i u & (j = i), \\ a_{j0} \triangleleft u & (j \neq i). \end{cases}$$

We remark that each element of $RQ(T_i)$ can be written as $a_{j0} \triangleleft u$ with some j $(1 \le j \le n)$ and $u \in RG_j(L)$. The projection π_i with the action $\langle l_i \rangle \curvearrowright RQ(T_i)$ satisfies the condition (E1) of a central extension but does not satisfy the condition (E2) in general. Indeed, although $\langle l_i \rangle$ acts freely and transitively on a fiber $\pi_i^{-1}(a_{i0} \triangleleft u)$, it acts trivially on a fiber $\pi_i^{-1}(a_{j0} \triangleleft u)$ if $j \neq i$. However, we can define a 2-cocycle $\theta_i \in \mathrm{Hom}(C_2^{Q,qt}(RQ(L)), \langle l_i \rangle)$ associated with a section $s : RQ(L) \to RQ(T_i)$ so that $s(a) * s(b) = \theta_i(a,b) \cdot s(a * b)$ if $a = a_{i0} \triangleleft u$ with some $u \in RG_i(L)$, and $\theta_i(a,b) = 0$ otherwise. It is routine to check that the class $[\theta_i]^{qt} \in H^2_{Q,qt}(RQ(L); \langle l_i \rangle)$ does not depend on the choice of s. By definition, we have $\langle [\theta_i]^{qt} \,|\, \mathrm{id} \,|\, [K_j]^{qt} \rangle = l_i^{\delta_{ij}}$, where δ_{ij} denotes the Kronecker delta. It means that $[K_i]^{qt} \neq 0$ for $i = i_1, i_2, \ldots i_m$ and $\langle [K_{i_1}]^{qt} \rangle \oplus \langle [K_{i_2}]^{qt} \rangle \oplus \cdots \oplus \langle [K_{i_m}]^{qt} \rangle$ is a subgroup of $H_2^{Q,qt}(RQ(L))$.

It is easy to see that $H_1^{Q,qt}(RQ(L))$ is freely generated by $[(a_{10})]^{qt}, [(a_{20})]^{qt}, \ldots, [(a_{n0})]^{qt}$. Thus, for each abelian group G, $H^2_{Q,qt}(RQ(L); G)$ is isomorphic to $\mathrm{Hom}(H_2^{Q,qt}(RQ(L)), G)$ by the universal coefficient theorem. We let

$$G = H_2^{Q,qt}(RQ(L)) / \langle [K_{i_1}]^{qt} \rangle \oplus \langle [K_{i_2}]^{qt} \rangle \oplus \cdots \oplus \langle [K_{i_m}]^{qt} \rangle$$

and $[\theta]^{qt} : H_2^{Q,qt}(RQ(L)) \to G$ be the projection. Then, by Lemma 4.5, we have a central extension $E : G \curvearrowright G \times_\theta RQ(L) \to RQ(L)$ associated with a representative θ of $[\theta]^{qt}$. By definition, $\langle [\theta]^{qt} \,|\, \mathrm{id} \,|\, [K_i]^{qt} \rangle = 0$ for all i $(1 \le i \le n)$. Therefore, by Corollary 4.7, we have a homomorphism $s : RQ(L) \to G \times_\theta RQ(L)$ which is a lift of the identity map of $RQ(L)$. Since s is a section of E, $[\theta]^{qt}$ should be the zero map by Lemma 4.5 again. It means that G is trivial, i.e., $H_2^{Q,qt}(RQ(L)) = \langle [K_{i_1}]^{qt} \rangle \oplus \langle [K_{i_2}]^{qt} \rangle \oplus \cdots \oplus \langle [K_{i_m}]^{qt} \rangle$. □

In the light of Theorem 4.1, we can completely determine which components of a link L are trivial up to link-homotopy by computing $H_2^{Q,qt}(RQ(L))$.

Remark 4.8. Consider the projection $[\zeta_i] : H_2^{Q,qt}(RQ(L)) \to \langle [K_i]^{qt} \rangle$, which sends $[K_j]^{qt}$ to $([K_i]^{qt})^{\delta_{ij}}$, for each $i = i_1, i_2, \ldots, i_m$. Then, in the light of Lemma 4.5, we have a central extension $E_i : \langle [K_i]^{qt} \rangle \curvearrowright \langle [K_i]^{qt} \rangle \times_{\zeta_i} RQ(L) \to RQ(L)$ associated with a representative ζ_i of $[\zeta_i]$. Further, by Proposition 4.6, we have $\widetilde{\mathrm{id} \circ \pi}(a_{i0} \triangleleft l_i) = [K_i]^{qt} \cdot \widetilde{\mathrm{id} \circ \pi}(a_{i0})$. Thus the order of $[K_i]^{qt}$ should divide that of l_i, if l_i has finite order. On the other hand, we have the homomorphism $[\theta_i] : H_2^{Q,qt}(RQ(L)) \to \langle l_i \rangle$ sending $[K_j]^{qt}$ to $l_i^{\delta_{ij}}$. Therefore, the cardinality of $\langle [K_i]^{qt} \rangle$ coincides with that of $\langle l_i \rangle$.

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