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ON FOCK-GONCHAROV COORDINATES OF THE ONCE-PUNCTURED TORUS GROUP

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1. Introduction

In their seminal paper [3], Fock and Goncharov defined *positive representations* of the fundamental group of a surface S into a split semi-simple real Lie group G (e.g. $\mathrm{PSL}(n,\mathbb{R})$). They showed that the space of positive representations satisfies properties similar to those of the Teichmüller space: a positive representation is faithful, has discrete image in G, and the moduli space of positive representations is diffeomorphic to $\mathbb{R}^{-\chi(S)\dim G}$. In fact, when $G=\mathrm{PSL}(2,\mathbb{R})$, the space of positive representations coincides with the Teichmüller space. They showed that the space of positive representations coincides with the Hitchin component [9] in the representation space of $\pi_1(S)$ into G. It should be mentioned here that Labourie introduced in [11] the notion of *Anosov representations*, whose moduli space coincides with the Hitchin component and the space of positive representations [11], [8].

When the Lie group is $\mathrm{PSL}(n,\mathbb{R})$ and an ideal triangulation of S is fixed, Fock and Goncharov defined two types of invariants for positive representations: 'vertex functions' and 'edge functions'. A vertex function is also called a *triple ratio*, which we will use in this note. They showed that these invariants give a set of coordinates of positive representations. (Their coordinates are also defined for more general representations into $\mathrm{PSL}(n,\mathbb{C})$.) The Fock-Goncharov coordinates are extensively studied: there are generalizations to 3-manifolds groups [1], [6], [5]; the McShane identities are studied in [12]; Fenchel-Nielsen type coordinates for the Hitchin component in [2]. In [10], I and Xin Nie give a parametrization of $\mathrm{PGL}(n,\mathbb{C})$-representations of a surface group as an analogue of the Fenchel-Nielsen coordinates.

In this note, I will explain Fock-Goncharov coordinates and give an explicit construction of matrix generators for once-punctured torus group, in terms of Fock-Goncharov coordinates.

2. Flags

Let $\mathrm{GL}(n,\mathbb{C})$ be the general linear group of $n\times n$ complex matrices. We define two subgroups B and U by

$$B=\left\{\begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ O & & * \end{pmatrix}\right\},\quad U=\left\{\begin{pmatrix} 1 & & * \\ & \ddots & \\ O & & 1 \end{pmatrix}\right\}.$$

The center of $\mathrm{GL}(n,\mathbb{C})$ is isomorphic to $\mathbb{C}^*$, the set of diagonal matrices with the same diagonal entries. We let $\mathrm{PGL}(n,\mathbb{C})=\mathrm{GL}(n,\mathbb{C})/\mathbb{C}^*$. We have a short exact sequence $1\to\mathbb{Z}/n\mathbb{Z}\to\mathrm{SL}(n,\mathbb{C})\to\mathrm{PGL}(n,\mathbb{C})\to 1$.

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A (complete) *flag* in $\mathbb{C}^n$ is a sequence of subspaces

$$\{0\} = V^0 \subsetneq V^1 \subsetneq V^2 \subsetneq \cdots \subsetneq V^n = \mathbb{C}^n.$$

We denote the set of all flags by $\mathcal{F}_n$. $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{PGL}(n,\mathbb{C})$ act naturally on $\mathcal{F}_n$ from the left.

We represent $X \in \mathrm{GL}(n,\mathbb{C})$ by n column vectors as

$$X = \begin{pmatrix} x^1 & x^2 & \cdots & x^n \end{pmatrix}$$

where $x^i = {}^t(x^i_1, \ldots, x^i_n)$ are column vectors. By setting $X^i = \mathrm{span}_{\mathbb{C}}\{x^1, \ldots, x^i\}$, we obtain a flag $\{0\} \subset X^1 \subsetneq \cdots \subsetneq X^n$ from an element of $\mathrm{GL}(n,\mathbb{C})$. Thus we have a map from $\mathrm{GL}(n,\mathbb{C})$ to $\mathcal{F}_n$. Since an upper triangular matrix acts from the right as

$$(1) \qquad X \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ & \ddots & \vdots \\ O & & b_{nn} \end{pmatrix} = \begin{pmatrix} b_{11}x^1 & b_{12}x^1 + b_{22}x^2 & \ldots & b_{1n}x^1 + \cdots + b_{nn}x^n \end{pmatrix},$$

the map induces a map $\mathrm{GL}(n,\mathbb{C})/B \to \mathcal{F}_n$. We can easily show that this is bijective and equivariant with respect to the left action of $\mathrm{GL}(n,\mathbb{C})$. Thus we can identify $\mathcal{F}_n$ with $\mathrm{GL}(n,\mathbb{C})/B$. We can also identify $\mathcal{F}_n$ with $\mathrm{PGL}(n,\mathbb{C})/B$ where we also denote by B for the quotient in $\mathrm{PGL}(n,\mathbb{C})$ by abuse of notation. We let $\mathcal{AF}_n = \mathrm{GL}(n,\mathbb{C})/U$ and call an element of $\mathcal{AF}_n$ an *affine flag*. We have the following short exact sequence:

$$\begin{array}{ccccccc} 1 \to B/U & \to & \mathcal{AF}_n & \to & \mathcal{F}_n & \to 1. \\ & & \| & & \| & \\ & & \mathrm{GL}(n,\mathbb{C})/U & & \mathrm{GL}(n,\mathbb{C})/B & \end{array}$$

Example 2.1. When $n = 2$, $\mathcal{F}_n$ can be identified with the set of lines in $\mathbb{C}^2$. In other words, $\mathcal{F}_2$ is the projective line $\mathbb{C}P^1$. If we regard $\mathbb{C}P^1$ as $\mathbb{C} \cup \{\infty\}$, $\mathrm{PGL}(2,\mathbb{C})$ acts on $\mathbb{C}P^1$ by linear fractional transformations and the stabilizer at ∞ is the subgroup B of upper triangular matrices. Thus we have $\mathcal{F}_2 = \mathbb{C}P^1 \cong \mathrm{PGL}(2,\mathbb{C})/B$.

3. Triples of flags

We will describe the moduli space of configurations of 'generic' n-tuples of flags.

Definition 3.1. Let $(X_1, \ldots, X_k)$ be a k-tuple of flags. We fix a matrix representative $X_i = (x^1_i \cdots x^n_i) \in \mathrm{GL}(n,\mathbb{C})$ for each i. A k-tuple of flags $(X_1, \ldots, X_k)$ is called *generic* if

$$(2) \qquad \det(x^1_1 \cdots x^{i_1}_1 x^1_2 \cdots x^{i_2}_2 \cdots x^1_k \cdots x^{i_k}_k) \neq 0$$

for any $0 \leq i_1, \ldots, i_k \leq n$ satisfying $i_1 + i_2 + \cdots + i_k = n$.

We remark that the genericity does not depend on the choice of the matrix representatives. Moreover the determinant in (2) is a well-defined complex number if $X_1, \ldots, X_k \in \mathcal{AF}_n$ (recall (1)). We denote the determinant by $\det(X_1^{i_1} X_2^{i_2} \ldots X_k^{i_k})$ for a k-tuple of affine flags. In this note, we only consider generic triples or quadruples of flags.

Let (X, Y, Z) be a generic triple of $\mathcal{F}_n$. We fix lifts of X, Y, Z to $\mathcal{AF}_n$. For a triple (i, j, k) of integers satisfying $0 \leq i, j, k \leq n$ and $i + j + k = n$, we denote $\Delta^{i,j,k} = \det(X^i Y^j Z^k)$. Consider a big triangle subdivided into n^2 small triangles as in Figure 1. Such a triple

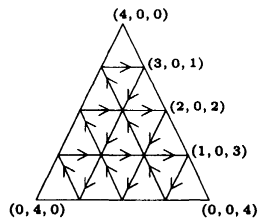

FIGURE 1. A subdivision into n^2 triangles ($n = 4$).

(i, j, k) corresponds to a vertex of the subdivided triangle. For an interior vertex (i, j, k) (in other words $1 \leq i, j, k \leq n-1$ and $i+j+k=n$), the *triple ratio* is defined by

$$T_{i,j,k}(X,Y,Z) = \frac{\Delta^{i+1,j,k-1}\Delta^{i-1,j+1,k}\Delta^{i,j-1,k+1}}{\Delta^{i+1,j-1,k}\Delta^{i,j+1,k-1}\Delta^{i-1,j,k+1}}.$$

We show a graphical representation of $T_{i,j,k}(X,Y,Z)$ in Figure 2. Each factor of the numerator (resp. denominator) corresponds to a vertex colored by black (resp. white) in Figure 2. We remark that $T_{i,j,k}(X,Y,Z)$ does not depend on the choice of the matrix representatives. By definition, we have

$$T_{i,j,k}(X,Y,Z) = T_{j,k,i}(Y,Z,X) = T_{k,i,j}(Z,X,Y), \tag{3}$$

$$T_{i,j,k}(X,Y,Z) = \frac{1}{T_{i,k,j}(X,Z,Y)}, \tag{4}$$

$$T_{i,j,k}(X,Y,Z) = T_{i,j,k}(AX,AY,AZ), \tag{5}$$

for any generic triple $X, Y, Z \in \mathcal{F}_n$ and $A \in \mathrm{PGL}(n, \mathbb{C})$.

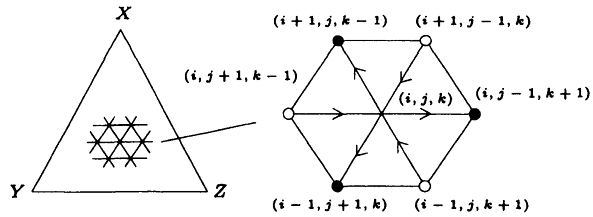

FIGURE 2. The black (resp. white) vertices correspond to the factors of the numerator (resp. denominator) of the triple ratio.

If we denote

$$\mathrm{Conf}_k(\mathcal{F}_n) = \mathrm{GL}(n, \mathbb{C})\backslash\{(X_1, \ldots, X_k) \mid \text{generic } k\text{-tuple of } \mathcal{F}_n\},$$

$T_{i,j,k}$ are functions on $\mathrm{Conf}_3(\mathcal{F}_n)$ by (5). Moreover, we have the following theorem.

Theorem 3.2 (Fock-Goncharov). *A point of* $\mathrm{Conf}_3(\mathcal{F}_n)$ *is completely determined by the* $\frac{(n-1)(n-2)}{2}$ *triple ratios. In particular,* $\mathrm{Conf}_3(\mathcal{F}_n) \cong (\mathbb{C}^*)^{(n-1)(n-2)/2}$.

This theorem follows from the existence of the following normal form of a generic triple of flags.

Lemma 3.3. *Let (X, Y, Z) be a generic triple of $\mathcal{F}_n$. Then there exists a unique $A \in \mathrm{GL}(n, \mathbb{C})$ and upper triangular matrices B_1, B_2, B_3 up to scalar multiplication such that*

$$AXB_1 = \begin{pmatrix} 1 & & O \\ & \ddots & \\ O & & 1 \end{pmatrix}, \quad AYB_2 = \begin{pmatrix} O & & 1 \\ & \iddots & \\ 1 & & O \end{pmatrix}, \quad AZB_3 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & & O \\ \vdots & & \ddots & \\ 1 & * & & 1 \end{pmatrix}.$$

This means that the lower triangular part of AZB_3 gives a set of complete invariants for configurations of generic triples of flags. We will later give a brief sketch of the proof of Lemma 3.3, which gives an explicit construction of the matrix A. Combining with the following lemma, we complete the proof of Theorem 3.2.

Lemma 3.4. *Each entry of the lower triangular part of AZB_3 in Lemma 3.3 is written by a Laurent polynomial of the triple ratios $T_{i,j,k}(X, Y, Z)$.*

This can be proved by induction. Probably the Laurent polynomial might be a polynomial. Here are some examples for small n.

Example 3.5. When $n = 3$, let $T = T_{1,1,1}(X, Y, Z)$, then we have the following normal form:

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & T+1 & 1 \end{pmatrix}. \tag{6}$$

In fact, we have

$$T_{1,1,1}(X, Y, Z) = \frac{\det \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) \det \left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{array}\right) \det \left(\begin{array}{c|cc} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & T+1 \end{array}\right)}{\det \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right) \det \left(\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right) \det \left(\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & T+1 \end{array}\right)} = T.$$

When $n = 4$, let $T_{ijk} = T_{i,j,k}(X, Y, Z)$, then we have the following normal form:

$$X = I_4, \quad Y = C_4, \quad Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & T_{121}+1 & 1 & 0 \\ 1 & (T_{211}+1)T_{121}+1 & (T_{112}+1)T_{211}+1 & 1 \end{pmatrix},$$

where I_4 is the identity matrix and C_4 is the counter diagonal matrix with all counter diagonal entries 1.

Sketch of proof of Lemma 3.3. First we show that for a generic triple of flags (X, Y, Z), there exists a unique matrix $A \in \mathrm{GL}(n, \mathbb{C})$ such that

$$AX = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ O & & * \end{pmatrix}, \quad AY = \begin{pmatrix} O & & * \\ & \iddots & \vdots \\ * & \cdots & * \end{pmatrix}, \quad AZ = \begin{pmatrix} 1 & \\ \vdots & * \\ 1 & \end{pmatrix}.$$

We need to find a matrix $A = (a_{ij})$ satisfying

$$\begin{aligned} a_{i1}x_1^j + a_{i2}x_2^j + \cdots + a_{in}x_n^j &= 0, \quad (j < i) \\ a_{i1}y_1^j + a_{i2}y_2^j + \cdots + a_{in}y_n^j &= 0, \quad (j < n-i+1) \\ a_{i1}z_1^1 + a_{i2}z_2^1 + \cdots + a_{in}z_n^1 &= 1. \end{aligned}$$

This system of linear equations is equivalent to the matrix equation

$$\begin{pmatrix} x_1^1 & \dots & x_n^1 \\ \vdots & & \vdots \\ x_1^{i-1} & \dots & x_n^{i-1} \\ y_1^1 & \dots & y_n^1 \\ \vdots & & \vdots \\ y_1^{n-i} & \dots & y_n^{n-i} \\ z_1^1 & \dots & z_n^1 \end{pmatrix} \begin{pmatrix} a_{i1} \\ \vdots \\ a_{in} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad i = 1, \dots, n. \tag{7}$$

Since (X, Y, Z) is generic, we can show that the $n \times n$-matrix in the above equation is invertible. So we have a unique solution $A \in M(n, \mathbb{C})$. We can show that $\det A \neq 0$ by genericity.

Multiplying an upper triangular matrix from the right, we can eliminate the upper right (or lower right) triangular part of a matrix. This completes the proof of Lemma 3.3. □

From the proof of Lemma 3.3, we have the following proposition.

Proposition 3.6. (1) *Let $X, Y \in \mathcal{F}_n$ and $z \in \mathbb{C}P^{n-1}$ be a generic triple, and $X', Y' \in \mathcal{F}_n$ and $z' \in \mathbb{C}P^{n-1}$ another generic triple. Then there exists a unique matrix $A \in \mathrm{PGL}(n, \mathbb{C})$ such that*

$$AX = X', \quad AY = Y', \quad Az = z'.$$

(2) *Let $X, Y \in \mathcal{F}_n$ and $z \in \mathbb{C}P^{n-1}$ be a generic triple and $T_{i,j,k}$ be nonzero complex numbers for i, j, k satisfying $1 \leq i, j, k \leq n-1$ and $i + j + k = n$. Then there exists a unique flag Z such that $Z^1 = z$ and $T_{i,j,k}(X, Y, Z) = T_{i,j,k}$.*

4. Quadruples of flags

Let X, Z be affine flags and y, t be non-zero n-dimensional vectors. We say that (X, Z, y) is generic if $\det(X^k Z^{n-k-1} y) \neq 0$ for $k = 0, \dots, n-1$. If (X, Z, y) and (X, Z, t) are generic, we define the *edge function* for $i = 1, \dots, n-1$ by

$$\delta_i(X, y, Z, t) = \frac{\det(X^{i-1} y Z^{n-i}) \det(X^i Z^{n-i-1} t)}{\det(X^i y Z^{n-i-1}) \det(X^{i-1} Z^{n-i} t)}. \tag{8}$$

We show a graphical representation of $\delta_i(X, y, Z, t)$ in Figure 3. We can easily check that $\delta_i(X, y, Z, t)$ is well-defined for $X, Z \in \mathcal{F}_n$ and $y, t \in \mathbb{C}P^{n-1}$. By definition, we have

$$\delta_i(X, y, Z, t) = \frac{1}{\delta_i(X, t, Z, y)}, \tag{9}$$

$$\delta_i(X, y, Z, t) = \delta_{n-i}(Z, t, X, y), \tag{10}$$

$$\delta_i(X, y, Z, t) = \delta_i(AX, Ay, AZ, At), \tag{11}$$

for any $A \in \mathrm{PGL}(n, \mathbb{C})$. For a generic quadruple $X, Y, Z, T \in \mathcal{F}_n$, we simply denote

$$\delta_i(X, Y, Z, T) = \delta_i(X, Y^1, Z, T^1).$$

By (11), $\delta_i(X,Y,Z,T)$ are functions on $\mathrm{Conf}_4(\mathcal{F}_n)$. For a generic quadruple (X,Y,Z,T), we have $\frac{(n-1)(n-2)}{2}$ triple ratios for each (X,Y,Z) and (X,Z,T) and $(n-1)$ edge functions. These $(n-1)(n-2)+(n-1)=(n-1)^2$ invariants completely determine a point of $\mathrm{Conf}_4(\mathcal{F}_n)$. First we show the following proposition.

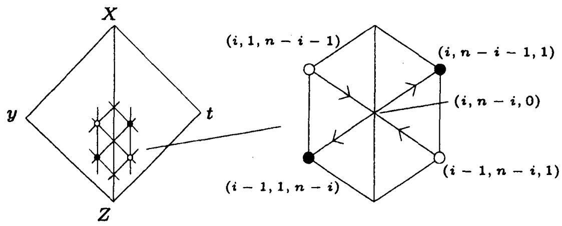

FIGURE 3. The black (resp. white) vertices correspond to the factors of the numerator (resp. denominator) of the edge function.

Proposition 4.1. *Let $X, Z \in \mathcal{F}_n$ and $y \in \mathbb{C}P^{n-1}$ such that the triple (X,Z,y) is generic. For any $d_1,\dots,d_{n-1} \in \mathbb{C}^*$, there exists a unique $t \in \mathbb{C}P^{n-1}$ such that*

$$\delta_i(X,y,Z,t) = d_i, \quad i = 1,\dots,n-1.$$

In fact, by (11) and Proposition 3.6 (1), we can assume that the triple (X,Z,y) is of the form

$$X = \begin{pmatrix} 1 & & O \\ & \ddots & \\ O & & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} O & & 1 \\ & \iddots & \\ 1 & & O \end{pmatrix}, \quad y = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{12}$$

We denote the identity matrix of size i by I_i and the counter diagonal matrix of size i with counter diagonal entries 1 by C_i. We let $t = [t_1 : \cdots : t_n] \in \mathbb{C}P^{n-1}$, then we have

$$\delta_i(X,y,Z,t) = \frac{\left|\begin{array}{c|c|c} I_{i-1} & \vdots & O \\ \hline O & 1 & O \\ \hline O & \vdots & C_{n-i} \end{array}\right| \cdot \left|\begin{array}{c|c|c} I_i & O & \vdots \\ \hline O & O & t_{i+1} \\ \hline O & C_{n-i-1} & \end{array}\right|}{\left|\begin{array}{c|c|c} I_i & \vdots & O \\ \hline O & 1 & O \\ \hline O & \vdots & C_{n-i-1} \end{array}\right| \cdot \left|\begin{array}{c|c|c} I_{i-1} & O & \vdots \\ \hline O & O & t_i \\ \hline O & C_{n-i} & \vdots \end{array}\right|} = -\frac{t_{i+1}}{t_i}.$$

Thus t is uniquely determined by $d_1,\dots,d_{n-1}$.

Corollary 4.2. *A point (X,Y,Z,T) of $\mathrm{Conf}_4(\mathcal{F}_n)$ is uniquely determined by $T_{i,j,k}(X,Y,Z)$, $T_{i,j,k}(X,Z,T)$ and $\delta_i(X,Y,Z,T)$.*

In fact, (X,Y,Z) is uniquely determined by $T_{i,j,k}(X,Y,Z)$ by Theorem 3.2. Then $T^1 \in \mathbb{C}P^{n-1}$ is determined by $\delta_i(X,Y,Z,T)$ by Proposition 4.1, and then $T \in \mathcal{F}_n$ is determined by $T_{i,j,k}(X,Z,T)$ by Proposition 3.6 (2). We remark that the quadruple (X,Y,Z,T) determined by arbitrary given $T_{i,j,k}(X,Y,Z)$, $T_{i,j,k}(X,Z,T)$ and $\delta_i(X,Y,Z,T)$ might not be generic but the triples (X,Y,Z) and (X,Z,T) are generic. (If we further assume 'positivity' of triple ratios and edge functions, then the quadruple must be generic.) By a similar argument, we can show that a configuration of generic k flags is uniquely determined by some triple ratios and edge functions.

Example 4.3. When $n = 2$, we observed in Example 2.1 that $\mathcal{F}_2$ is nothing but $\mathbb{C}P^1$. So we assume that $X, Z \in \mathbb{C}P^1$. In this identification, the normalization (12) corresponds to
$$X = [1:0] = \infty, \quad Z = [0:1] = 0, \quad y = [1:1] = 1.$$
Then we have $\delta_1(\infty, 1, 0, t) = -t$. (See Figure 4.) Thus if we define the cross ratio by
$$[x_0 : x_1 : x_2 : x_3] = \frac{x_3 - x_0}{x_3 - x_1}\frac{x_2 - x_1}{x_2 - x_0},$$
we have $\delta_1(X, y, Z, t) = -[X : Z : y : t]$.

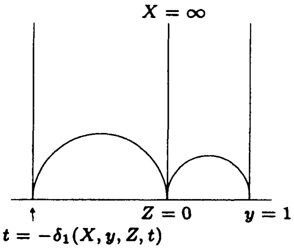

FIGURE 4

5. Fock-Goncharov coordinates

We will use triple ratios and edge functions to give a parametrization of $\mathrm{PGL}(n, \mathbb{C})$-representations of a surface group.

Let S be an orientable surface with at least one puncture. We assume that S admits a hyperbolic metric. An *ideal triangle* is a triangle with the vertices removed. An *ideal triangulation* of S is a system of disjointly embedded arcs on S which decomposes S into ideal triangles $\Delta_1, \dots, \Delta_N$, see Figure 5. (If S is a surface of genus g with p punctures, then $N = 4g - 4 + 2p$.) We denote the universal cover of S by $\widetilde{S}$, which can be identified with the hyperbolic plane $\mathbb{H}^2$. The ideal triangulation of S lifts to an ideal triangulation of $\widetilde{S}$. Each ideal vertex of an ideal triangle of $\widetilde{S}$ defines a point on the ideal boundary $\partial\mathbb{H}^2$. Let $\partial\widetilde{S} \subset \partial\mathbb{H}^2$ be the set of these ideal points. The fundamental group $\pi_1(S)$ acts on the universal cover $\widetilde{S}$ by deck transformations. It also acts on the ideal triangulation of $\widetilde{S}$ and the ideal boundary $\partial\widetilde{S}$.

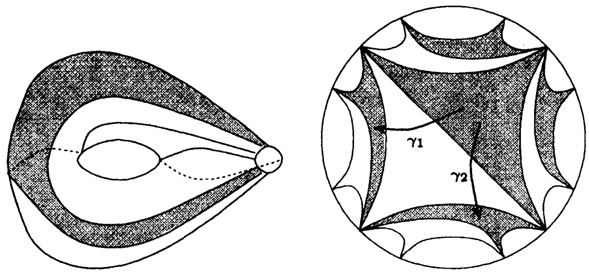

FIGURE 5

Let $\rho : \pi_1(S) \to \mathrm{PGL}(n, \mathbb{C})$ be a representation. A map $f : \partial\widetilde{S} \to \mathcal{F}_n$ is called a *developing map* for ρ if it is ρ-equivariant i.e. it satisfies $f(\gamma x) = \rho(\gamma) f(x)$ for $x \in \partial\widetilde{S}$ and $\gamma \in \pi_1(S)$. The representation ρ is recovered from the developing map as follows. Fix an ideal triangle of $\widetilde{S}$, and denote its ideal vertices by v_1, v_2, v_3. Since f is ρ-equivariant, we have $f(\gamma v_i) = \rho(\gamma) f(v_i)$ for any $\gamma \in \pi_1(S)$ and $i = 1, 2, 3$. By Proposition 3.6 (1), $\rho(\gamma)$ is uniquely determined by these data as an element of $\mathrm{PGL}(n, \mathbb{C})$.

Let Δ be an ideal triangle of the ideal triangulation of S. We take a lift of Δ to $\widetilde{S}$. Then the ideal vertices of the triangle are mapped to a triple of flags by f. If the triple is generic, we can define the triple ratios for Δ. Since f is ρ-equivariant and by (5), the triple ratios do not depend on the choice of the lift. We can similarly define the edge functions for each edge of the ideal triangulation. Thus, if S is a surface of genus g with p punctures, we have $(4g-4+2p)\frac{(n-1)(n-2)}{2}$ triple ratio parameters and $(6g-6+3p)(n-1)$ edge functions. Altogether we have $(n^2-1)(2g-2+p)$ parameters. These parameters completely determine f and hence ρ up to conjugacy. In fact, we can reconstruct f from these parameters. First we choose one ideal triangle in $\widetilde{S}$ and denote the ideal vertices by v_1, v_2, v_3. Then take arbitrary $X_1, X_2 \in \mathcal{F}_n$ and $x_3 \in \mathbb{C}P^{n-1}$. Define $f(v_i) = X_i$ for $i = 1, 2$. By Proposition 3.6 (2), there exists unique $f(v_3) \in \mathcal{F}_n$ such that $f(v_3)^1 = x_3$ and the triple ratios $T_{i,j,k}(f(v_1), f(v_2), f(v_3))$ are the same as the prescribed ones. Let (v_1, v_2, v_4) be the ideal triangle of $\widetilde{S}$ adjacent to (v_1, v_2, v_3). By Proposition 4.1, $f(v_4)^1 \in \mathbb{C}P^{n-1}$ is uniquely determined by the edge functions $\delta_i(f(v_1), f(v_3)^1, f(v_2), f(v_4)^1)$. Again by Proposition 3.6 (2), $f(v_4) \in \mathcal{F}_n$ is determined by the triple ratios $T_{i,j,k}(f(v_1), f(v_2), f(v_4))$. Iterating these steps, $f : \widetilde{S} \to \mathcal{F}_n$ is uniquely determined by these data. If we change the first choice of $X_1, X_2 \in \mathcal{F}_n$ and $x_3 \in \mathbb{C}P^{n-1}$, then the result differs by a conjugation. The conjugating element is explicitly given by Proposition 3.6 (1). This system of triple ratio and edge function parameters are called *Fock-Goncharov coordinates.*

6. Once-punctured torus case

Let S be a once punctured torus. Fix an ideal triangulation of S as in Figure 5. We take a system of generators γ_1, γ_2 of $\pi_1(S)$ as in the right of Figure 5. We give the explicit representation $\rho : \pi_1(S) \to \mathrm{PGL}(n, \mathbb{C})$ when $n = 3$ parametrized by Fock-Goncharov coordinates.

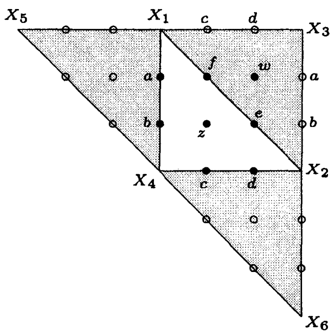

FIGURE 6

Figure 6 shows a part of the universal cover $\widetilde{S}$. We let z, w be the triple ratios for the two ideal triangles and a, b, c, d, e, f be the edge functions for the three edges as indicated in Figure 6. Each X_i in Figure 6 indicates the flag corresponding to the ideal vertex.

First we fix

$$X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad X_4^1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

By (4), we have $z = T_{1,1,1}(X_1, X_4, X_2) = (T_{1,1,1}(X_1, X_2, X_4))^{-1}$. From the normal form (6), we have

$$X_4 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1+1/z & 1 \end{pmatrix}.$$

Next we compute X_5^1. Put $X_5^1 = [s_1 : s_2 : s_3]$. By the definition (8), we have

$$a = \delta_2(X_1, X_5^1, X_4, X_2^1) = \frac{\begin{vmatrix} 1 & s_1 & 1 \\ 0 & s_2 & 1 \\ 0 & s_3 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & s_1 \\ 0 & 1 & s_2 \\ 0 & 0 & s_3 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}} = \frac{s_2 - s_3}{s_3},$$

$$b = \delta_1(X_1, X_5^1, X_4, X_2^1) = \frac{s_1/z - s_2(1+1/z) + s_3}{s_2 - s_3}.$$

Solving these equations, we have $X_5^1 = [s_1 : s_2 : s_3] = [abz + az + a + 1 : a + 1 : 1]$. Similarly we have

$$X_3^1 = [1 : -e : ef], \quad X_6^1 = [cdz : cdz + cz : cdz + cz + c + 1].$$

Next we determine X_3 in $\mathcal{F}_n$. We have

$$X_1 = I_3, \quad X_2 = C_3, \quad X_3 = \begin{pmatrix} 1 & * & * \\ -e & * & * \\ ef & * & * \end{pmatrix},$$

where I_3 and C_3 are defined as in Example 3.5. Since this triple is obtained from the normal form of (X_1, X_2, X_3) by multiplication by a diagonal matrix with diagonal entries $(1, -e, ef)$, we have

$$X_3 = \begin{pmatrix} 1 & 0 & 0 \\ -e & -e & 0 \\ ef & ef(1+w) & ef \end{pmatrix}.$$

The matrix $\rho(\gamma_1)$ maps the triple (X_1^1, X_2, X_3) to (X_5^1, X_4, X_1). Decompose $\rho(\gamma_1)$ into two matrices as

$$(X_2, X_3, X_1^1) \xrightarrow{A} (I_3, C_3, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}) \xleftarrow{B} (X_4, X_1, X_5^1),$$

each of which is calculated explicitly by (7). After some computation, we have

$$\rho(\gamma_1) = \begin{pmatrix} (1 + a(z+1) + abz)efw & f(w+1) + afw(z+1) & 1 \\ (1+a)efw & f(w+1) + afw & 1 \\ efw & f(w+1) & 1 \end{pmatrix}.$$

Similarly, since $\rho(\gamma_2)$ maps (X_1, X_2^1, X_3) to (X_4, X_6^1, X_2), we obtain

$$\rho(\gamma_2) = \begin{pmatrix} cdefwz & cdf(w+1)z & cdz \\ cdefwz & cfz + cdf(w+1)z & cz + cdz \\ cdefwz & cf(z+1) + cdf(w+1)z & 1 + c(z+1) + cdz \end{pmatrix}.$$

We end this note by drawing some pictures of the images of developing maps. If we restrict the coordinates to real numbers, we obtain a $\mathrm{PGL}(3,\mathbb{R})$-representation. A $\mathrm{PGL}(3,\mathbb{R})$ representation preserving a convex set in $\mathbb{R}P^2$ is called a *convex projective representation.* In [4], Fock and Goncharov showed that, when all triple ratios and edge functions are positive, the associated $\mathrm{PGL}(3,\mathbb{R})$-representation is convex projective. We remark that Goldman gave a parametrization of convex projective structures in [7]. Figures 7, 8 and 9 are drawn in local coordinates of $\mathbb{R}P^2$ given by

$$[x : y : z] \mapsto \left(\frac{z-y}{x+z}, \frac{x-y}{x+z}\right).$$

In particular, $X_1^1 = [1 : 0 : 0]$ maps to $(0,1)$, $X_2^1 = [0 : 0 : 1]$ to $(0,0)$ and $X_4^1 = [1 : 1 : 1]$ to $(0,0)$. I only drew triangles developed by the products of $\rho(\gamma_1)$ and $\rho(\gamma_2)$ whose word lengths within 4 by using Sage [13]. I remark that these pictures might miss large triangles in the developed images.

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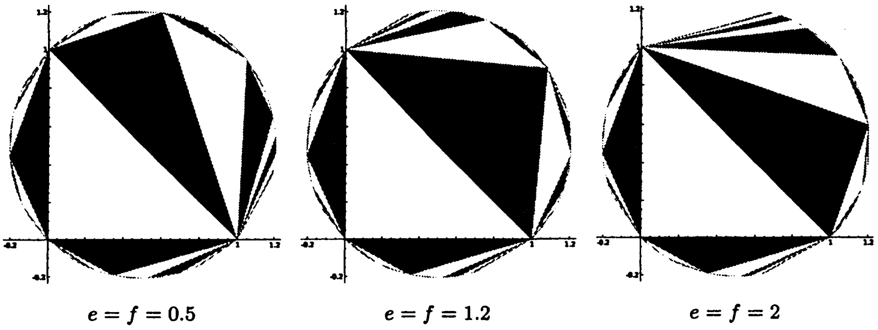

FIGURE 7. $a = b = c = d = 1.2$, $z = w = 1$. (These correspond to Fuchsian representations, so the developed images are in a round disk.)

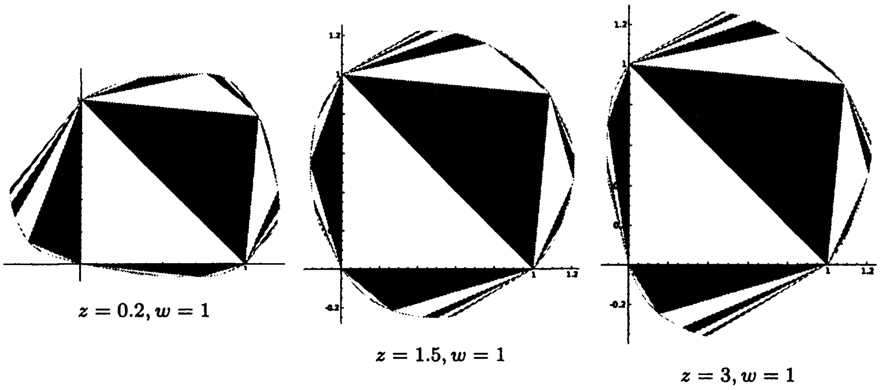

FIGURE 8. $a = b = c = d = e = f = 1.2$.

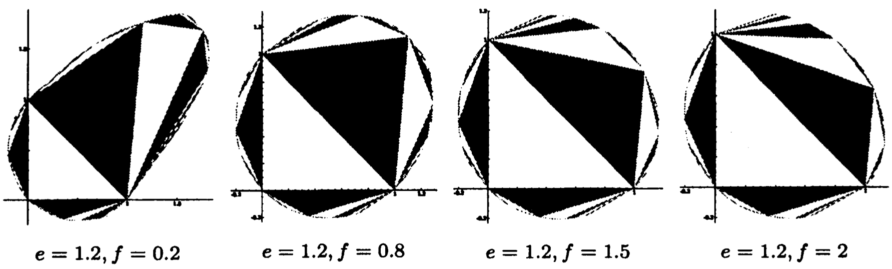

FIGURE 9. $a = b = c = d = 1.2$, $z = w = 1$.