数理解析研究所講究録
第 1836 巻 2013 年 81-87

END INVARIANTS OF HECKOID GROUPS FOR 2-BRIDGE LINKS

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1. Introduction

By extending the concept of a geometrically infinite end of a Kleinian group, Bowditch [4] introduced the notion of the end invariants of a type-preserving $SL(2,\mathbb{C})$-representation of the fundamental group $\pi_1(\boldsymbol{T})$ of the once-punctured torus $\boldsymbol{T}$. Tan, Wong and Zhang [23, 24] extended this notion (with slight modification) to an arbitrary $SL(2,\mathbb{C})$-representation of $\pi_1(\boldsymbol{T})$. In [12], we gave an explicit description of the sets of end invariants of the $SL(2,\mathbb{C})$-characters of the once-punctured torus corresponding to the holonomy representation of hyperbolic 2-bridge link groups. The purpose of this note is to announce the result obtained in [14] which explicitly describes the sets of end invariants of the $SL(2,\mathbb{C})$-characters of the once-punctured torus corresponding to the holonomy representation of Heckoid groups (Theorem 4.1).

2. Bowditch, Tan-Wong-Zhang end invariants

Motivated by the definition of the end of a geometrically infinite end of a Kleinian group, Bowditch [4] introduced the notion of the end invariants of an arbitrary type-preserving $PSL(2,\mathbb{C})$-representation of $\pi_1(\boldsymbol{T})$. Tan, Wong and Zhang [23, 24] extended this notion (with slight modification) to an arbitrary $PSL(2,\mathbb{C})$-representation of $\pi_1(\boldsymbol{T})$. To describe this, let $\mathcal{C}$ be the set of free homotopy classes of essential simple loops on $\boldsymbol{T}$. Then $\mathcal{C}$ is identified with $\hat{\mathbb{Q}}$, the vertex set of the Farey tessellation $\mathcal{D}$, by the following rule $s\mapsto\beta_s$, where β_s is the image of a line in $\mathbb{R}^2-\mathbb{Z}^2$ of slope s in $\boldsymbol{T}=(\mathbb{R}^2-\mathbb{Z}^2)/\mathbb{Z}^2$. The projective lamination space $\mathcal{PL}$ of $\boldsymbol{T}$ is then identified with $\hat{\mathbb{R}}:=\mathbb{R}\cup\{\infty\}$ and contains $\mathcal{C}$ as the dense subset of rational points.

Definition 2.1. Let ρ be a $\mathrm{PSL}(2,\mathbb{C})$-representation of $\pi_1(\boldsymbol{T})$.
(1) An element $X\in\mathcal{PL}$ is an *end invariant* of ρ if there exists a sequence of distinct elements $X_n\in\mathcal{C}$ such that $X_n\to X$ and that $\{|\mathrm{tr}\rho(X_n)|\}_n$ is bounded from above.
(2) $\mathcal{E}(\rho)$ denotes the set of end invariants of ρ.

In the above definition, it should be noted that $|\mathrm{tr}\rho(X_n)|$ is well-defined though $\mathrm{tr}\rho(X_n)$ is defined only up to sign. Note also that the condition that $\{|\mathrm{tr}\rho(X_n)|\}_n$ is bounded from above is equivalent to the condition that the (real) hyperbolic translation lengths of the isometries $\rho(X_n)$ of $\mathbb{H}^3$ are bounded from above. So, if ρ is a faithful discrete type-preserving representation and ν is the end invariant of a geometrically infinite end of the quotient hyperbolic manifold, then ν is an end invariant of ρ in the sense of the above definition.

Tan, Wong and Zhang [23, 24] showed that $\mathcal{E}(\rho)$ is a closed subset of $\mathcal{PL}$ and proved various interesting properties of $\mathcal{E}(\rho)$, including a characterization of those representations ρ with $\mathcal{E}(\rho)=\emptyset$ or $\mathcal{PL}$, generalizing results of Bowditch [4]. They also proposed an

Received December 8, 2012.

interesting conjecture [24, Conjecture 1.8] concerning possible homeomorphism types of $\mathcal{E}(\rho)$. The following is a modified version of the conjecture which Tan [22] informed to the authors.

Conjecture 2.2. Suppose $\mathcal{E}(\rho)$ has at least two accumulation points. Then either $\mathcal{E}(\rho) = \mathcal{PL}$ or a Cantor set of $\mathcal{PL}$.

They constructed a family of representations ρ which have Cantor sets as $\mathcal{E}(\rho)$, and proved the following supporting evidence to the conjecture (see [24, Theorem 1.7]).

Theorem 2.3. *Let* $\rho : \pi_1(\boldsymbol{T}) \to \mathrm{SL}(2,\mathbb{C})$ *be* discrete *in the sense that the set* $\{\mathrm{tr}(\rho(X)) \,|\, X \in \mathcal{C}\}$ *is discrete in* $\mathbb{C}$. *Then if* $\mathcal{E}(\rho)$ *has at least three elements, then* $\mathcal{E}(\rho)$ *is either a Cantor set of* $\mathcal{PL}$ *or all of* $\mathcal{PL}$.

However, the above theorem does not describe the set $\mathcal{E}(\rho)$ explicitly. In [12], we gave an explicit description of the sets of end invariants of the $SL(2,\mathbb{C})$-characters of the once-punctured torus corresponding to the holonomy representation of hyperbolic 2-bridge link groups. In this note, we announce a result obtained in [14] which explicitly describes the sets of end invariants of the $SL(2,\mathbb{C})$-characters of the once-punctured torus corresponding to the holonomy representation of Heckoid groups (Theorem 4.1). These give an infinite family of representations ρ for which $\mathcal{E}(\rho)$ are explicitly described Cantor sets.

3. Heckoid orbifold $S(r;n)$ and Heckoid group $G(r;n)$

For a rational number $r \in \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$, let $K(r)$ be the 2-bridge link of slope r, which is defined as the sum $(S^3, K(r)) = (B^3, t(\infty)) \cup (B^3, t(r))$ of rational tangles of slope ∞ and r. The common boundary $\partial(B^3, t(\infty)) = \partial(B^3, t(r))$ of the rational tangles is identified with the *Conway sphere* $(\boldsymbol{S}^2, \boldsymbol{P}) := (\mathbb{R}^2, \mathbb{Z}^2)/H$, where H is the group of isometries of the Euclidean plane $\mathbb{R}^2$ generated by the π-rotations around the points in the lattice $\mathbb{Z}^2$. Let $\boldsymbol{S}$ be the 4-punctured sphere $\boldsymbol{S}^2 - \boldsymbol{P}$ in the link complement $S^3 - K(r)$. Any essential simple loop in $\boldsymbol{S}$, up to isotopy, is obtained as the image of a line of slope $s \in \hat{\mathbb{Q}}$ in $\mathbb{R}^2 - \mathbb{Z}^2$ by the covering projection onto $\boldsymbol{S}$. The (unoriented) essential simple loop in $\boldsymbol{S}$ so obtained is denoted by α_s. We also denote by α_s the conjugacy class of an element of $\pi_1(\boldsymbol{S})$ represented by (a suitably oriented) α_s. The loops α_∞ and α_r bound disks in $B^3 - t(\infty)$ and $B^3 - t(r)$, respectively. Thus the *link group* $G(K(r)) = \pi_1(S^3 - K(r))$ is obtained as follows:

$$G(K(r)) = \pi_1(S^3 - K(r)) \cong \pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty, \alpha_r \rangle\rangle \cong \pi_1(B^3 - t(\infty))/\langle\langle \alpha_r \rangle\rangle.$$

For each rational number r and an integer $n \geq 2$, the *even Heckoid orbifold of index* n *for the* 2*-bridge link* $K(r)$ is the 3-orbifold $\boldsymbol{S}(r;n)$, such that the underlying space $|\boldsymbol{S}(r;n)|$ is the exterior, $E(K(r)) = S^3 - \mathrm{int}\, N(K(r))$, of $K(r)$, and that the singular set is the lower tunnel of $K(r)$ (i.e., the core tunnel of $(B^3, t(\infty))$ in the sense of [10, p.360]), where the index of the singularity is n (see Figure 1). We call the orbifold fundamental group $\pi_1(\boldsymbol{S}(r;n))$ the *Heckoid group of index* n *for* $K(r)$, and denote it by $G(r;n)$. Since the loop α_r is isotopic to a meridional loop around the lower tunnel, the even Hekoid group $G(r;n) = \pi_1(\boldsymbol{S}(r;n))$ $(n \geq 2)$ is obtained as follows:

$$G(r;n) = \pi_1(\boldsymbol{S}(r;n)) \cong \pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty, \alpha_r^n \rangle\rangle \cong \pi_1(B^3 - t(\infty))/\langle\langle \alpha_r^n \rangle\rangle.$$

The announcement by Agol [1] and the announcement made in the second author's joint work with Akiyoshi, Wada and Yamashita in [2, Section 3 of Preface] suggest that

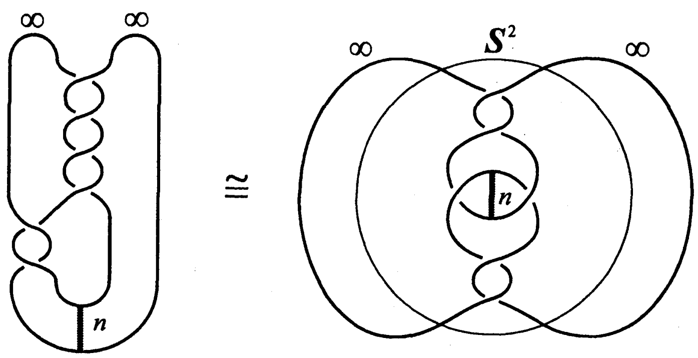

FIGURE 1. The Heckoid orbifold $\boldsymbol{S}(r;n)$. The labels ∞ indicate the parabolic loci. Here $(S^3, K(r)) = (B^3, t(\infty)) \cup (B^3, t(r))$ with $r = [4,2] = 2/9$, where $(B^3, t(r))$ and $(B^3, t(\infty))$, respectively, are the inside and the outside of the bridge sphere $\boldsymbol{S}^2$. The lower tunnel is the core tunnel of $(B^3, t(r))$.

the group $G(r;n)$ makes sense even when n is a half-integer greater than 1. We refer to [14, Definition 3.2] for the definition of the group $G(r;n)$ and the corresponding orbifold $\boldsymbol{S}(r;n)$ when n is a non-integral half-integer greater than 1. Roughly speaking, $\boldsymbol{S}(r;n)$ is defined to be a $\mathbb{Z}/2\mathbb{Z}$-covering of a certain orbifold $\boldsymbol{O}(r;m)$, with $m = 2n$, which is obtained from the quotient of $K(r)$ by the natural $(\mathbb{Z}/2\mathbb{Z})^2$-symmetry (see Figure 2 for the case when $K(r)$ is a knot). We call them the *odd Heckoid orbifold* and the *odd Heckoid group*, respectively, of index n for $K(r)$. A topological description of an odd Heckoid orbifold is given by [14, Proposition 5.3 and Figures 5 and 6].

Remark 3.1. Our terminology is slightly different from that of Riley [20], where $G(r;n)$ is called the Heckoid group of index "m" for $K(r)$ with $m = 2n$. The Heckoid orbifold $\boldsymbol{S}(r;n)$ and the Heckoid group $G(r;n)$ are *even* or *odd* according to whether Riley's index $m = 2n$ is even or odd.

The following theorem was anticipated in [20] and is contained in [1] without proof.

Theorem 3.2. *For a rational number r and an integer or a half-integer $n > 1$, the Heckoid group $G(r;n)$ is isomorphic to a geometrically finite Kleinian group generated by two parabolic transformations.*

A proof of this theorem is given in [14, Section 6] by using the orbifold theorem for pared orbifolds [3, Theorem 8.3.9] (cf. [5, 8]). As noted in [1], the proof is analogous to the arguments in [7, Proof of Theorem 9].

By this theorem and the topological description of odd Heckoid orbifolds ([14, Proposition 5.3]), we obtain the following proposition, which shows a significant difference between odd and even Heckoid groups (see [14, Section 6]).

Proposition 3.3. *Any odd Heckoid group is not a one-relator group.*

4. End invariants of even Heckoid groups

For a rational number r and an integer $n \geq 2$, let $\rho_{r,n}$ be the PSL(2, $\mathbb{C}$)-representation of $\pi_1(\boldsymbol{S})$ obtained as the composition

$$\pi_1(\boldsymbol{S}) \to \pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty, \alpha_r^n \rangle\rangle \cong G(r;n) \to \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2,\mathbb{C}),$$

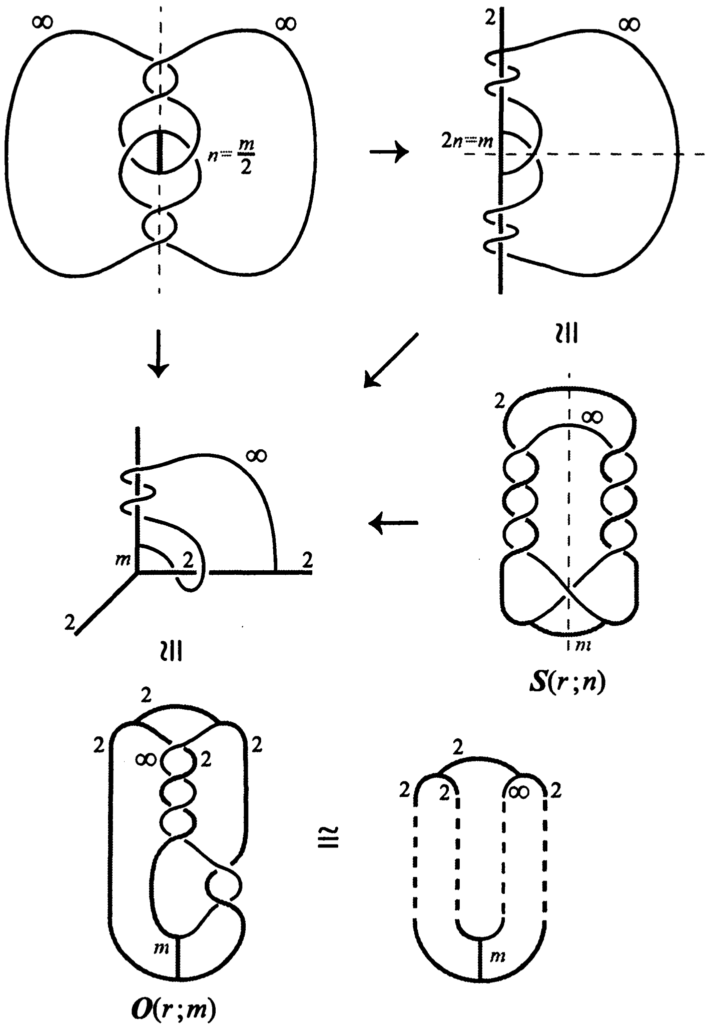

FIGURE 2. The case when $K(r)$ is a knot and $m = 2n > 1$ is an odd integer. Here $r = 2/9 = [4, 2]$. The odd Heckoid orbifold $\boldsymbol{S}(r; n)$ (middle right) is a $\mathbb{Z}/2\mathbb{Z}$-covering of $\boldsymbol{O}(r; m)$ (lower left). The upper left figure is not an orbifold, but is a hyperbolic cone manifold. The odd Heckoid orbifold $\boldsymbol{S}(r; n)$ is the quotient of the cone manifold by the π-rotation around the axis containing the singular set.

where the last homomorphism is the holonomy representation of the pared hyperbolic orbifold $\boldsymbol{S}(r; n)$.

Now, let $\boldsymbol{O}$ be the orbifold $(\mathbb{R}^2 - \mathbb{Z}^2)/\hat{H}$ where $\hat{H}$ is the group generated by π-rotations around the points in $(\frac{1}{2}\mathbb{Z})^2$. Note that $\boldsymbol{O}$ is the orbifold with underlying space a once-punctured sphere and with three cone points of cone angle π. The surfaces $\boldsymbol{T}$ and $\boldsymbol{S}$, respectively, are $\mathbb{Z}/2\mathbb{Z}$-covering and $(\mathbb{Z}/2\mathbb{Z})^2$-covering of $\boldsymbol{O}$, and hence their fundamental groups are identified with subgroups of the orbifold fundamental group $\pi_1(\boldsymbol{O})$ of indices 2 and 4, respectively. The PSL$(2, \mathbb{C})$-representation $\rho_{r,n}$ of $\pi_1(\boldsymbol{S})$ extends, in a unique way, to that of $\pi_1(\boldsymbol{O})$ (see [2, Proposition 2.2]), and so we obtain, in a unique way, a

PSL(2, $\mathbb{C}$)-representation of $\pi_1(\boldsymbol{T})$ by restriction. We continue to denote it by $\rho_{r,n}$. The following theorem, which determines the set $\mathcal{E}(\rho_{r,n})$, is obtained by [16].

Theorem 4.1. *For a non-integral rational number r and an integer $n \geq 2$, the set $\mathcal{E}(\rho_{r,n})$ of end invariants of $\rho_{r,n}$ is equal to the limit set $\Lambda(\Gamma(r;n))$ of the group $\Gamma(r;n)$.*

5. Simple loops on bridge spheres of Heckoid orbifolds

Let $\mathcal{D}$ be the *Farey tessellation*, that is, the tessellation of the upper half space $\mathbb{H}^2$ by ideal triangles which are obtained from the ideal triangle with the ideal vertices $0, 1, \infty \in \hat{\mathbb{Q}}$ by repeated reflection in the edges. Then $\hat{\mathbb{Q}}$ is identified with the set of the ideal vertices of $\mathcal{D}$. For each $r \in \hat{\mathbb{Q}}$, let Γ_r be the group of automorphisms of $\mathcal{D}$ generated by reflections in the edges of $\mathcal{D}$ with an endpoint r. It should be noted that Γ_r is isomorphic to the infinite dihedral group and that the region bounded by two adjacent edges of $\mathcal{D}$ with an endpoint r is a fundamental domain for the action of Γ_r on $\mathbb{H}^2$. For an integer m, let $C_r(m)$ be the group of automorphisms of $\mathcal{D}$ generated by the parabolic transformation, centered on the vertex r, by m units in the clockwise direction.

For r a rational number and n an integer or a half-integer greater than 1, let $\Gamma(r;n)$ be the group generated by Γ_∞ and $C_r(2n)$. Suppose that r is not an integer. Then $\Gamma(r;n)$ is the free product $\Gamma_\infty * C_r(2n)$ having a fundamental domain, R, shown in Figure 3. Here, R is obtained as the intersection of fundamental domains for Γ_∞ and $C_r(2n)$, and so R is bounded by the following two pairs of Farey edges:

(1) the pair of adjacent Farey edges with an endpoint ∞ which cuts off a region in $\bar{\mathbb{H}}^2$ containing r, and
(2) a pair of Farey edges with an endpoint r which cuts off a region in $\bar{\mathbb{H}}^2$ containing ∞, such that one edge is the image of the other by a generator of $C_r(2n)$.

Let $\bar{I}(n;r)$ be the union of two closed intervals in $\partial\mathbb{H}^2 = \hat{\mathbb{R}}$ obtained as the intersection of the closure of R with $\partial\mathbb{H}^2$. Note that there is a pair $\{r_1, r_2\}$ of boundary points of $\bar{I}(n;r)$ such that r_2 is the image of r_1 by a generator of $C_r(2n)$. Set $I(n;r) = \bar{I}(n;r) - \{r_i\}$ with $i = 1$ or 2. Note that $I(n;r)$ is the disjoint union of a closed interval and a half-open interval, except for the special case when $r \equiv \pm 1/p \pmod{\mathbb{Z}}$.

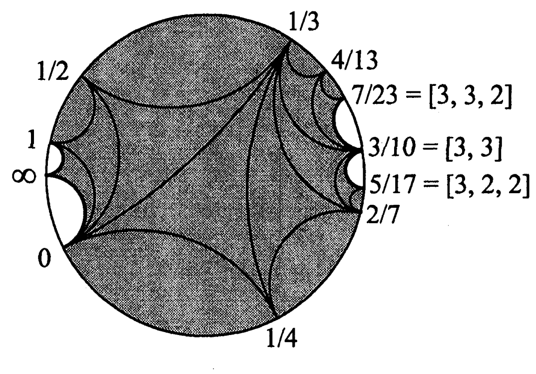

FIGURE 3. A fundamental domain of $\Gamma(r;n)$ in the Farey tessellation (the shaded domain) for $r = 3/10 = \cfrac{1}{3+\cfrac{1}{3}} =: [3,3]$ and $n = 2$

The following theorem proved in [14] is the starting point of all the results which we announce in this note.

Theorem 5.1. *Suppose that r is a non-integral rational number and that n is an integer or a half integer greater than* 1. *Then, for any* $s \in \hat{\mathbb{Q}}$, *there is a unique rational number* $s_0 \in I(r;n) \cup \{\infty, r\}$ *such that s is contained in the* $\Gamma(r;n)$*-orbit of* s_0. *Moreover* α_s *is homotopic to* α_{s_0} *in* $\boldsymbol{S}(r;n)$. *In particular, if* $s_0 = \infty$, *then* α_s *is null-homotopic in* $\boldsymbol{S}(r;n)$.

Theorem 5.2 is proved in [14], and Theorems 5.3 and 5.4 will be proved in [15].

Theorem 5.2. *Suppose that r is a non-integral rational number and that n is an integer with* $n \geq 2$. *Then the loop* α_s *is null-homotopic in* $\boldsymbol{S}(r;n)$ *if and only if s belongs to the* $\Gamma(r;n)$*-orbit of* ∞. *In other words, if* $s \in I(r;n) \cup \{r\}$, *then* α_s *is not null-homotopic in* $\boldsymbol{S}(r;n)$.

Theorem 5.3. *Suppose that r is a non-integral rational number and that n is an integer with* $n \geq 2$. *For two rational numbers s and* s', *the simple loops* α_s *and* $\alpha_{s'}$ *are homotopic in* $\boldsymbol{S}(r;n)$ *if and only if s and* s' *belong to the same* $\Gamma(r;n)$*-orbit. In other words, for distinct* $s, s' \in I(r;n) \cup \{\infty, r\}$, *the simple loops* α_s *and* $\alpha_{s'}$ *are not homotopic in* $\boldsymbol{S}(r;n)$.

Theorem 5.4. *Suppose that r is a non-integral rational number and that n is an integer with* $n \geq 2$. *Then the following hold.*

(1) *The loop* α_s *is peripheral in* $\boldsymbol{S}(r;n)$ *if and only if s belongs to the* $\Gamma(r;n)$*-orbit of* ∞.
(2) *The loop* α_s *is torsion in* $\boldsymbol{S}(r;n)$ *if and only if s belongs to the* $\Gamma(r;n)$*-orbit of* ∞ *or r*.

In other words, there is no rational number $s \in I(r;n)$ *for which the simple loop* α_s *is peripheral or torsion in* $\boldsymbol{S}(r;n)$.

In the above theorem, we say that α_s is *peripheral* or *torsion* if the conjugacy class α_s is represented by a (possibly trivial) parabolic or elliptic transformation, respectively, when we identify $G(r;n)$ with a Kleinian group generated by two parabolic transformations.

These theorems are proved by using the small cancellation theory [17]. Please see [13] for basic ideas of the proof. Theorem 4.1 is proved by using these theorems, Bowditch's results [4] and the discreteness of marked length spectrum of geometrically finite hyperbolic 3-manifolds, as in [12, Section 8].

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