数理解析研究所講究録
第 1836 巻 2013 年 94-103

GEOMETRIC IDENTITIES

GREG MCSHANE

1. Introduction

This text is a complement to the talk I gave at Hakone in May 2012. I would like to thank the organizers for invitation to participate in a workshop in such an idyllic setting. I should also like to thank Martin Bridgeman, Dany Caligari and Hidetoshi Masai for many useful discussions. Finally, we thank Sadayoshi Kojima for all his help and hospitality whilst visiting Tokyto Institute of Technology.

1.1. **History.** Roughly speaking a geometric identity for a Fuchsian group, or more generally a Kleinian group, expresses some fundamental quantity as a series whose terms depend on the values of the lengths of the closed geodesics in the quotient space.

Let us begin by giving a very brief chronology of the development of these identities. In the early 90s Ara Basmajian produced an identity which, in the simplest case, gives the length of a totally geodesic boundary component of a hyperbolic surface as a sum over lengths of orthogeodesics. At about the same time the author gave an identity which calculated the area of a cusp region as a sum over the length of the boundaries of embedded pairs of pants. These identities found applications and have been extended by Bowditch, Sakuma et al and Tan, Wong, Zhang to a variety of different settings. Around 2008 Bridgeman discovered an identity which gives the area of a surface as a sum over lengths of orthogeodesics. With Kahn he went on to extend this to obtain the volume of higher dimensional manifolds with totally geodesic boundary as a series. Calegari gave a related but different construction and with Masai the author shows that both constructions give the same identity whatever the dimension. Then, in 2010, Luo and Tan discovered an identity which gives the area of a surface as a sum over the length of the boundaries of embedded pairs of pants.

1.2. **The underlying idea.** All of the identities have the following in common: they are proved using hyperbolic geometry to decompose some set X associated to the surface into countably many pieces and then calculating the volume (or area or length depending on the dimension) of the pieces. To obtain Basmajian's and McShane's identities the set X is the totally geodesic boundary, whilst for Bridgeman-Kahn,Calegari and Luo-Tan X is the unit tangent bundle.

In this manuscript we will discuss in detail the constructions of Bridgeman-Kahn and Calegari.

1.3. **Bridgeman-Kahn and Calegari.** Bridgeman-Kahn-Calegari formulae give the volume of M^n, a compact hyperbolic n-manifold with totally geodesic boundary, in terms of the orthospectrum of the manifold. Bridgeman-Kahn's formula is:

$$\mathrm{vol}_{n-1}(\mathbb{S}^{n-1}) \times \mathrm{vol}_n(M^n) = \sum_{\alpha*} \mathrm{vol}(\mathcal{B}_{\ell(\alpha*)})$$

Received December 30, 2012.

where the sum is over all orthogeodesics $\alpha*$ and $\mathcal{B}_{\ell(\alpha*)}$ is a certain subset of the unit tangent bundle of the convex hull of a pair of disjoint totally geodesic n-1 dimensional hyperplanes in $\mathbb{H}^n$. Calegari's formula has the same form but the set $\mathcal{B}_{\ell(\alpha*)}$ is replaced by a quite different set $\mathcal{C}_{\ell(\alpha*)}$. Both methods are thus based on decomposing the unit tangent bundle into countably many pieces, each of which is naturally associated to a unique orthogeodesic.

In two dimensions the volume of each piece turns out to be the Rogers' dilogarithm of a simple function of the ortholength [3], [6]. This case is of particular interest since the deformation theory of convex surfaces yields functional relations for the dilogarithm. However, as one sees from the formula below, in three dimensions the volume of each piece can be written in terms of the ortholength and its exponential. The deformation theory of hyperbolic 3 manifolds which have totally geodesic boundary is trivial and no functional relations are to be expected.

In dimensions greater than 3 it is possible to give an explicit closed formula for the term in the sum although it seems that (see [9]) in even dimensions one should use Bridgeman-Kahn's construction whilst in odd dimensions Calegari's construction is more suitable.

1.4. **Explicit formulae.** The Basmajian identity for a compact hyperbolic 3-manifold M with totally geodesic boundary ∂M, one has

$$-\chi(\partial M) = \sum_{\alpha^*} \frac{4}{e^{\ell(\alpha^*)} - 1}$$

where the sum is over all orthogeodesics α^*. Compare this with Bridgeman-Kahn

$$2\mathrm{vol}_3(M) = \sum_{\alpha^*} \frac{\ell(\alpha^*) + 1}{e^{2\ell(\alpha^*)} - 1}.$$

Thus both the volume of the 3-manifold and the area of the boundary are determined by the orthospectrum. Moreover, these quantities are written as series over the orthospectra and the terms are expressed using just usual functions.

For completeness we give McShane's identity for the punctured torus

$$\sum_{\alpha} \frac{2}{1 + e^{\ell(\alpha)}} = 1$$

where the sum is over all closed simple geodesics.

1.5. **Statement of results.** In this manuscript we give a new proof of:

Theorem 1.1. *$\mathcal{B}_l$ and $\mathcal{C}_l$ have the same volume when $n = 2$.*

This result was proven by Calegari in [7] by computing the volume for $\mathcal{C}_l$ as a function of l and comparing with the analogous expression obtained by Bridgeman for $\mathcal{B}_l$. Here we give a different proof which is entirely geometric being based on the non existence of geodesic bigons in non positive curvature. As such our proof should be valid in pinched strictly negative curvature once the definition of $\mathcal{B}_l$ and $\mathcal{C}_l$ have been suitably modified (the reader is left to check the details). In [9], we prove that the two volumes are the same whatever the dimension using another quite different technique.

Our proof requires an analysis of the geometry of a class of surfaces called crowns. Recall that a crown (see [8]) is a complete convex hyperbolic surface of finite area homeomorphic to an annulus. In passing we determine the orthospectrum of a crown (Theorem 5.1 and Lemma 5.2) and use this to give another proof of a result of Lewin (Corollary 3).

2. Orthogeodesics

Informally, the orthospectrum of the surface S is the set of lengths of common perpendiculars between (not necessarily distinct) boundary components.

2.1. **Orthospectrum.**

We define the orthospectrum in two different settings: for a generalized ideal polygon in $\mathbb{H}$ and for a general surface with geodesic boundary.

2.1.1. *Orthospectrum of a generalized ideal polygon.* Recall that an ideal polygon is the convex hull of a finite set of points $X = \{x_1, x_2 \dots x_n\} \subset \partial\mathbb{H}$ More generally let, $X \subset \partial\mathbb{H}$ be a closed, nowhere dense subset and note that the complement X^c is an open subset consisting of countably many intervals. The convex hull $C(X)$ of X is a closed convex subset bounded by countably many geodesics α_k, one for each interval in X^c. We refer to $C(X)$ as a *generalized ideal polygon*, the α_k are the *sides* of $C(X)$ and we say that two sides α_j, α_k are *adjacent* if they are asymptotic.

The *orthospectrum of* $C(X)$ is the collection of distances $d_{\mathbb{H}}(\alpha_j, \alpha_k)$ where α_j, α_k are distinct, non adjacent sides. Note that $d_{\mathbb{H}}(\alpha_j, \alpha_k)$ is realised by the length of the unique common perpendicular between α_j, α_k which we refer to as the *orthogeodesic associated to this pair of sides.*

The numbers $d_{\mathbb{H}}(\alpha_j, \alpha_k)$ can be determined explicitly as functions of the x_i as follows. Suppose, without loss of generality, that α_n has endpoints x_n, x_1 and that if $k \neq n$ then α_k has endpoints x_k, x_{k+1}. Then (see Beardon [2] for details)

$$\tanh^2\left(\frac{1}{2}d_{\mathbb{H}}(\alpha_j, \alpha_k)\right) = \frac{(x_j - x_k)}{(x_{j+1} - x_k)}\frac{(x_{j+1} - x_{k+1})}{(x_j - x_{k+1})}.$$

In [3] Bridgeman calculates the orthospectrum of a regular ideal n-gon. If $n \geq 5$ is an odd integer the orthospectrum consists of the numbers l_m, $m = 2 \dots (n-1)/2$ counted n times, where l_m is defined by

$$\cosh\left(\frac{l_m}{2}\right) = \frac{\sin(\frac{m\pi}{n})}{\sin(\frac{\pi}{n})}.$$

The case $n = 3$ corresponds to an ideal triangle which has exactly 3 pairwise adjacent sides so that the orthospectrum is empty.

2.1.2. *Orthospectrum of a convex surface.* We now consider a not necessarily compact hyperbolic surface S of finite volume with non-empty geodesic boundary ∂S. Our surface S is obtained as the quotient of a convex subset $C(X) \subset \mathbb{H}$ by a group of orientation preserving isometries Γ. For example, the limit set Λ of Γ is a non empty Γ-invariant closed, nowhere dense subset of $\partial\mathbb{H}$ and in this case S can be identified with the quotient of the convex hull $C(\Gamma)$ of the limit set by Γ. In fact any non empty Γ-invariant closed subset of $\partial\mathbb{H}$ contains Λ the union of Λ and the Γ-orbit of some finite subset of points *not* in Λ.

The set of orthogeodesics of the surface is just the set of Γ-orbits of orthogeodesics of $C(X)$ and the orthospectrum is the corresponding collection of lengths. As an example consider the orbifold S obtained as a quotient of the regular n-gon, $n \geq 5$ odd, by the group of rotations contained in its group of symmetries. From Bridgeman's work cited in the previous paragraph one sees that the orthospectrum of S is the set l_m, $m = 2 \dots (n-1)/2$ with each number now counted just once.

2.1.3. *Enumerating the orthospectrum.* The orthospectrum of a finite volume surface S can be computed algorithmically. For example, if the surface has a single totally geodesic boundary component and $H < \Gamma$ is a subgroup generated by a simple loop around the boundary, β say, then it suffices to enumerate the cosets of H in Γ. This can be done efficiently using a finite state automaton (see [10]) for details. If g_kH, $g_k \in \Gamma$ be a complete repetion free list of cosets then orthospectrum is computed using the cross ratios of the endpoints of the axes of β and $g_k\beta g_k^{-1}$.

2.1.4. *Iso orthospectral surfaces.* The spectrum of lengths of closed geodesics of a hyperbolic surfaces behaves quite subtly under taking finite covers see for example [12]. The problem of finding pairs of non isometric isospectral surfaces was solved (see Buser [5] for background), in partcular Sunada [13] gave a construction based on pairs of almost conjugate subgroups.

The behavior of the orthospectrum is much simpler:

Lemma 2.1. *Let $X \to Y$ be a n-fold cover then the orthospectrum of X is just that of Y but with all the multiplicties multiplied by n.*

Proof: Since X covers Y they have the same universal cover $U \subset \mathbb{H}$ and there are groups of orientation preserving isometries Γ_X and Γ_Y such that

$$X = U/\Gamma_X, Y = U/\Gamma_Y.$$

Since X is an n-fold cover of Y we have

$$\Gamma_Y = \sqcup_{k=1}^{n} \Gamma_X g_k$$

for any choice of coset representatives g_k. It is easy to check that if $\alpha*$ is a common perpendicular to sides of U then its Γ_Y-orbit decomposes into exactly n of the Γ_X-orbits and the lemma follows.

□

Corollary 1. There are surfaces X, X', 2-fold covers of a pair of pants Y which are not isometric but have the same orthospectrum.

Proof: Let Y be a pair of pants with boundary geodesics of lengths $1, 1, 2$ There is a 2-fold cover X of Y with boundary lengths $2, 2, 2, 2$ and another X' with lengths $1, 1, 2, 4$. These surfaces cannot be isometric since boundary curves are sent to boundary curves by an isometry. However, by the lemma they have the same orthospectrum.

□

3. The unit tangent bundle

We denote $p : T\mathbb{H}^n \to \mathbb{H}^n$ the canonical map that associates to a tangent vector a point in the base. Let A be an isometry (diffeomorphism) of $\mathbb{H}^n$ then it induces a diffeomorphism of the tangent bundle which we continue to denote by A.

If $v \in T\mathbb{H}^n$ is a (non zero) tangent vector then

$$\gamma_v : \mathbb{R} \to \mathbb{H}^n$$

is the unique geodesic parameterised by arclength such that $\dot{\gamma}_v(0)$ is a positive multiple of v. The geodesic γ_v determines a pair of distinct points $\gamma_v(\pm\infty)$ in the ideal boundary of $\mathbb{H}^n$. Observe that the map

$$\begin{array}{rcl} v & \mapsto & \gamma_v(-\infty) \\ T\mathbb{H}^n & \to & \partial\mathbb{H}^n \end{array}$$

is continuous and, in particular, the preimage of any measurable subset of $\partial\mathbb{H}^n$ is a measurable subset of the tangent bundle.

Whenever we speak of a geodesic α in $\mathbb{H}^n \cup \partial\mathbb{H}^n$ we mean the union of a geodesic α and its ideal endpoints $\alpha^{\pm}$.

As discussed in [4], the unit tangent bundle $T_1\mathbb{H}^n$ has a standard volume form Ω, which is just the product of the standard volume forms on $\mathbb{H}^n$ and S^{n-1}. To obtain an explicit formula for $d\Omega$, we shall try to parametrize unit tangent vectors by triples

$$(x, y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}.$$

Consider the upper half space model of $\mathbb{H}^n$ so that the ideal boundary is identified with $\mathbb{R}^{n-1} \cup \{\infty\}$. A point $v \in T_1\mathbb{H}^n$ determines a unique directed geodesic γ_v and so an ordered pair of points $(\gamma_v(-\infty), \gamma_v(\infty))$ in the ideal boundary $\mathbb{R}^{n-1} \cup \{\infty\}$ and, provided neither of these points is ∞, we may set $(x, y) = (\gamma_v(-\infty), \gamma_v(\infty))$. The last coordinate $t \in \mathbb{R}$ is the signed hyperbolic length between the highest point of γ_v and $p(v)$. Our parametrization is defined on a open dense subset of $T_1\mathbb{H}^n$ and it is easy to check that the complement has measure zero, so we may ignore its contribution when we compute volumes in $T_1\mathbb{H}^n$. With this parametrization, we have

$$d\Omega = \frac{2dV(x)dV(y)dt}{|x - y|^{2n-2}},$$

where $dV(x) = dx_1 dx_2 \cdots dx_{n-1}$ for $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^{n-1}$.

4. Decompositions of the tangent bundle and probabilities

Probably the easiest convex finite volume hyperbolic surfaces to study are the ideal polygons. In particular, the orthospectrum of an ideal polygon is finite and easy to compute. Surprisingly the study of the orthospectrum in this very simple case provides useful information: in [3] Bridgeman discusses the associated orthospectra and derives many of the classical identities satisfied by Roger's dilogarithm.

4.1. **Ideal polygons.** Let $P \subset \mathbb{H}$ be an ideal polygon and T^1P be the set of unit tangent vectors v such that $p(v) \in P$. Let $A(P)$ denote the area of P and note that the total volume of T^1P is $2\pi A(P)$.

If $v \in T^1P$ then the geodesic γ_v

- either intersects a pair of sides of P
- or has at least one endpoint at an ideal vertex of P.

The set of vectors such that $\gamma_v(\infty) = x \in \partial\mathbb{H}$ is a closed, co dimension 1 subvariety of the unit tangent bundle and so has measure 0. Since P has only finitely many vertices, it follows that the set of vectors v such that γ_v does not intersect a pair of sides has measure zero. Thus we have a decomposition of a subset of full measure of T^1P into pieces $\mathcal{B}(\alpha, \beta)$ labelled by pairs of sides $\alpha \neq \beta$ of the polygon P.

Recall that the sides of an ideal polygon are disjoint complete geodesics and that a pair of sides of an ideal polyhedron are adjacent if the underlying geodesics are asympotic so that they share a common endpoint in the ideal boundary. Configurations of pairs of complete geodesics are essentially determined up to isometry by a cross ratio of the four endpoints (see Beardon [2]). It follows that the probability that a random geodesic meets a given pair of sides of an ideal polygon can be expressed as a function of the associated cross ratio.

Theorem 2 (Bridgeman [3]). Let α, β be a pair of sides of an ideal polyhedron P then the probability that γ_v meets α and β is

- $\frac{1}{\pi A(P)}\mathcal{L}\left(\frac{(\alpha^- - \alpha^+)}{(\beta^+ - \alpha^+)}\frac{(\beta^+ - \alpha^-)}{(\alpha^- - \alpha^-)}\right)$ if α, β are non adjacent
- $\frac{1}{\pi A(P)}\frac{\pi^2}{6}n$ if α, β are adjacent.

Further,

$$\pi A(P) = \sum_{\alpha^*} \mathcal{L}\left(\frac{1}{\cosh^2(\ell(\alpha *)/2)}\right) + n\frac{\pi^2}{6}$$

where the sum is over all common perpendiculars $\alpha*$.

Note that in fact

$$\frac{(\alpha^- - \alpha^+)\,(\beta^+ - \alpha^-)}{(\beta^+ - \alpha^+)\,(\alpha^- - \alpha^-)} = \frac{1}{\cosh^2(\ell(\alpha *)/2)}$$

where $\alpha*$ is the common perpendicular between α and β so the probability depends on an ortholength.

4.2. **Decompositions of the unit tangent bundle.** From the construction in the previous paragraph we obtain a decomposition of a subset of full measure of T^1P into pieces labelled by pairs of sides of the polygon P. To calculate the probabilities in Bridgeman's theorem one has to determine the volume of certain subsets of the unit tangent bundle of $\mathbb{H}$ defined by pairs of disjoint geodesics. In fact, it is sufficient to do this for an ideal quadrilateral. Let $l > 0$ and β, α be a pair of disjoint geodesics in $\mathbb{H} \cup \partial\mathbb{H}$ such that the length of the common perpendicular is l. The convex hull of β and α is an ideal quadrilateral $\mathcal{Q}$ (see Figure 1).

4.2.1. *Bridgeman's set.* With the above notation we define:
$\mathcal{B}_l := \mathcal{B}(\alpha, \beta)$ to be the set of unit vectors v tangent to geodesic segments joining α to β. More formally, it is the set of $v \in p^{-1}(\mathcal{Q})$ satisfying

(1) the ray $\gamma_v(\mathbb{R}_+)$ meets β,
(2) the ray $\gamma_v(\mathbb{R}_-)$ meets α.

This set is the intersection of two open sets of the unit tangent bundle so is measurable.

4.2.2. *Calegaris's set.* Subsequently, Calegari introduced a different decomposition:
$\mathcal{C}_l := \mathcal{C}(\alpha, \beta)$ is the set of unit vectors v such that

(1) the ray $\gamma_v(\mathbb{R}_+)$ meets β,
(2) the point $p(v)$ is in the chimney (see below) of the quadrilateral $\mathcal{Q}$.

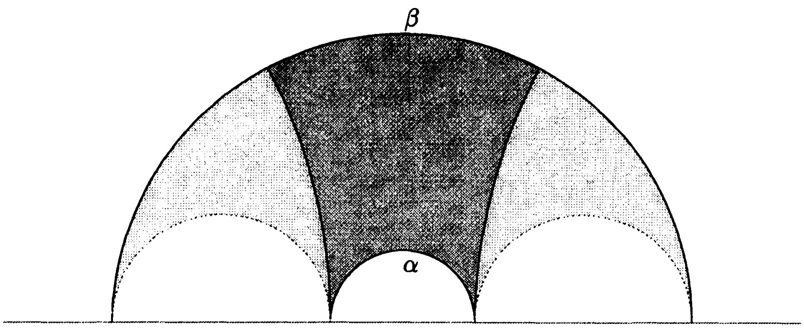

Figure 1: The quadrilateral Q and its chimney.

The *chimney* is the dark subset of the ideal quadrilateral in Figure 1 it is the convex hull of α and the nearest point retraction of α to β. Following Calegari, we say that the *top of the chimney* is α and *the base of the chimney* is the nearest point retraction of α to β. The chimney is a convex quadrilateral with the top and the base forming a pair of sides.

5. Crowns and spikes

Crowns form a class of surfaces for which one can give a closed form for the orthospectrum. The reason for this is that the fundamental group of a crown is isomorphic to $\mathbb{Z}$ and so lifts to the universal cover are indexed by integers.

A *crown* (Figure 2)is a complete convex hyperbolic surface of finite area homeomorphic to an annulus. The boundary of the crown consists of a single closed geodesic, which we denote β, and finitely many disjoint complete geodesics $\alpha_i, i = 1, \dots n$. One sees easily from the definition that a crown is non compact and further that the ends consist of spikes. A *spike* is a portion of the surface isometric to a region between two asymptotic geodesics in the hyperbolic plane. There are n spikes, that is, exactly the same number of spikes as complete geodesics α_i.

From the Gauss-Bonnet formula the of an n spiked crown is πn. and so the volume of the unit tanget bundle is $2\pi^2 n$.

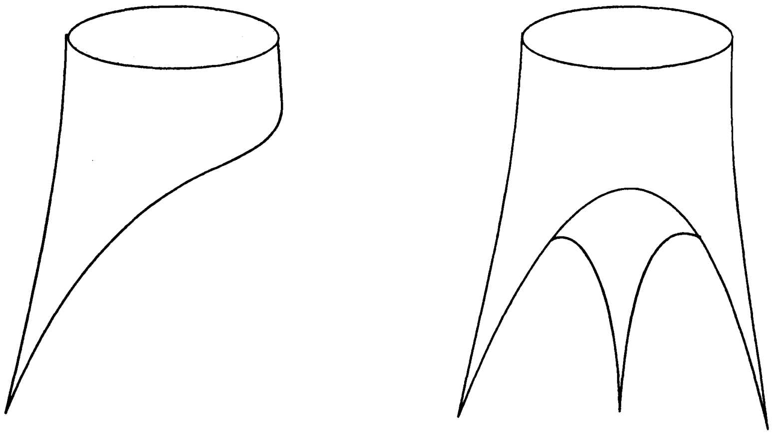

Figure 2: A pair of crowns.

5.1. **The single spiked crown.** Let $\lambda > 1$ and S be the crown with a single spike and a boundary geodesic β of length $\log(\lambda)$.

We begin by finding a subset of $\mathbb{H}$ isometric to the universal cover of S. Let $\tilde{S} \subset \mathbb{H}$ denote the convex hull of $\{0,\infty\} \sqcup \{\lambda^k,\ k \in \mathbb{Z}\} \subset \partial\mathbb{H}$. Observe that $\tilde{S}$ is a generalized ideal polygon, invariant under the hyperbolic isometry $T : z \mapsto \lambda^k z$. Further, the chimney contained in the ideal quadrilateral $1, \lambda, 0, \infty$ is a fundamental domain for the group generated by T. It follows that the universal cover of S can be identified with $\tilde{S}$.

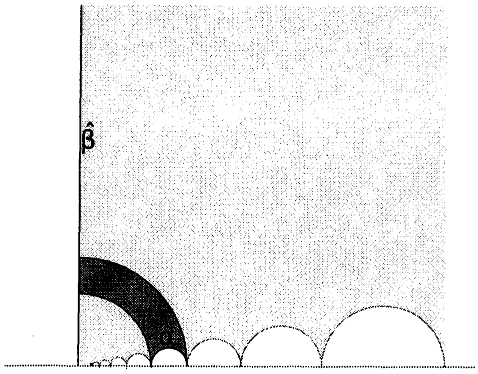

Figure 3: The universal cover of a crown in light grey with a chimney in darker grey.

Theorem 5.1. *The orthospectrum of the single spiked crown is the set $l_k, k \geq 2$ where l_k satisfies*

$$\cosh\left(\frac{l_k}{2}\right) = \frac{\sinh(\frac{k\ell_\beta}{2})}{\sinh(\frac{\ell_\beta}{2})}$$

and l_∞ satisfying

$$\cosh\left(\frac{l_\infty}{2}\right) = \frac{1}{\sinh(\frac{\ell_\beta}{2})}.$$

Proof: Let $\lambda > 1$ and S be the crown with a single spike and a boundary geodesic β of length $\log(\lambda)$. The orthospectrum is easy to compute since we have determined the universal cover of S. From the preceding discussion $\tilde{S}$ has a distinguished side $0, \infty$ and sides λ^k, λ^{k+1} for $k \in \mathbb{Z}$. The distiguished side is Γ-invariant and Γ acts transitively on the other sides. The set of orthogeodesics of $\tilde{S}$ consists of

- the perpendiculars to $0, \infty$ and λ^k, λ^{k+1}
- the perpendiculars to λ^k, λ^{k+1} and λ^m, λ^{m+1}

Using the transitivity of the action one sees that the orthospectrum of S consists of

- the length l_∞ of the perpendicular $0, \infty$ to $1, \lambda$
$$\cosh^2\left(\frac{l_\infty}{2}\right) = \frac{(\infty - 1)}{(\lambda - 1)}\frac{(\lambda - 0)}{(\infty - 0)} = \frac{\lambda}{(\lambda - 1)^2} = \frac{1}{\sinh^2(\frac{\ell_\beta}{2})}.$$
- the lengths l_k of perpendiculars $1, \lambda$ to λ^k, λ^{k+1}
$$\cosh^2\left(\frac{l_k}{2}\right) = \frac{(\lambda^k - 1)}{(\lambda - 1)}\frac{(\lambda - \lambda^{k+1})}{(\lambda^k - \lambda^{k+1})} = \frac{(\lambda^k - 1)^2\lambda}{(\lambda - 1)^2\lambda^k} = \frac{\sinh^2(\frac{k\ell_\beta}{2})}{\sinh^2(\frac{\ell_\beta}{2})}.$$

□

5.2. **Lewin's identity and crowns.** A very special crown is the punctured ideal monogon which can be obtained by identifying two sides of an ideal triangle. We compute it's spectrum and relate it to an identity first proved by Lewin. We note that Bridgeman [3] gave a different proof using his computation of the orthospectrum of a regular ideal n-gon and a limiting argument. Since the punctured monogon is a limit of crowns as the length of the boundary geodesic goes to 0, that it is possible to deduce this from Lemma 5.1 using an analogous argument to Bridgeman's, however, we give a direct proof using the universal cover.

Lemma 5.2. *The orthospectrum of the punctured monogon is the set $l_k, k \geq 2$ where l_k satisfies*

$$\cosh\left(\frac{l_k}{2}\right) = k^2$$

Proof: Let $\tilde{S} \subset \mathbb{H}$ denote the convex hull of the integers $\mathbb{Z} \subset \partial\mathbb{H}$. The polygon $\tilde{S}$ is invariant under $T : z \mapsto z + 1$ since $\mathbb{Z}$ is invariant by this translation. Observe that the ideal triangle with vertices $0, 1$ and ∞ is a fundamental domain for the group generated by T and it follows that the universal cover of S can be identified with $\tilde{S}$.

One now computes the orthospectrum as follows. Each side of $\tilde{S}$ is a geodesic joining pairs of consecutive integers. The geodesic $[0, 1]$ joining 0 and 1 is a lift of the the boundary geodesic α and so every orthogeodesic lifts to a perpendicular between this geodesic and another side of $\tilde{S}$, that is, a geodesic $[k, k + 1]$ joining the integers $k, k + 1$. It follows immediately that the orthospectrum is the set of distances $d_{\mathbb{H}}([0, 1], [k, k + 1])$ where

$$\cosh^2(d_{\mathbb{H}}([0,1],[k,k+1])) = \frac{(k-0)}{(1-0)}\frac{(1-(k+1))}{(k-(k+1))} = k^2.$$

□

It is easy to check that the set of tangent vectors such that γ_v does not meet a pair of sides is measure zero and so one obtains:

Corollary 3. (Lewin's identity)

$$\sum_k \mathcal{L}\left(\frac{1}{k^2}\right) = \frac{\pi^2}{6}.$$

5.3. **Proof of main theorem.** In this section, we show the following theorem:

The Bridgeman set $\mathcal{B}_l$ and the Calegari set $\mathcal{C}_l$ have the same area.

Proof: Let S be a single spiked crown with closed boundary geodesic β and let α denote the other boundary component. Consider γ a maximal geodesic on S. Both endpoints of γ cannot be on β since this implies the existence of a geodesic bigon, bounded by a lift of γ and a lift of β, in the universal cover. but this is forbidden in non positive curvature. Thus

- either both endpoints of γ are on α
- or there is one endpoint on α and the other on β.
- or γ is a geodesic meeting β and asymptotic to α.

This means that the unit tangent bundle of the interior of S decomposes as $X_1 \sqcup X_2 \sqcup X_3$ where X_k is the set of vectors v such that γ_v is respectively one of three types of geodesic listed above.

Since Calegari's chimney β is a fundamental domain for action of the covering group on $\mathbb{H}$ the set $p^{-1}(\beta)$ is a fundamental domain for action on $T^1\mathbb{H}$. It follows that any $v \in X_1$ has exactly one lift $\tilde{v} \in p^{-1}(\beta)$ and by the preceding discussion $\tilde{v} \in \mathcal{C}_l$. The map p preserves the measure and so

$$\mathrm{vol}_3(X_1) = \mathrm{vol}_3(\mathcal{C}_l).$$

Likewise every $v \in X_1$ has exactly one lift $\tilde{v} \in \mathcal{B}_l$ and

$$\mathrm{vol}_3(X_1) = \mathrm{vol}_3(\mathcal{B}_l),$$

and the result follows

□

6. Closing remarks and question

In this text we have given a brief survey of some of geometric identities and the orthospectrum of hyperbolic manifolds with a particular emphasis on recent results of Bridgeman-Kahn and Calegari. There are many questions still open. In particular:

- Are these the only possible identities?
- By developping the ideas of Paragraph 2.1.4 it is not difficult to give examples of pairs of surfaces with the same orthospectrum but different spectra of lengths of closed geodesics. It is natural to ask: does the spectrum of lengths of closed geodesics determine the orthospectrum?
- Is a partial converse to Lemma 2.1 true: if two surfaces with the same orthospectrum are they necessarily commensurable?

References

[1] A. Basmajian, *The orthogonal spectrum of a hyperbolic manifold.* Amer. J. Math. **115** (1993), no. 5, 1139–1159.

[2] A.F. Beardon , *The geometry of discrete groups*, Springer-Verlag, cop. 1983

[3] M. Bridgeman, *Orthospectra of Geodesic Laminations and Dilogarithm Identities on Moduli Space.* Geom. Topol. **15** (2011), no. 2, 707–733.

[4] M. Bridgeman and J. Kahn, *Hyperbolic volume of n-manifolds with geodesic boundary and orthospectra*, Geometric and Functional Analysis, **20**, Issue 5 2010.

[5] P. Buser, *Geometry and spectra of compact Riemann surfaces*, Birkheuser, 1992

[6] D. Calegari, *Chimneys, leopard spots, and the identities of Basmajian and Bridgeman*, Algebr. Geom. Topol. **10** (2010), no. 3, 1857–1863.

[7] D. Calegari, *Bridgeman's Orthospectrum Identity*, Topol. Proc. **38** (2011), 173–179.

[8] A. Casson and S Bleiler, *Automorphisms of surfaces after Nielsen and Thurston*, Cambridge University Press, 1988

[9] H. Masai and G.McShane, *Equidecomposability, volume formulae and orthospectra*, Preprint.

[10] G.McShane, J.R. Parker, and I. Redfern *Drawing limit sets of Kleinian groups using finite state automata*, Experiment. Math. **3** (1994), 153–170.

[11] J.Ratcliffe, *Foundations of Hyperbolic Manifolds*, Graduate Texts in Mathematics **149**, 2006.

[12] P. Sarnak, *Class numbers of indefinite binary quadratic forms.*, J. Number Theory **15** (1982), 229–247.

[13] T. Sunada , *Riemannian coverings and isospectral manifolds*, Ann. of Math. **121** (1985), 169–186.

Institut Fourier, University of Grenoble
E-mail address: `mcshane@ujf-grenoble.fr`