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# LOCAL AND INFINITESIMAL RIGIDITY OF REPRESENTATIONS OF HYPERBOLIC THREE MANIFOLDS

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ABSTRACT. We discuss local and infinitesimal rigidity for finite dimensional representations of hyperbolic three manifolds. We are motivated by the fact that some of the representations have a geometric interpretation, though we discuss it in a general setting.

## 1. INTRODUCTION

Let $M^3$ be a closed, compact, hyperbolic and orientable three-manifold. Fix a lift of its holonomy representation

$$\widetilde{\mathrm{hol}}:\pi_1(M^3)\to SL_2(\mathbf{C}).$$

Let $G$ denote a (real or complex) Lie group and let

$$\sigma: SL_2(\mathbf{C})\to G$$

be a linear representation, that does not need to be holomorphic. For simplicity, we shall assume that $\sigma$ is irreducible.

**Question 1.1.** *Is $\sigma\circ\widetilde{\mathrm{hol}}:\pi_1(M^3)\to G$ locally rigid?*

In order to properly define local rigidity, we consider the variety of representations

$$\hom(\pi_1(M^3),G),$$

which naturally embeds in $G\times\cdots\times G$, by considering the image of the elements in a (finite) generating set. Then we define:

**Definition 1.2.** A representation $\rho:\pi_1(M^3)\to G$ is *locally rigid* if a neighborhood of $\rho$ in $\hom(\pi_1(M^3),G)$ consist only of representations that are conjugate to $\rho$.

We are interested in the stronger notion of infinitesimal rigidity. For this we consider the Lie algebra equipped with the adjoint action, that we denote $\mathfrak{g}_{Ad\rho}$.

**Definition 1.3.** A representation $\rho:\pi_1(M^3)\to G$ is said to be *infinitesimally rigid* if

$$H^1(\pi_1(M^3),\mathfrak{g}_{Ad\rho})=0.$$

Infinitesimal rigidity is stronger than local rigidity, as $H^1(\pi_1(M^3),\mathfrak{g}_{Ad\rho})$ may be viewed as the tangent space to the variety of representations up to conjugacy. We shall discus this later in Section 4. It is natural to arise the following question:

**Question 1.4.** *Is $\sigma\circ\widetilde{\mathrm{hol}}:\pi_1(M^3)\to G$ infinitesimally rigid?*

The answer will vary for different choices of $G$. To describe the possibilities, we need to recall the classification of irreducible representations of $SL_2(\mathbf{C})$. This will be done in Section 2, before we want to discuss some motivating examples.

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**Example 1.5.** Consider $\sigma$ to be the identity. Hence deformations of the representation correspond to deformations of the hyperbolic structure, cf. [41, 14]. By Mostow's theorem [35], it is rigid (globally and locally), but infinitesimal rigidity is given by a theorem of Weil that we recall next [43].

**Theorem 1.6** (Weil infinitesimal rigidity [43]). *If $M^3$ is a closed hyperbolic three manifold, then*

$$H^1(\pi_1(M^3); \mathfrak{sl}_2(\mathbf{C})_{Ad_{\widetilde{\mathrm{hol}}}}) = 0.$$

Weil proved this theorem in dimension three and higher. When the manifold is non-compact, there is a deformation space coming from the ends of the manifold, that we shall discuss in Section 12

**Example 1.7.** Consider the representation

$$\sigma : SL_2(\mathbf{C}) \to SO(3,1),$$

which induces an isomorphism between $PSL_2(\mathbf{C})$ and $SO_0(3,1)$. The notation

$$\rho_{1,1} = \sigma \circ \widetilde{\mathrm{hol}} : \pi_1(M^3) \to SO(3,1)$$

will be clear later. Notice that $H^1(\pi_1(M^3), \mathfrak{so}(3,1)_{Ad\rho_{1,1}}) = 0$ by Weil infinitesimal rigidity. Then embed $SO(3,1)$ in $SL_4(\mathbf{R})$, so that rigidity of the representation in $SL_4(\mathbf{R})$ means rigidity of the induced real *projective* structure.

**Definition 1.8.** One says that $M^3$ is *projectively rigid* if $\rho_{1,1}$ is rigid as representation in $SL_4(\mathbf{R})$, and $M^3$ is *infinitesimally projectively rigid* if

$$H^1(\pi_1(M^3); \mathfrak{sl}_4(\mathbf{R})_{Ad_{\rho_{1,1}}}) = 0.$$

Cooper, Long, and Thistlethwaite compute in [17] the deformation space of projective structures for a large number of hyperbolic three manifolds. They show that all possibilities can occur: infinitesimally projectively rigid, projectively rigid but not infinitesimally, and projectively non rigid (that they call flexible).

Historically, one of the first to study projective structures was Benzécri in the 1960's [10]. Kac and Vinberg [42] gave the first examples of such deformations. Koszul [29] and Goldman later generalized these examples. Johnson and Millson provided deformations of the canonical projective structure by means of bending along totally geodesic surfaces [25]. Examples of deformations for Coxeter orbifolds have been obtained by Benoist [8], Choi [16], and Marquis [31]. See the survey by Benoist [9] and references therein for more results on convex projective structures.

With Heusener, we have proved in [24] the existence of infinitely many hyperbolic manifolds that are infinitesimally projectively rigid.

**Example 1.9.** Next consider the embedding

$$\mathrm{Isom}(\mathbf{H}^3) \hookrightarrow \mathrm{Isom}(\mathbf{H}^4)$$

and ask whether its composition with the holonomy is rigid here or not. This is equivalent to the study of deformations of the flat conformal structure, as $\mathrm{Isom}(\mathbf{H}^4)$ is the group of Möbius transformations of $S^3 = \partial_\infty \mathbf{H}^3$. We may view them also as quasifuchsian structures.

Here we mention again the construction of Johnson and Millson on bending along totally geodesic surfaces [25], but also the results on rigidity by Kapovich, Scannell and Francaviglia and myself on (infinitesimal, local and global) rigidity of such structures [19, 26, 39, 40]. Also Apanasov [3, 5], Apanasov and Tetenov [4], and Bart and Scannell [7] have constructed deformations that do not correspond to bending.

The paper is addressed to readers in low dimensional topology and geometry and I do not assume any background in representation theory. Some of the statements are well known in representation theory, and most of the proofs are given or sketched here. There are of course a lot of results presented here that are known, but to my knowledge, some of them where not previously known in the literature.

The paper is organized as follows. In Section 2 we recall the classification of finite dimensional representations of $SL_2(\mathbf{C})$, and we look at those that are real. The main results are then stated in Section 3. In Section 4 we recall some known facts on the tangent space of the varieties of representations and cohomology required for the proofs, basically Weil's construction. Then we need two main tools for proving local rigidity. The first one is Raghunathan's vanishing theorem, that will be recalled in Section 5. The second tool is to decompose the Lie algebras as irreducible modules, in order to apply Raghunathan's vanishing. This decomposition is done in Sections 6, 7 and 8. Next we discuss real representations in Section 9, including the projective structures. This also concerns complex hyperbolic structures in Section 10 and conformally flat ones in Section 11. Finally, Section 12 is devoted to noncompact hyperbolic three manifolds of finite type.

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## 2. Finite dimensional representations of $SL_2(\mathbf{C})$

Given $n \geq 0$, consider

$$V_{n,0} = \{P(X,Y) \in \mathbf{C}[X,Y] \mid P \text{ homogeneous and } \deg P = n\}.$$

Then $SL_2(\mathbf{C})$ acts on $V_{n,0}$ as follows:

$$\begin{array}{rcl} SL_2(\mathbf{C}) \times V_{n,0} & \to & V_{n,0} \\ (A,P) & \mapsto & P \circ A^t \end{array}$$

where $A^t$ denotes the transpose of $A$. Notice that instead of the transpose one can consider the inverse, as transposing and taking the inverse are conjugate operations in $SL_2(\mathbf{C})$. Next define

$$V_{n_1,n_2} = V_{n_1,0} \otimes \overline{V_{n_2,0}}$$

where the bar denotes complex conjugation. We have:

$$\dim_{\mathbf{C}} V_{n_1,n_2} = (n_1+1)(n_2+1).$$

The corresponding representation is denoted by

$$\mathrm{Sym}_{n_1,n_2} : SL_2(\mathbf{C}) \to \mathrm{Aut}_{\mathbf{C}} V_{n_1,n_2}.$$

The automorphisms in the image of $\mathrm{Sym}_{n_1,n_2}$ have determinant one

$$\mathrm{Sym}_{n_1,n_2} : SL_2(\mathbf{C}) \to SL_{(n_1+1)(n_2+1)}(\mathbf{C}).$$

This gives the classification of finite dimensional representations (cf. [28]):

**Theorem 2.1.** *Every irreducible and finite dimensional representation of $SL_2(\mathbf{C})$ is equivalent to $\mathrm{Sym}_{n_1,n_2}$ for some (unique) pair of integers $n_1, n_2 \geq 0$*

The idea of the proof is to classify the representations of the (real) Lie algebra $\mathfrak{sl}_2(\mathbf{C})$. To do so, one classifies the holomorphic representation of its complexification

$$\mathfrak{sl}_2(\mathbf{C}) \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{sl}_2(\mathbf{C}) \oplus \mathfrak{sl}_2(\mathbf{C}).$$

Holomorphic irreducible representations of $\mathfrak{sl}_2(\mathbf{C})$ are classified by a weight, a nonnegative integer that is the largest eigenvalue of a semisimple element of $\mathfrak{sl}_2(\mathbf{C})$. Hence irreducible representations of $\mathfrak{sl}_2(\mathbf{C})$ are classified by a pair of nonnegative integers.

For example $\mathrm{Sym}_{0,0}$ is the trivial representation, $\mathrm{Sym}_{1,0}$ the tautological one, and $\mathrm{Sym}_{0,1}$, its complex conjugate. We will see later that $\mathrm{Sym}_{1,1}$ is the complexification of the isomorphism of (real) Lie groups between $PSL(2, \mathbf{C})$ and $SO(3,1)$, as the orientation preserving isometry group of hyperbolic space.

The group $SL_{(n_1+1)(n_2+1)}(\mathbf{C})$ may be too large to have rigidity, for this we remark that $\mathrm{Sym}_{n_1,n_2}$ preserves a bilinear form. We start by viewing the determinant as a skew (antisymmetric) bilinear form:

$$\begin{array}{rcl} \det : \mathbf{C}^2 \times \mathbf{C}^2 & \to & \mathbf{C} \\ \binom{a}{c}, \binom{b}{d} & \mapsto & \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \end{array}$$

which is invariant by the action of $SL_2(\mathbf{C})$. Since $V_{n.0}$ is the $n$-th symmetric power of $\mathbf{C}^2 \cong V_{1,0}$, taking symmetric powers and tensor products, it induces a bilinear form:

$$\Phi : V_{n_1,n_2} \times V_{n_1,n_2} \to \mathbf{C}.$$

This form is $\mathrm{Sym}_{n_1,n_2}$-invariant, nondegenerate and

$$\begin{cases} \text{symmetric} & \text{if } n_1 + n_2 \text{ is even,} \\ \text{skew} & \text{if } n_1 + n_2 \text{ is odd.} \end{cases}$$

Thus

$$\mathrm{Sym}_{n_1,n_2} : SL_2(\mathbf{C}) \to G = \begin{cases} SO((n_1+1)(n_2+1), \mathbf{C}) & \text{if } n_1 + n_2 \text{ is even,} \\ Sp(\frac{(n_1+1)(n_2+1)}{2}, \mathbf{C}) & \text{if } n_1 + n_2 \text{ is odd.} \end{cases}$$

We may look also for representations with real image. Let $SO(p,q) \subset SL_{p+q}(\mathbf{R})$ denote the special real orthogonal group of signature $p, q$.

**Proposition 2.2.** *The image of $\mathrm{Sym}_{n,n}$ is contained in $SO(p,q)$, with*

$$p = \frac{n^2 + 3n + 2}{2} \quad \text{and} \quad q = \frac{n^2 + n}{2}.$$

Notice that $p + q = (n+1)^2$. For instance the image of $\mathrm{Sym}_{1,1}$ is contained in $SO(3,1)$ and in fact it induces an isomorphism between $PSL_2(\mathbf{C})$ and the identity component of $SO(3,1)$, both the isometry group of hyperbolic space. Also the image of $\mathrm{Sym}_{2,2}$ is contained in $SO(6,3)$.

## 3. Rigidity and non-rigidity results

Let $\rho_{n_1,n_2}$ denote the representation
(1)

$$\rho_{n_1,n_2} = \mathrm{Sym}_{n_1,n_2} \circ \widetilde{\mathrm{hol}} : \pi_1(M^3) \to G = \begin{cases} SO((n_1+1)(n_2+1), \mathbf{C}) & \text{if } n_1+n_2 \text{ is even} \\ Sp(\frac{(n_1+1)(n_2+1)}{2}, \mathbf{C}) & \text{if } n_1+n_2 \text{ is odd.} \end{cases}$$

**Theorem 3.1** (Infinitesimal rigidity in $G$). *Let $M^3$ be a closed, oriented, and hyperbolic three manifold and let $\rho_{n_1,n_2} : \pi_1(M^3) \to G$ be as in* (1). *Then*

$$H^1(\pi_1(M^3), \mathfrak{g}_{Ad\rho_{n_1,n_2}}) = 0.$$

**Corollary 3.2.** *Under the hypothesis of Theorem 3.1, $\rho_{n_1,n_2}$ is rigid in* $\hom(\pi_1(M^3), G)$.

The fact that $\rho_{n_1,n_2}$ is rigid in $\hom(\pi_1(M^3), G)$ does not mean that it is rigid in

$$\hom(\pi_1(M^3), SL_{(n_1+1)(n_2+1)}(\mathbf{C})).$$

This is described by the following two results.

**Theorem 3.3.** *Let $M^3$ be a closed, oriented hyperbolic three manifold. For $n \geq 1$, $\rho_{n,0}$ and $\rho_{0,n}$ are infinitesimally rigid (and rigid) in* $\hom(\pi_1(M^3), SL_{n+1}(\mathbf{C}))$*:*

$$H^1(\pi_1(M^3), \mathfrak{sl}_{n+1}(\mathbf{C})_{Ad\rho_{n,0}}) = H^1(\pi_1(M^3), \mathfrak{sl}_{n+1}(\mathbf{C})_{Ad\rho_{0,n}}) = 0.$$

**Theorem 3.4.** *Let $M^3$ be a closed, oriented hyperbolic three manifold. Assume that $n_1, n_2 \geq 1$ and that $M^3$ contains a totally geodesic surface. Then*

$$H^1(\pi_1(M^3), \mathfrak{sl}_{(n_1+1)(n_2+1)}(\mathbf{C})_{Ad\rho_{n_1,n_2}}) \neq 0.$$

*Moreover $\rho_{n,n}$ is nonrigid in $SL_{(n+1)^2}(\mathbf{C})$.*

Notice that for some manifolds $\rho_{n_1,n_2}$ can still be rigid in $SL_{(n_1+1)(n_2+1)}(\mathbf{C})$. This is the case for manifolds that are projectively rigid for $n_1 = n_2 = 2$. Some other representations for those manifolds are rigid because of the following:

**Proposition 3.5.** *Let $M^3$ be as above and assume that $n = \min(n_1, n_2) \geq 1$. Then*

$$H^1(\pi_1(M^3), \mathfrak{sl}_{(n_1+1)(n_2+1)}(\mathbf{C})_{Ad\rho_{n_1,n_2}}) \cong H^1(\pi_1(M^3), \mathfrak{sl}_{(n+1)^2}(\mathbf{C})_{Ad\rho_{n,n}}).$$

*Thus $\rho_{n_1,n_2}$ is infinitesimally rigid in $\hom(\pi_1(M^3), SL_{(n_1+1)(n_2+1)}(\mathbf{C}))$ if and only if $\rho_{n,n}$ is infinitesimally rigid in $\hom(M^3, SL_{(n+1)^2}(\mathbf{C}))$.*

Recall from Proposition 2.2 that the image of $\rho_{n,n}$ is contained in $SO(p,q)$ with

$$p = \frac{n^2+3n+2}{2} \quad \text{and} \quad q = \frac{n^2+n}{2}.$$

From Theorems 3.1 and 3.4, since

$$\mathfrak{so}((n+1)^2, \mathbf{C}) \cong \mathfrak{so}(p,q) \otimes_{\mathbf{R}} \mathbf{C} \quad \text{and} \quad \mathfrak{sl}_{(n+1)^2}(\mathbf{C}) \cong \mathfrak{sl}_{(n+1)^2}(\mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C}$$

we obtain:

**Corollary 3.6.** *Let $M^3$ be as above. For $n \geq 1$,*

$$H^1(\pi_1(M^3), \mathfrak{so}(p,q)_{Ad\rho_{n,n}}) = 0.$$

*In particular $\rho_{n,n}$ is rigid in $\hom(\pi_1(M^3), SO(p,q))$. If in addition $M^3$ contains a totally geodesic surface, then $\rho_{n,n}$ is nonrigid in $\hom(\pi_1(M^3), SL_{(n+1)^2}(\mathbf{R}))$.*

**Proposition 3.7.** *Let $M^3$ be as above. For $n \geq 1$, $\rho_{n,n} : \pi_1(M^3) \to X(M^3, SO(p,q))$ is infinitesimally rigid with coefficients $\mathfrak{sl}_{(n+1)^2}(\mathbf{R})$ iff it is so with coefficients $\mathfrak{su}(p,q)$.*

As a particular case of Proposition 3.5 we get:

**Corollary 3.8.** *Let $M^3$ be as above. Then for $n \geq 1$, $M^3$ is infinitesimally projectively rigid iff $\rho_{n,1}$ is infinitesimally rigid in $\hom(\pi_1(M^3), SL_{2(n+1)}(\mathbf{C}))$.*

We finally discuss the noncompact case. Assume that $M^3$ is a topologically finite hyperbolic manifold. This means that it has a finite number of ends. By the solution of Marden's conjecture [1, 13] the ends are either cusps (homeomorphic to $T^2 \times [0, +\infty)$) or have infinite volume, homeomorphic to $F_g^2 \times [0, +\infty)$, where $F_g^2$ is a surface of genus $g \geq 2$. In particular it has a compactification consisting in adding boundary surfaces.

The variety of characters is denoted by $X(M^3, G)$. Since this paper only deals with local rigidity and local deformations, we may assume that $X(M^3, G)$ is locally the quotient of $\hom(\pi_1(M^3), G)/G$, where $G$ acts by conjugation.

**Theorem 3.9.** *Let $M^3$ be a topologically finite, hyperbolic, and orientable three manifold. Let $\rho_{(n_1,n_2)} : \pi_1(M^3) \to G$ be as in Theorem 3.1 or 3.3. Then the character $[\rho_{(n_1,n_2)}]$ is a smooth point of $X(M^3, G)$. Moreover, If $\partial \overline{M}^3$ is the union of $k$ tori and $l$ surfaces of genus $g_1, \ldots, g_l \geq 2$, and $N \geq 1$, then the local dimension of $X(M^3, G)$ is*

$$k \operatorname{rank} G + \sum (g_i - 1) \dim G.$$

## 4. Tangent spaces and cohomology

In [43] André Weil showed that the tangent space at the variety of representations can be identified to the space of group cocycles, and the tangent space to the orbit by conjugation to the subspace of coboundaries.

Here $\Gamma$ denotes a finitely generated group, though we are mainly interested in $\Gamma = \pi_1(M^3)$.

For a representation

$$\rho : \Gamma \to G$$

the adjoint representation on the Lie algebra is denoted by

$$Ad_\rho : \Gamma \to \operatorname{Aut} \mathfrak{g}.$$

Recall that the space of group cocycles is

$$Z^1(\Gamma, \mathfrak{g}_{Ad_\rho}) = \{ d : \Gamma \to \mathfrak{g} \mid d(\gamma_1\gamma_2) = d(\gamma_1) + Ad_{\rho(\gamma_1)} d(\gamma_2),\ \forall \gamma_1, \gamma_2 \in \Gamma \},$$

and the subspace of group coboundaries:

$$B^1(\Gamma, \mathfrak{g}_{Ad_\rho}) = \{ d_a : \Gamma \to \mathfrak{g} \mid \exists a \in \mathfrak{g} \text{ s.t. } d_a(\gamma) = (Ad_{\rho(\gamma)} - 1)a,\ \forall \gamma \in \Gamma \}.$$

The group cohomology is then

$$H^1(\Gamma, \mathfrak{g}_{Ad_\rho}) = Z^1(\Gamma, \mathfrak{g}_{Ad_\rho}) / B^1(\Gamma, \mathfrak{g}_{Ad_\rho}).$$

We view the Zariski tangent space to an algebraic variety as the space of germs of paths that satisfy the equations up to first order. Thus, in the variety of representations, a Zariski tangent vector is represented by a first order deformation. Namely a path of representations $\rho_t : \Gamma \to G$ that satisfies

$$\begin{cases} \rho_0 = \rho \\ \rho_t(\gamma_1\gamma_2) = \rho_t(\gamma_1)\rho_t(\gamma_2) + O(t^2), & \forall \gamma_1, \gamma_2 \in \Gamma. \end{cases}$$

*Weil's construction* assigns to such a first order (or infinitesimal) deformation the cocycle

$$(2) \qquad \begin{array}{rcl} \Gamma & \to & \mathfrak{g} \\ \gamma & \mapsto & \frac{d}{dt}\rho_t(\gamma)\rho_0(\gamma^{-1})\big|_{t=0} \end{array}.$$

**Theorem 4.1** (Weil's construction). *The map (2) defines an isomorphism between the Zariski tangent space to the variety of representations at $\rho$ and the space of group cocycles:*

$$T^{Zar}_\rho \hom(\Gamma, G) \cong Z^1(\Gamma, \mathfrak{g}_{Ad_\rho}).$$

*In addition, this isomorphism maps the Zariski tangent space to an orbit by conjugation $G\rho$ to the space of coboundaries:*

$$T^{Zar}_\rho G\rho \cong B^1(\Gamma, \mathfrak{g}_{Ad_\rho}).$$

Observe that when we have infinitesimal rigidity, we have $B^1(\Gamma, \mathfrak{g}_{Ad_\rho}) = Z^1(\Gamma, \mathfrak{g}_{Ad_\rho})$, thus the inclusion $G\rho \subset \hom(\pi_1(M^3), G)$ induces an isomorphism of tangent spaces. In fact one can prove

**Corollary 4.2.** *If $\rho$ is semisimple and $H^1(\Gamma, \mathfrak{g}_{Ad_\rho}) = 0$, then $\rho$ is locally rigid.*

**Definition 4.3.** A linear representation $\rho : \pi_1(M^3) \to G \subset GL_N(\mathbf{C})$ is called *simple* if $\mathbf{C}^N$ has no proper invariant subspaces, and it is called *semisimple* if it is the direct sum of simple ones.

**Remark 4.4.** For cocompact manifolds, the representations $\rho_{n_1,n_2} : \pi_1(M^3) \to G$ are simple, because $\mathrm{Sym}_{n_1,n_2}$ is irreducible and $\widetilde{hol}(\pi_1(M^3))$ is Zariski dense in $SL_2(\mathbf{C})$. This always holds true for any $M^3$ which is not Fuchsian nor elementary.

A stronger formulation is the following one. We may think of the variety of characters $X(\Gamma, G)$ as (locally) the quotient $\hom(\Gamma, G)/G$, in neighbourhoods of semisimple points.

**Corollary 4.5.** *If $\rho$ is semisimple then*

$$T^{Zar}_\rho X(\Gamma, G) \cong H^1(\Gamma, \mathfrak{g}_{Ad_\rho}).$$

See [30] for a proof of Theorem 4.1 and Corollaries 4.2 and 4.5.

Now the strategy will be to decompose the $SL_2(\mathbf{C})$-module $\mathfrak{g}_{Ad_\rho}$ into irreducible representations $V_{n_1,n_2}$ and to use Raghunathan's vanishing theorem in cohomology. We start with Raghunatan's theorem in the next section, then in Sections 6, 7 and 8 we study the decompositions of $\mathfrak{g}_{Ad\rho}$.

## 5. Raghunathan's vanishing theorem

By Corollary 4.5, we are interested in computing $H^1(\pi_1(M^3), \mathfrak{g}_{Ad_\rho})$. After decomposing $\mathfrak{g}_{Ad_\rho}$ into irreducible modules, we must compute $H^1(\pi_1(M^3), V_{n_1,n_2})$. The key result is the following:

**Theorem 5.1** (Raghunathan's vanishing [37]). *Let $M^3$ be a compact hyperbolic three manifold. If $n_1 \neq n_2$ then*

$$H^1(\pi_1(M^3), V_{n_1,n_2}) = 0.$$

This theorem is proved using de Rham cohomology. Thus let $E_{n_1,n_2}$ denote the flat bundle with fibre $V_{n_1,n_2}$ and monodromy $\rho_{n_1,n_2}$:

$$V_{n_1,n_2} \to E_{n_1,n_2} \to M.$$

Let $\Omega^p(M^3, E_{n_1,n_2})$ denote the $p$-forms on $M^3$ valued on $E_{n_1,n_2}$. By de Rham's theorem, the cohomology of

$$(\Omega^p(M^3, E_{n_1,n_2}), d)$$

is isomorphic to the group cohomology $H^*(\pi_1(M^3), V_{n_1,n_2})$.

There is a natural Hermitian product in the bundle $E_{n_1,n_2}$ denoted by $\langle,\rangle$. Let also $\Delta$ denote the Laplacian. Then Raghunathan proved his vanishing theorem as a consequence of the following:

**Lemma 5.2** ([37, 38]). *Let $M^3$ be a hyperbolic three manifold, and assume that $n_1 \neq n_2$. Then there exists a constant $C > 0$ such that every $\omega \in \Omega^p(M^3, E_{n_1,n_2})$ with compact support satisfies*

$$\langle \Delta\omega, \omega\rangle > c\langle \omega, \omega\rangle.$$

Since by Hodge theorem every cohomology class in a compact manifold is represented by a harmonic form (i.e. a form $\omega$ satisfying $\Delta\omega = 0$), Lemma 5.2 immediately implies Theorem 5.1.

The property of Lemma 5.2 is called *strong acyclicity* by Bergeron and Venkatesh in [11], and it is used to compute the asymptotic behaviour of Reidemeister torsion or homology torsion under coverings.

When $M^3$ is not compact, Lemma 5.2 gives a vanishing theorem, due to Matsushima-Murakami [32] and Andreotti-Vesentini [2]:

**Theorem 5.3.** *Let $M^3$ be a hyperbolic three manifold, and assume that $n_1 \neq n_2$. Then every closed form $\omega \in \Omega^p(M^3, E_{n_1,n_2})$ that is $L^2$ (square summable) is exact.*

This theorem will be used in Section 12 for discussing the situation for noncompact manifolds.

It is normal to ask what happens when $n_1 = n_2$. This has been discussed by Millson, who proved in [34] a more general result that implies:

**Proposition 5.4** (Millson [34]). *Let $M^3$ be a compact, orientable, hyperbolic three manifold. Assume that $M^3$ contains a totally geodesic surface, then*

$$H^1(M^3, V_{n,n}) \neq 0.$$

We discuss its proof in Section 9. This is related to bending.

Notice also that there exist manifolds for which $H^1(M^3, V_{n,n}) = 0$ for $n = 1, 2$. When $n = 1$ those are conformally flat manifolds, and for $n = 2$ those are projectively rigid. It has been proved by Kapovich [26] and Scannell [40] (improved by Francaviglia and myself [19]) that almost all Dehn fillings in a hyperbolic two bridge not are conformally flat. Moreover, we showed with Heusener that infinitely many Dehn fillings on the figure eight knot exterior are projectively rigid [24].

**Question 5.5.** *Is there any manifold $M^3$ for which $H^1(M^3, V_{n,n}) = 0$ for every $n \geq 1$?*

A manifold for which the question would have a positive answer would satisfy all possible rigidity properties.

## 6. Decomposing holomorphic representations

Once we have Theorem 5.1, in order to compute the cohomology of $\mathfrak{g}_{Ad_\rho}$ the next step is to decompose it as sum of $V_{n_1,n_2}$.

We start with some preliminaries in the holomorphic case, i.e. $n_2 = 0$. Recall that

$$V_{n,0} = \{P(X,Y) \in \mathbf{C}[X,Y] \mid P \text{ homogeneous and } \deg P = n\}.$$

As vector space, we view $V_{n,0}$ as its own tangent space and we consider the action of the Lie algebra

$$\mathfrak{sl}_2(\mathbf{C}) \curvearrowright V_{n,0}.$$

Consider the standard basis for $\mathfrak{sl}_2(\mathbf{C})$:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We also write, for $i = 0, \dots, n$,

$$e_i = X^{n-i}Y^i$$

so that

$$\{e_0, e_1, \dots, e_n\}$$

is a basis for $V_{n,0}$.

A straightforward computation gives that the $e_i$ are eigenvectors for $h$:

$$h \cdot e_i = (n - 2i)e_i$$

Those are the weights, and the maximal weight of the representation is $n$. We also may compute

$$f \cdot e_i = (n - i)e_{i+1} \tag{3}$$

$$g \cdot e_i = ie_{i-1} \tag{4}$$

with the convention that $e_{-1} = e_{n+1} = 0$.

**Proposition 6.1** (Clebsch-Gordan formula).

$$V_{n,0} \otimes V_{n,0} = \bigoplus_{i=0}^{n} V_{2i,0}.$$

Though the proof is well known, we give it in order to understand the decompositions of $\mathfrak{g}$ that we give later.

*Proof.* The idea in representation theory is to look at the roots, namely at the eigenvectors and eigenvalues of the action of $h$. Consider the basis

$$\{e_i \otimes e_j\}_{0 \le i,j \le n}$$

for $V_{n,0} \otimes V_{n,0}$. Knowing that $h \cdot e_i = (n - 2i)e_i$, we have:

$$h \cdot (e_i \otimes e_j) = (h \cdot e_i) \otimes e_j + e_i \otimes (h \cdot e_j) = 2(n - i - j)e_i \otimes e_j.$$

Thus the eigenvalues of the action of $h$ are given by the following table:

| $\otimes$ | $e_0$ | $e_1$ | $\cdots$ | $e_n$ |
|---|---|---|---|---|
| $e_0$ | $2n$ | $2n-2$ | | $0$ |
| $e_1$ | $2n-2$ | $2n-4$ | $\cdots$ | $-2$ |
| $e_2$ | $2n-4$ | $2n-6$ | | $-4$ |
| $\vdots$ | $\vdots$ | | | $\vdots$ |
| $e_n$ | $0$ | $-2$ | | $-2n$ |

The largest eigenvalue is $2n$, which means that $V_{2n,0}$ has to appear once in the decomposition into irreducible factors. The next largest eigenvalue is $2n-2$, which appears twice, one for $V_{2n,0}$ and the other must be for $V_{2n-2,0}$. Notice that by looking at the action of $f$ and $g$, we can describe the eigenvectors: since $e_0 \otimes e_0$ is the eigenvector of eigenvalue $2n$ in $V_{2n,0}$, $f(e_0 \otimes e_0)$ is the eigenvector in $V_{2n,0}$ of eigenvalue $2n-2$. In addition, the eigenvector in $V_{2n-2,0}$ must lie in the kernel of $g$. More explicitly, $e_0 \otimes e_1$ and $e_1 \otimes e_0$ span the eigenspace with eigenvalue $2n-2$, and:

$$f \cdot (e_0 \otimes e_0) = (f \cdot e_0) \otimes e_0 + e_0 \otimes (f \cdot e_0) = n(e_0 \otimes e_1 + e_1 \otimes e_0) \in V_{2n,0}.$$

In addition, since $g \cdot e_0 = 0$ and $g \cdot e_1 = e_0$:

$$g \cdot (e_0 \otimes e_1 - e_1 \otimes e_0) = e_0 \otimes (g \cdot e_1) - (g \cdot e_1) \otimes e_0 = 0$$

therefore $e_0 \otimes e_1 - e_1 \otimes e_0 \in V_{2n-2}$. Without explicitly describing the eigenspaces, the argument can be carried out to conclude the lemma. □

We can already apply Clebsch-Gordan decomposition to $\mathfrak{sl}_{n+1}(\mathbf{C})$. Since $V_{n,0}^* \cong V_{n,0}$ we deduce that

$$\mathfrak{gl}_{n+1}(\mathbf{C})_{Ad\rho_{n,0}} \cong V_{n,0}^* \otimes V_{n,0} \cong V_{n,0} \otimes V_{n,0} = \bigoplus_{i=0}^{n} V_{2i,0}. \tag{5}$$

In addition, since

$$\mathfrak{gl}_{n+1}(\mathbf{C})_{Ad\rho_{n,0}} \cong \mathfrak{sl}_{n+1}(\mathbf{C})_{Ad\rho_{n,0}} \oplus \mathbf{C} \cong \mathfrak{sl}_{n+1}(\mathbf{C})_{Ad\rho_{n,0}} \oplus V_{0,0}, \tag{6}$$

we deduce

$$\mathfrak{sl}_{n+1}(\mathbf{C})_{Ad\rho_{n,0}} \cong \bigoplus_{i=1}^{n} V_{2i,0}. \tag{7}$$

*Proof of Theorem 3.3.* By the decomposition in Equation (7), the cohomology splits

$$H^1(M^3, \mathfrak{sl}_{n+1}(\mathbf{C})_{Ad\rho_{n,0}}) \cong \bigoplus_{i=1}^{n} H^1(M^3, V_{2i,0}).$$

Now, since $M^3$ is closed and $i \geq 1$. Raghunathan's vanishing applies to conclude that $H^1(M^3, \mathfrak{sl}_{n+1}(\mathbf{C})_{Ad\rho_{n,0}}) = 0$. □

## 7. Decomposing according to the bilinear product

We recall the invariant bilinear form

$$\Phi : V_{n,0} \otimes V_{n,0} \to \mathbf{C}.$$

For $n = 1$, $\Phi$ is just the determinant, so it has matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since $V_{n,0}$ is the $n$-th symmetric product of $V_{1,0}$, the matrix of $\Phi$ on $V_{n,0}$ is

$$J = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & -1 & 0 \\ \vdots & & & \vdots \\ (-1)^n & \cdots & 0 & 0 \end{pmatrix}$$

which is antisymmetric for $n$ odd and symmetric for $n$ even. The Lie algebra of the subgroup $G$ of $J$-isometries then is

$$\mathfrak{g} = \{a \in \mathfrak{gl}_{n+1}(\mathbf{C}) \mid a^t J + Ja = 0\}.$$

In fact we need to compute the $J$-antisymmetric part and the $J$-symmetric part.

**Definition 7.1.** We say that $a \in \mathfrak{gl}_{n+1}(\mathbf{C})$ is:

- $J$-symmetric if $a^t J - Ja = 0$, and
- $J$-antisymmetric if $a^t J + Ja = 0$.

The Lie algebra $\mathfrak{gl}_{n+1}(\mathbf{C})$ is the direct sum of its $J$-symmetric and its $J$-antisymmetric part. Since $J$ is preserved by $\rho_{n,0}$, the $J$-symmetric and $J$-antisymmetric part are preserved, thus the irreducible factors in the decomposition (5),

$$\mathfrak{gl}_{n+1}(\mathbf{C})_{Ad\rho_{n,0}} \cong \bigoplus_{i=0}^{n} V_{2i,0},$$

are either $J$-symmetric or $J$-antisymmetric.

**Proposition 7.2.** *Let $V_{2i,0}$ be one of the irreducible factors in the decomposition* (5) *of $\mathfrak{gl}_{n+1}(\mathbf{C})_{Ad\rho_{n,0}}$. Then:*

- *$V_{2i,0}$ is $J$-symmetric if $i$ is even,*
- *$V_{2i,0}$ is $J$-antisymmetric if $i$ is odd.*

To prove the proposition, we first need the following lemma, whose proof is a straightforward computation:

**Lemma 7.3.** *The endomorphism*

$$\begin{array}{rcl} \mathfrak{sl}_{n+1}(\mathbf{C}) & \to & \mathfrak{sl}_{n+1}(\mathbf{C}) \\ a & \mapsto & J^{-1}a^t J \end{array}$$

*(where $a^t$ denotes the transpose) has the following expression in coordinates*

$$(a_{i,j})_{ij} \mapsto ((-1)^{i+j} a_{n-j,n-i})_{ij}.$$

Notice that up to sign this endomorphism is the symmetry with respect to the antidiagonal. As a consequence of the lemma, the matrices in $\mathfrak{gl}_{n+1}(\mathbf{C})$ satisfy:

- $a \in \mathfrak{gl}_{n+1}(\mathbf{C})$ is $J$-symmetric iff
$$a_{i,j} = (-1)^{i+j} a_{n-j,n-i}, \qquad \forall i,j = 0,\dots,n.$$
- $a \in \mathfrak{gl}_{n+1}(\mathbf{C})$ is $J$-antisymmetric iff
$$a_{i,j} = (-1)^{i+j+1} a_{n-j,n-i}, \qquad \forall i,j = 0,\dots,n.$$

Consider the antidiagonal of such a matrix, namely when $i + j = n$. Then:

- If $a \in \mathfrak{gl}_{n+1}(\mathbf{C})$ is $J$-symmetric then
$$a_{i,n-i} = (-1)^n a_{i,n-i}, \qquad \forall i = 0,\dots,n.$$
- If $a \in \mathfrak{gl}_{n+1}(\mathbf{C})$ is $J$-antisymmetric then
$$a_{i,n-i} = (-1)^{n+1} a_{i,n-i}, \qquad \forall i = 0,\dots,n.$$

Thus we deduce:

**Remark 7.4.**

- When $n$ is even the antidiagonal belongs to the $J$-symmetric part.

- When $n$ is odd, the antidiagonal belongs to the $J$-antisymmetric part.

*Proof of Proposition 7.2.* We look at the weights in the proof of Proposition 6.1. Here we must care of the ordering and the fact that we work with he dual in the tensor product

$$\mathfrak{gl}_{n+1}(\mathbf{C}) = V_{n,0} \otimes V_{n,0}^*.$$

If we use the bilinear form for the isomorphism $V_{n,0} \cong V_{n,0}^*$, the vector $e_i = X^{n-i}Y^i$ is mapped to $\pm e_{n-i} = \pm X^i Y^{n-i}$, as $X$ and $Y$ are dual up to sign. Thus the weight of $e_i^*$ is minus the weight of $e_i$. The eigenvectors of $h$ in $\mathfrak{gl}_{n+1}(\mathbf{C}) = V_{n,0} \otimes V_{n,0}^*$ are precisely $e_i \otimes e_j^*$, namely the entries of a matrix, and the weights are given by the following table:

| $\otimes$ | | $e_0^*$ $(-n)$ | $e_1^*$ $(-n+2)$ | $e_2^*$ $(-n+4)$ | $\cdots$ $\cdots$ | $e_{n-1}^*$ $(n-2)$ | $e_n^*$ $(n)$ |
|---|---|---|---|---|---|---|---|
| $e_0$ | $(n)$ | $0$ | $2$ | $4$ | | $2n-2$ | $2n$ |
| $e_1$ | $(n-2)$ | $-2$ | $0$ | $2$ | $\cdots$ | $2n-4$ | $2n-2$ |
| $e_2$ | $(n-4)$ | $-4$ | $-2$ | $0$ | | $2n-6$ | $2n-4$ |
| $\vdots$ | | | $\vdots$ | | | | $\vdots$ |
| $e_{n-1}$ | $(-2n+2)$ | $-2n+2$ | $-2n+4$ | $-2n+6$ | $\cdots$ | $0$ | $2$ |
| $e_n$ | $(-n)$ | $-2n$ | $-2n+2$ | $-2n+4$ | | $-2$ | $0$ |

By Lemma 7.3, it suffices to describe the weights on the upper left triangle of this matrix (i.e. above the antidiagonal) or the lower right triangle (i.e. below the antidiagonal). Moreover for $n$ even the antidiagonal goes to the symmetric part and for $n$ odd it goes to the antisymmetric one. Notice that by symmetry, being upper left of lower right is not relevant, what makes the difference is whether the antidiagonal is contained or not. Thus we shall use the notation *large* triangle and *small* triangle according to whether it contains the antidiagonal or not.

For $n = 1$, the weights of $\mathfrak{gl}_2(\mathbf{C})$ are

$$\begin{matrix} 0 & 2 \\ -2 & 0 \end{matrix}.$$

Since 1 is odd the large triangle goes to the $J$-antisymmetric part, and the small one to the symmetric part. The triangles are:

$$\begin{matrix} 0 & 2 \\ -2 & \end{matrix} \quad \text{and} \quad 0.$$

The weights of the $J$-antisymmetric part (the large triangle) are precisely the weights $\{-2, 0, 2\}$ of $V_{2,0}$, and the for small one are $\{0\}$, namely $V_{0,0}$.

For $n = 2$, the weights of $\mathfrak{gl}_3(\mathbf{C})$ are a matrix that we may view as obtained from the previous one by adding a bottom row and a right most column

$$\begin{matrix} 0 & 2 & \mathbf{4} \\ -2 & 0 & \mathbf{2} \\ \mathbf{-4} & \mathbf{2} & \mathbf{0} \end{matrix}$$

Since 2 is even, the antidiagonal goes to the $J$-symmetric part, that we assume lower right. The decomposition is:

$$\begin{matrix} 0 & 2 \\ -2 & \end{matrix} \quad \text{and} \quad \begin{matrix} & & \mathbf{4} \\ & 0 & \mathbf{2} \\ \mathbf{-4} & \mathbf{2} & \mathbf{0} \end{matrix}.$$

Thus the $J$-antisymmetric part for $n=2$ is the same as for $n=1$, but to the $J$-symmetric part we have added the weights in boldface, that are precisely those of $V_{4,0}$. Thus the $J$-antisymmetric part is $V_{2,0}$ and the $J$-symmetric part is $V_{0,0} \oplus V_{4,0}$.

For $n=3$, we view the weights of $\mathfrak{sl}_3(\mathbf{C})$ as obtained from those of $\mathfrak{sl}_2(\mathbf{C})$ by adding a top row and a leftmost column:

$$\begin{array}{cccc} \mathbf{0} & \mathbf{2} & \mathbf{4} & \mathbf{6} \\ \mathbf{-2} & 0 & 2 & 4 \\ \mathbf{-4} & -2 & 0 & 2^{\cdot} \\ \mathbf{-6} & -4 & 2 & 0 \end{array}$$

Now the antidiagonal goes to the $J$-antisymmetric part. Thus the decomposition of triangles is

$$\begin{array}{cccc} \mathbf{0} & 2 & 4 & \mathbf{6} \\ \mathbf{-2} & 0 & 2 & \\ \mathbf{-4} & -2 & & \\ \mathbf{-6} & & & \end{array} \quad \text{and} \quad \begin{array}{ccc} & & 4 \\ & 0 & 2^{\cdot} \\ -4 & 2 & 0 \end{array}$$

Thus we have just added the weights of $V_{6,0}$ to the $J$-antisymmetric part. Therefore the $J$-symmetric part is $V_{0,0} \oplus V_{4,0}$ and the $J$-antisymmetric part is $V_{2,0} \oplus V_{6,0}$.

As we increase the $n$ of $\rho_{n,0}$ we repeat this pattern:

- When $n$ is even, the weights of $\mathfrak{gl}_{n+1}(\mathbf{C})$ are obtained by adding on the right and the bottom the weights of $V_{2n}$ to those of $\mathfrak{gl}_n(\mathbf{C})$. As the antidiagonal goes to the symmetric part, the weights of $V_{2n}$ are added to the symmetric part.
- When $n$ is odd, the weights of $\mathfrak{gl}_{n+1}(\mathbf{C})$ are obtained by adding on the left and the top the weights of $V_{2n}$ to those of $\mathfrak{gl}_n(\mathbf{C})$. Now the antidiagonal goes to the $J$-antisymmetric part, hence the weights of $V_{2n}$ are added to the $J$-antisymmetric part, while the $J$-symmetric part remains the same.

This proves inductively that the decomposition of $\mathfrak{gl}_{n+1}(\mathbf{C})$ into $J$-symmetric and $J$-antisymmetric parts correspond to factors $V_{2i,0}$ with $i$ even and odd respectively. This proves the lemma. □

## 8. Decomposing representations in general

Now we have all ingredients to compute cohomology groups we are interested in. Recall that the image of $\rho_{n_1,n_2}$ is contained in

$$G = \begin{cases} SO((n_1+1)(n_2+1), \mathbf{C}) & \text{if } n_1+n_2 \text{ is even} \\ Sp(\frac{(n_1+1)(n_2+1)}{2}, \mathbf{C}) & \text{if } n_1+n_2 \text{ is odd.} \end{cases}$$

The Lie algebra of $G$ is

$$\mathfrak{g} = \begin{cases} \mathfrak{so}((n_1+1)(n_2+1), \mathbf{C}) & \text{if } n_1+n_2 \text{ is even} \\ \mathfrak{sp}(\frac{(n_1+1)(n_2+1)}{2}, \mathbf{C}) & \text{if } n_1+n_2 \text{ is odd.} \end{cases}$$

**Proposition 8.1.** *For $\mathfrak{sl}_{(n_1+1)(n_2+1)}(\mathbf{C})$ we have*

$$\mathfrak{sl}_{(n_1+1)(n_2+1)}(\mathbf{C})_{Ad\rho_{n_1,n_2}} = \bigoplus_{\substack{0\leq i\leq n_1 \\ 0\leq j\leq n_2 \\ (i,j)\neq(0,0)}} V_{2i,2j}.$$

*For $\mathfrak{g}$ as above, we have*

$$\mathfrak{g}_{Ad\rho_{n_1,n_2}} = \bigoplus_{\substack{0\leq i\leq n_1\\ 0\leq j\leq n_2\\ i+j \text{ odd}}} V_{2i,2j}.$$

*Proof.* Since

$$\mathrm{Sym}_{n_1,n_2} = \mathrm{Sym}_{n_1,0} \otimes \mathrm{Sym}_{0,n_2}$$

the decomposition of $\mathfrak{sl}_{(n_1+1)(n_2+1)}(\mathbf{C})_{Ad\rho_{n_1,n_2}}$ follows from (5). To get the decomposition of $\mathfrak{g}_{Ad\rho_{n_1,n_2}}$, we notice that it is the sum of factors in the decomposition of $\mathfrak{sl}_{(n_1+1)(n_2+1)}(\mathbf{C})$ that are $J$-antisymmetric. Since the form on $V_{n_1,n_2}$ is also a tensor product, this is a straightforward consequence of Proposition 7.2. □

Now we can already prove some of the results of the introduction.

*Proof of Theorem 3.1.* By Proposition 8.1

$$H^1(M^3, \mathfrak{g}_{Ad\rho_{n_1,n_2}}) = \bigoplus_{\substack{0\leq i\leq n_1\\ 0\leq j\leq n_2\\ i+j \text{ odd}}} H^1(M^3, V_{2i,2j}).$$

Since $i+j$ is odd in this summation, $i \neq j$ and by Raghunathan's vanishing theorem (Theorem 5.1) we have

$$H^1(M^3, V_{2i,2j}) = 0.$$

Hence

$$H^1(M^3, \mathfrak{g}_{Ad\rho_{n_1,n_2}}) = 0,$$

which proves the theorem. □

*Proof of Proposition 3.5.* By Proposition 8.1

$$\mathfrak{sl}_{(m+1)(n+1)}(\mathbf{C})_{Ad\rho_{m,n}} = \bigoplus_{\substack{0\leq i\leq m\\ 0\leq j\leq n}} V_{2i,2j}.$$

Since $H^*(M^3, V_{2i,2j}) = 0$ by Raghunathan's vanishing theorem, assuming $m \leq n$, we get:

$$H^1(M^3, \mathfrak{sl}_{(m+1)(n+1)}(\mathbf{C})_{Ad\rho_{m,n}}) = \bigoplus_{0\leq i\leq m} H^1(M^3, V_{2i,2i});$$

hence

$$H^1(M^3, \mathfrak{sl}_{(m+1)(n+1)}(\mathbf{C})_{Ad\rho_{m,n}}) \cong H^1(M^3, \mathfrak{sl}_{(m+1)^2}(\mathbf{C})_{Ad\rho_{m,m}}).$$

Namely the value of $n$ is not relevant provided it is larger or equal than $m$, which proves the proposition. □

## 9. Real representations

We consider now the representation

$$V_{n,n} = V_{n,0} \times V_{0,n} = V_{n,0} \times \overline{V_{n,0}},$$

which is invariant under complex conjugation. Hence we may take its real part:

$$W_n := \{P(X,Y,\overline{X},\overline{Y}) \in V_{n,n} \mid \overline{P(X,Y,\overline{X},\overline{Y})} = P(X,Y,\overline{X},\overline{Y})\}$$

which is invariant, namely it is a real representation.

We start by looking at the behaviour of the bilinear form:

**Proposition 9.1.** *The bilinear form $\Phi$ restricted to $W_n$ takes real values and has signature*

$$(p,q) = \left(\frac{n^2+3n+2}{2}, \frac{n^2+n}{2}\right).$$

**Remark 9.2.** Notice that for $n=1$, $(p,q)=(3,1)$ and in fact this gives the isomorphism

$$PSL_2(\mathbf{C}) \cong \mathrm{Isom}^+(\mathbf{H}^3) \cong SO_0(3,1).$$

*Proof.* We consider the following three families of elements:

(8) $$X^kY^{n-k}\overline{X^kY^{n-k}}, \qquad \text{for } k=0,\dots n;$$

(9) $$X^kY^{n-k}\overline{X^lY^{n-l}} + X^lY^{n-l}\overline{X^kY^{n-k}}, \qquad \text{for } k,l=0,\dots n,\ k\neq l;$$

(10) $$i\left(X^kY^{n-k}\overline{X^lY^{n-l}} - X^lY^{n-l}\overline{X^kY^{n-k}}\right), \qquad \text{for } k,l=0,\dots n,\ k\neq l.$$

Their union is a basis for $W_n$, and $\Phi$ takes real values on them (notice that elements in (10) are orthogonal to the ones in (8) and (9)).

We use these families to describe the signature. We group them in subspaces that are orthogonal and then we count their contribution to the signature.

- Assume first $n$ is *even*. We group the elements in (8), (9) and (10) as follows:
  (a) When $k=n/2$, the element of (8) is self dual. It contributes to the signature as
  $$(1,0).$$
  (b) When $k\neq n/2$, then the dual of an element in (8) is obtained by replacing $k$ by $n-k$. Thus we obtain $n/2$ blocks $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Hence their contribution to the signature is
  $$\left(\frac{n}{2}, \frac{n}{2}\right).$$
  (c) When $l+k=n$, then the $\frac{n}{2}$ elements of (9) are self dual, and so for (10) (notice that elements of (9) and (10) are orthogonal). Hence their contribution to the signature is
  $$(n,0).$$
  (d) Finally, when $l+k\neq n$, then the elements of (9) and their dual (obtained my replacing $k$ by $n-k$ and $l$ by $n-l$) give a block $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Similarly for elements of (10). In the previous items (a), (b) and (c) we have a total of $2n+1$ elements, hence we have $(n+1)^2-(2n+1)=n^2$ elements remaining. Their contribution to signature is therefore
  $$\left(\frac{n^2}{2}, \frac{n^2}{2}\right).$$

  Adding up all four contributions we get $\left(\frac{n^2+3n+2}{2}, \frac{n^2+n}{2}\right)$, as claimed.
- Assume now that $n$ is odd. The grouping is simpler, as the case $k=n/2$ does not occur:
  (e) The elements of (8) must be counted as in item (b) of the even case, as $k$ is never $n/2$. Thus we have $n+1$ elements that contribute
  $$\left(\frac{n+1}{2}, \frac{n+1}{2}\right).$$

(f) When $l+k=n$, then the $\frac{n+1}{2}$ elements of (9) are self dual, and so for (10), similarly as (c) in the even case. So their contribution to signature is

$$(n+1,0).$$

(g) Finally, when $l+k\neq n$, then elements of (9) and (10) have a contribution that must be computed as in item (d) in the even case. Here the number of elements is $(n+1)^2-2(n+1)=n^2-1$, so their contribution to signature is:

$$\left(\frac{n^2-1}{2},\frac{n^2-1}{2}\right).$$

Adding up all three contributions we obtain again $\left(\frac{n^2+3n+2}{2},\frac{n^2+n}{2}\right)$.

□

**Lemma 9.3.** *The module $W_n$ has a proper subspace where $SL_2(\mathbf{R})$ acts trivially.*

*Proof.* For $n=1$ this is a consequence that $W_1$ is the representation that identifies $PSL_2(\mathbf{C})$ with $SO(3,1)$. Hence the image of $SL_2(\mathbf{R})$ is contained in $SO(2,1)$ in the embedding

$$\begin{pmatrix} SO(2,1) & 0 \\ 0 & 1 \end{pmatrix} \subset SO(3,1).$$

Thus it acts trivially on a line. The invariant polynomial in $V_{1,1}$ can be given explicitly:

$$P(X,Y,\overline{X},\overline{Y})=X\overline{Y}-Y\overline{X}\in V_{1,1}.$$

Namely, for $A\in SL_2(\mathbf{R})$,

$$P\circ A^t=P.$$

Notice also that $i\,P\in W_1$. Now,

$$i^nP^n\in W_n\subset V_{n,n}$$

is a nontrivial element invariant by the action of $SL_2(\mathbf{R})$. □

*Proof of Corollary 3.6.* Notice that $V_{n,n}=W_n\otimes\mathbf{C}$ and that

$$\mathfrak{so}((n+1)^2,\mathbf{C})=\mathfrak{so}(p,q)\otimes\mathbf{C}$$

as $Ad_{\rho_{n,n}}$-modules. Thus from the infinitesimal rigidity for $\mathfrak{so}((n+1)^2,\mathbf{C})$,

$$H^1(M^3,\mathfrak{so}((n+1)^2,\mathbf{C}))=0,$$

which implies

$$H^1(M^3,\mathfrak{so}(p,q))=0;$$

namely infinitesimal rigidity in $SO(p,q)$.

To prove that it can be deformed in $SL_{(n+1)^2}(\mathbf{R})$, we use Lemma 9.3 and we construct bending. Namely, assume that the surface $F$ separates $M^3$ in two components $M_1$ and $M_2$. Then $\pi_1(M^3)$ is an amalgamated product

$$\pi_1(M^3)\cong\pi_1(M_1)*_{\pi_1(F)}\pi_1(M_2).$$

By Lemma 9.3, there exist a non trivial 1-parameter group $a_t\in SL_{(n+1)^2}(\mathbf{R})$ that commutes with the image of $\pi_1(F)$ (take for instance dilatations in the subspace invariant by the image of $F$, and normalize them to have determinant 1). Then define the deformation $\rho_t$ as:

$$\rho_t(\gamma) = \begin{cases} \rho(\gamma) & \text{for } \gamma \in \pi_1(M_1), \\ a_t\rho(\gamma)a_t^{-1} & \text{for } \gamma \in \pi_1(M_2). \end{cases}$$

This deformation is non trivial, $\rho_t$ is not conjugate to $\rho_0$ for $t \neq 0$, because the image of $\pi_1(M_i)$ in $SL_2(\mathbf{C})$ is Zariski closed (use Chen-Greenberg's theorem [15]) and $\mathrm{Sym}_{n,n}$ is irreducible. See [25] for details.

When $F$ does not separate $M^3$, we use the HNN structure of the group. Let $M_0$ be the result of cutting of $M^3$ along $F$, so that $\partial M_0$ consists of two copies of $F$, and $M^3 \setminus M_0 = F \times (0,1)$. Then

$$\pi_1(M^3) \cong \pi_1(M_0)*_{\pi_1(F)} = \pi_1(M^3) * \langle\tau\rangle / \langle i_{0*}(\gamma) = \tau i_{1*}(\gamma)\tau^{-1} \mid \gamma \in \pi_1(F)\rangle,$$

where $i_0, i_1 : \pi_1(F) \to \pi_1(M_0)$ are the inclusions at the boundary components of $M_0$. Again, by Lemma 9.3, there exist a non trivial 1-parameter group $a_t \in SL_{(n+1)^2}(\mathbf{R})$ that commutes with the image of $\pi_1(F)$ and define the deformation $\rho_t$ as:

$$\rho_t(\gamma) = \rho(\gamma) \text{ for } \gamma \in \pi_1(M_0),$$
$$\rho_t(\tau) = \rho(\tau).$$

Again $\rho_t$ is not conjugate to $\rho_0$ for $t \neq 0$, because the image of $\pi_1(M_i)$ in $SL_2(\mathbf{C})$ is Zariski closed and $\mathrm{Sym}_{n,n}$ is irreducible. See again [25] for details. □

Notice that the deformation also implies the infinitesimal deformability. In fact we may prove directly:

**Lemma 9.4.** *If $M^3$ contains a totally geodesic surface, then*

$$H^1(M^3, V_{n,n}) \neq 0$$

*for $n \geq 1$.*

Notice that this is equivalent to saying that

$$H^1(M^3, W_n) \neq 0,$$

as $V_{n,n} = W_n \otimes \mathbf{C}$. This is proved by Millson in [34] and we follow his proof.

*Proof.* By Lemma 9.3, $V_{n,n}$ has a subspace where $SL_2(\mathbf{R})$ acts trivially. Let $F \subset M^3$ be the totally geodesic subsurface of $M^3$. In particular its holonomy representation is contained in $PSL_2(\mathbf{R})$, and $V_{n,n}$ has nontrivial elements invariant by the action of $\pi_1 F$, thus:

$$H^0(F, V_{n,n}) \neq 0.$$

Now the proof follows from a Mayer-Vietoris argument. Assume first that $F$ separates $M^3$ into two components $M_1$ and $M_2$. Firstly the holonomy of $M_i$ is Zariski dense in $PSL_2(\mathbf{C})$ (use again Chen-Greenberg [15]) hence

$$H^0(M_1, V_{n,n}) = H^0(M_2, V_{n,n}) = 0.$$

Thus Mayer-Vietoris to the pair $(M_1, M_2)$ gives:

$$0 \to H^0(F, V_{n,n}) \to H^1(M^3, V_{n,n}),$$

which implies $H^1(M^3, V_{n,n}) \neq 0$.

When $F$ does not separate, the argument is similar. Namely, let $M_0$ be the result of cutting off $M^3$ along $F$, so that $M^3 = M_0 \cup (F \times [0,1])$ and $M_0 \cap (F \times [0,1]) = F \times \{0,1\}$. As before the holonomy of $M_0$ is Zariski dense in $PSL_2(\mathbf{C})$, hence

$$H^0(M_0, V_{n,n}) = 0.$$

Again Mayer-Vietoris gives

$$0 \to H^0(F, V_{n,n}) \to H^0(F, V_{n,n}) \oplus H^0(F, V_{n,n}) \to H^1(M^3, V_{n,n}),$$

so $H^1(M^3, V_{n,n}) \neq 0$. □

*Proof of Theorem 3.4.* Use Lemma 9.4 and Proposition 3.5. □

## 10. Complex hyperbolic structures

The real representation of previous section

$$\rho_{n,n} : \pi_1(M^3) \to SO(p,q)$$

may also be considered in the special unitary group by composing it with the natural embedding

$$\rho_{n,n} : \pi_1(M^3) \to SO(p,q) \subset SU(p,q).$$

Recall $\mathfrak{so}(p,q)$ is the subalgebra of $\mathfrak{sl}_{(n+1)^2}(\mathbf{R})$ consisting of matrices that are $J$-antisymmetric. If $\mathfrak{sl}_{(n+1)^2}(\mathbf{R})^{J-\mathrm{sym}}$ denotes the subspace of $J$-symmetric ones, then we have a decomposition of $\pi_1(M^3)$-modules:

$$\mathfrak{sl}_{(n+1)^2}(\mathbf{R})_{Ad\rho_{n,n}} = \mathfrak{so}(p,q)_{Ad\rho_{n,n}} \oplus \mathfrak{sl}_{(n+1)^2}(\mathbf{R})^{J-\mathrm{sym}}_{Ad\rho_{n,n}}.$$

If we now combine $J$ with complex conjugation we have that

$$\mathfrak{su}(p,q) = \{a \in \mathfrak{sl}_{(n+1)^2}(\mathbf{C}) \mid \overline{a^t} J = -Ja\}.$$

Taking real an imaginary parts, we obtain:

**Lemma 10.1.** *There is a natural isomorphism of $\pi_1(M^3)$-modules:*

$$\mathfrak{su}(p,q) = \mathfrak{so}(p,q) \oplus i\mathfrak{sl}_{(n+1)^2}(\mathbf{R})^{J-sym}.$$

**Corollary 10.2.** *There is a natural isomorphism of real vector spaces*

$$H^*(M^3, \mathfrak{sl}_{(n+1)^2}(\mathbf{R})_{Ad\rho_{n,n}}) \cong H^*(M^3, \mathfrak{su}(p,q)_{Ad\rho_{n,n}}).$$

In particular, for $n = 1$ we get $(p,q) = (3,1)$, thus:

**Corollary 10.3.** *The space of infinitesimal projective deformations of a hyperbolic three manifold is isomorphic to its space of infinitesimal complex hyperbolic deformations.*

We also have the following proposition (which was first noticed by Cooper, Long and Thistlethwaite [18]).

**Proposition 10.4.** *The following are equivalent:*

- $\rho_{n,n}$ *is a smooth point of* $\hom(M^3, SL_{(n+1)^2}(\mathbf{R}))$ ,
- $\rho_{n,n}$ *is a smooth point of* $\hom(M^3, SU(p,q))$,
- $\rho_{n,n}$ *is a smooth point of* $\hom(M^3, SL_{(n+1)^2}(\mathbf{C}))$.

*Proof.* We prove first the equivalence between $SL_{(n+1)^2}(\mathbf{R})$ and $SL_{(n+1)^2}(\mathbf{C})$. For this, notice that $\hom(M^3, SL_{(n+1)^2}(\mathbf{R}))$ is an algebraic variety embedded in $SL_{(n+1)^2}(\mathbf{R})^N$ – here $N$ is the number of generators of $\pi_1(M^3)$ – which in its turn is embedded in $\mathbf{R}^{N(n+1)^4}$. With this embedding, $\hom(M^3, SL_{(n+1)^2}(\mathbf{C}))$ is just the complexification of $\hom(M^3, SL_{(n+1)^2}(\mathbf{R}))$, and it is the zero set in $\mathbf{C}^{N(n+1)^4}$ of the same family of polynomials (with real coefficients) as $\hom(M^3, SL_{(n+1)^2}(\mathbf{R}))$. Thus being singular or not depends on whether we can find a subset polynomials of the right cardinality with nonzero Jacobian, and this does not change whether the ambient space is $\mathbf{R}^{N(n+1)^4}$ or $\mathbf{C}^{N(n+1)^4}$.

The other equivalence is proved similarly, as $SU(p,q)$ is a real form of $SL_{(n+1)^2}(\mathbf{C})$. According to Onishchik and Vinberg [36], there are complex coordinates for $SL_{(n+1)^2}(\mathbf{C})$ so that the intersection with $\mathbf{R}^{(n+1)^4}$ gives $SU(p,q)$. Otherwise, one can follow the transversality argument of Cooper, Long, and Thistlethwaite in [18, Theorem 2.2]. □

## 11. Conformally flat structures

Now we are interested in the embedding

$$SO(3,1) \subset SO(4,1).$$

Notice that we have the decomposition of $SO(3,1)$ modules of the Lie algebra

$$\mathfrak{so}(4,1) = \mathfrak{so}(3,1) \oplus V_{1,1}. \tag{11}$$

**Definition 11.1.** A closed hyperbolic 3-manifold $M^3$ has an *infinitesimally rigid flat conformal structure* if $H^1(M^3, V_{1,1}) = 0$.

By [25, 26], manifolds with a totally geodesic surface do not have an infinitesimally rigid flat conformal structure, due to bending. Apanasov [3, 5], Apanasov and Tetenov [4], and Bart and Scannell [7] construct conformally flat deformations that are not bending (they are called stamping).

Dehn fillings on hyperbolic 3-manifolds have been studied by Kapovich [26]. Subsequently, Scannell [40] and Francavliglia and myself [19], we have improved the results, using basically the ideas of Kapovich:

**Theorem 11.2** (Francaviglia-P. [19]). *Let $M^3$ be a compact and oriented 3-manifold such that* $\operatorname{int}(M)$ *is hyperbolic, with one cusp and of finite volume. Assume $\pi_1(M^3)$ is generated by two peripheral elements (e.g. $M^3$ is the exterior of a two bridge knot).*

*Then almost all Dehn fillings of $M^3$ have an infinitesimally rigid flat conformal structure.*

In [26] Kapovich conjectures that local rigidity is equivalent to not having an embedded fuchsian surface (not necessarily totally geodesic). He gives evidence for this conjecture in several cases. In [22] Goldman shows that a hyperbolic 3-manifold with such a surface is globally nonrigid (though local rigidity is not known).

## 12. Non compact three manifolds of finite type

Let $M^3$ be a noncompact hyperbolic three manifold of finite type. Thus $M^3$ is topologically and geometrically tame, by the proof of Marden's conjecture. It has finitely many ends and it admits a compactification $\overline{M}^3$ such that $\partial\overline{M}^3$ consists of finitely many surfaces of genus $g \geq 1$. Among them the surfaces that are torus correspond to cusps, and the other ends have infinite volume. We will not discuss whether these ends are geometrically finite or not.

We shall consider the following groups $G$ and representations $\rho : \pi_1(M^3) \to G$.

- $\rho = \rho_{n,0}$ or $\rho_{0,n}$ and $G = SL_{(n+1)}(\mathbf{C})$.
- $\rho = \rho_{n_1,n_2}$ with $n_1 + n_2$ even and $G = SO((n_1+1)(n_2+1), \mathbf{C})$.
- $\rho = \rho_{n_1,n_2}$ with $n_1 + n_2$ odd and $G = Sp(\frac{(n_1+1)(n_2+1)}{2}, \mathbf{C})$.

Then Theorem 3.9 can be restated as follows.

**Theorem 12.1.** *Let $M^3$, $\rho$ and $G$ be as above, and let $k$ be the number of cusps. Then $\rho$ is a smooth point of $X(M^3, G)$ of local dimension*

$$-\chi(\overline{M}^3) \dim G + k \operatorname{rank} G.$$

For $\rho_{1,0}$ and $G = SL_2(\mathbf{C})$, this result is due to Kapovich [27] (see also Bromberg [12]). For $\rho = \rho_{n,0}$ or $\rho_{0,n}$ and $G = SL_{n+1}(\mathbf{C})$, it was proved by Menal-Ferrer and myself in [33]. All other cases seem to be new.

**Corollary 12.2.** *Let $M^3$ be as above, $k$ the number of cusps and $n \geq 1$. Then $\rho_{n,n}$ is a smooth point of $X(M^3, SO(p,q))$ of local dimension*

$$-\chi(\overline{M}^3) \dim SO(p,q) + k \operatorname{rank} SO(p,q).$$

We shall use Theorem 5.3 to prove that all deformations come from the boundary. First we need to compute the cohomology of each boundary component.

Let $F$ be a component of $\partial M^3$, its cohomology will depend on whether it is a torus or it is a surface of genus $\geq 2$. We consider the restriction of the holonomy and the corresponding $\rho_{n_1,n_2}$ as in (1).

**Lemma 12.3.** *If $F$ has genus $g(F) \geq 2$ and $n_1 \neq n_2$, then*

$$\dim_{\mathbf{C}} H^i(F; V_{n_1,n_2}) = \begin{cases} -\chi(F)(n_1+1)(n_2+1) & \textit{for } i = 1, \\ 0 & \textit{otherwise.} \end{cases}$$

**Lemma 12.4.** *Assume that $n_1 \neq n_2$, then $V_{n_1,n_2}$ has no nontrivial elements that are fixed by $SL_2(\mathbf{R})$.*

*Proof of Lemma 12.4.* We prove it by contradiction and assume that such a nontrivial fixed element exists. Then the argument of Lemma 9.4 (that proves that the cohomology of a closed 3-manifold containing a totally geodesic surface with coefficients $V_{n,n}$ is non zero) would apply to $V_{n_1,n_2}$. Hence there would exist closed three manifolds whose cohomology with coefficients in $V_{n_1,n_2}$ is nontrivial, contradicting Theorem 3.1. □

*Proof of Lemma 12.3.* By Lemma 12.4 $H^0(F, V_{n_1,n_2}) = 0$. Then the lemma follows from Poincaré duality and Euler characteristic, as

$$\sum_i (-1)^i \dim H^i(F, V_{n_1,n_2}) = \chi(F) \dim V_{n_1,n_2}.$$

□

Using the decomposition of $\mathfrak{g}$ in Proposition 8.1, we get:

**Corollary 12.5.** *If $F$ has genus $g(F) \geq 2$ and $\rho$ and $G$ are as in Theorem 12.1, then*

$$\dim H^i(F, \mathfrak{g}_{Ad_\rho}) = \begin{cases} -\chi(F) \dim G & \textit{for } i = 1, \\ 0 & \textit{otherwise.} \end{cases}$$

**Lemma 12.6.** *Let $\rho$ and $G$ be as in Theorem 12.1. The restriction $\rho|_{\pi_1 F}$ is a smooth point of $X(F,G)$, of local dimension*

$$-\chi(F)\dim G.$$

*Proof.* This is a consequence of the fact that $H^2(F, \mathfrak{g}_{Ad_\rho}) = 0$. This is proved for instance by Goldman in [20]. Namely, this vanishing implies that every infinitesimal deformation in $H^1(F, \mathfrak{g}_{Ad_\rho})$ can be integrated (the obstructions to integration live in $H^2(F, \mathfrak{g}_{Ad_\rho})$). Therefore $H^1(F, \mathfrak{g}_{Ad_\rho})$ is not only the Zariski tangent space but the actual tangent space to $X(F,G)$. Since the Zariski and the actual tangent space are the same, we have smoothness and their dimension is the local dimension of the variety. □

We next proceed with cusps, and we start similarly, computing the fixed subspaces of $V_{n_1,n_2}$. In this computation we do not require $n_1 \neq n_2$. Let $T^2$ be a boundary component of $\partial \overline{M}$ of genus one (ie. corresponding to a cusp).

**Lemma 12.7.** *The subspace of elements in $V_{n_1,n_2}$ fixed by $\rho_{n_1,n_2}(\pi_1(T^2))$ has dimension*

$$(\dim V_{n_1,n_2})^{\rho_{n_1,n_2}(\pi_1(T^2))} = 1.$$

*Proof.* The (real) Zariski closure of the lift of the holonomy of $\pi_1(T^2)$ is a unipotent subgroup $U \subset SL_2(\mathbf{C})$, $U \cong \mathbf{C}$. Up to conjugacy,

$$U = \left\{ \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \middle| \, \tau \in \mathbf{C} \right\} \cong \mathbf{C}.$$

Since $U$ is the $\mathbf{R}$-Zariski closure of the holonomy of $\pi_1(T^2)$ and the action is polynomial, the subspaces of fixed elements is the same for $\rho_{n_1,n_2}(\pi_1(T^2))$ and for $\mathrm{Sym}_{n_1,n_2}(U)$:

$$(V_{n_1,n_2})^{\rho_{n_1,n_2}(\pi_1(T^2))} = (V_{n_1,n_2})^{\mathrm{Sym}_{n_1,n_2}(U)}.$$

Notice that the action of $U$ is equivalent to the action of $\mathbf{C}$ on polynomials $P \in V_{n_1,n_2}$:

$$P(X,Y,\overline{X},\overline{Y}) \mapsto P(X, Y+\tau X, \overline{X}, \overline{Y+\tau X}),$$

where $\tau \in \mathbf{C}$. Invariance implies that $P$ does not have terms on $Y$ and $\overline{Y}$, hence it belongs to the span of $X^{n_1}\overline{X}^{n_2}$, which is one dimensional. □

**Lemma 12.8.** *Let $G$ and $\rho$ be as above. The number of summands $V_{i,j}$ in the decomposition of $\mathfrak{g}_{Ad\rho}$ in Proposition 8.1 equals* $\mathrm{rank}(G)$.

This lemma follows from a straightforward computations, because

- $\mathrm{rank}\, SL_{r+1}(\mathbf{C}) = r$,
- $\mathrm{rank}\, SO(2r, \mathbf{C}) = r$,
- $\mathrm{rank}\, SO(2r+1, \mathbf{C}) = r$, and
- $\mathrm{rank}\, Sp(r, \mathbf{C}) = r$.

Lemma 12.8 may probably be consequence of a more general fact, but I am not aware of it.

Combining Lemmas 12.7 and 12.8, we deduce:

**Corollary 12.9.** *Let $G$ and $\rho$ be as above. The dimension of the subspace of fixed elements of the Lie algebra equals the rank:*

$$\dim \mathfrak{g}^{Ad\rho(\pi_1(T^2))} = \mathrm{rank}(G).$$

As in Lemma 12.3 and its corollary, using Corollary 12.9, Poincaré duality, and the Euler characteristic we get:

**Lemma 12.10.** *Let $G$ and $\rho$ be as above. Then*

$$\dim_{\mathbb{C}} H^i(T^2;\mathfrak{g}_{Ad\rho}) = \begin{cases} \operatorname{rank} G, & \text{for } i=0,2, \text{ and} \\ 2\operatorname{rank} G, & \text{for } i=1. \end{cases}$$

**Lemma 12.11.** *Let $\rho$ and $G$ be as in Theorem 12.1. The restriction $\rho|_{\pi_1(T^2)}$ is a smooth point of* $\hom(\pi_1(T^2),G)$.

*Proof.* Here the second cohomology does not vanish, and we must apply an argument different from the higher genus case. By the computations of dimensions of Lemma 12.10, we get

$$\begin{aligned}\dim Z^1(T^2,\mathfrak{g}_{Ad\rho}) = \dim H^1(T^2,\mathfrak{g}_{Ad\rho}) + B^1(T^2,\mathfrak{g}_{Ad\rho}) &= 2\operatorname{rank} G + (\dim G - \operatorname{rank} G)\\ &= \operatorname{rank} G + \dim G.\end{aligned}$$

On the other hand, if we want to compute the dimension of components of $\hom(\pi_1(T^2),G)$, we observe that one of the generators of $\pi_1(T^2)$ can be an arbitrary element of $G$, and the other element must commute with it. Thus the dimension of each component of $\hom(\pi_1(T^2),G)$ is bounded below by

$$\dim G + \operatorname{rank} G.$$

Thus

$$\begin{aligned}\dim G + \operatorname{rank} G \le \dim\left(\hom(\pi_1(T^2),G)\right) &\le \left(\dim T^{Zar}\hom(\pi_1(T^2),G)\right)\\ &= \dim Z^1(T^2,\mathfrak{g}_{Ad\rho}) = \dim G + \operatorname{rank} G,\end{aligned}$$

which gives equality of dimensions and smoothness. □

*Proof of Theorem 12.1.* Given a Zariski tangent vector $v \in Z^1(M^3,\mathfrak{g}_{Ad\rho_n})$, we have to show that it is integrable, i.e. that there is a path in the variety of representations whose tangent vector is $v$. For this, we use the algebraic obstruction theory, see [21, 23]. There exist an infinite sequence of obstructions that are cohomology classes in the second cohomology group, each obstruction being defined only if the previous one vanishes. These are related to the analytic expansion in power series of a deformation of a representation, and to Kodaira's theory of infinitesimal deformations. Our aim is to show that this infinite sequence vanishes. This gives a formal power series, that does not need to converge, but this is sufficient for $v$ to be a tangent vector by a theorem of Artin [6] (see [23] for details). We do not give the explicit construction of these obstructions, we just use that they are natural and that they live in the second cohomology group.

By Theorem 5.3 and since all $V_{n_1,n_2}$ that appear in the decomposition of $\mathfrak{g}_{Ad\rho}$ satisfy $n_1 \neq n_2$, for $p=1,2$ we have

$$\ker(H^p(\overline{M}^3,\mathfrak{g}_{Ad\rho}) \to H^p(\partial\overline{M}^3,\mathfrak{g}_{Ad\rho})) = 0, \tag{12}$$

because each cohomology class in this kernel is represented by a closed form in $M^3$ with compact support, in particular $L^2$. By looking at the long exact sequence of the pair

$$H^2(\overline{M}^3;\mathfrak{g}_{Ad\rho}) \cong H^2(\partial\overline{M}^3;\mathfrak{g}_{Ad\rho}). \tag{13}$$

Now, $H^2(\partial\overline{M}^3;\mathfrak{g}_{Ad\rho})$ decomposes as the sum of the connected components of $\partial\overline{M}^3$. If $F_g$ has genus $g\ge 2$ then $H^2(F_g;\mathfrak{g}_{Ad\rho})=0$. Thus, only the components of $\partial\overline{M}^3$ that are tori appear in $H^2(\partial\overline{M}^3;\mathfrak{g}_{Ad\rho})$. By Lemma 12.11 and naturality, the obstructions vanish when restricted to $H^2(T^2;\mathfrak{g}_{Ad\rho})$, hence they vanish in $H^2(M;\mathfrak{g}_{Ad\rho})$ by the isomorphism (13). This proves smoothness.

To compute the local dimension, by Corollary 4.5 this local dimension equals the dimension of $H^1(\overline{M}^3,\mathfrak{g}_{Ad\rho})$. By the injectivity of the maps in Equation (12), the long exact sequence in cohomology of the pair gives a short exact sequence

$$0\to H^1(\overline{M}^3;\mathfrak{g}_{Ad\rho})\to H^1(\partial\overline{M}^3;\mathfrak{g}_{Ad\rho})\to H^2(\overline{M}^3,\partial\overline{M}^3;\mathfrak{g}_{Ad\rho})\to 0.$$

Since $H^1(\overline{M}^3;\mathfrak{g}_{Ad\rho})$ is Poincaré dual to $H^2(\overline{M}^3,\partial\overline{M}^3;\mathfrak{g}_{Ad\rho})$,

$$\dim H^1(\overline{M}^3;\mathfrak{g}_{Ad\rho})=\frac{1}{2}\dim H^1(\partial\overline{M}^3;\mathfrak{g}_{Ad\rho}).$$

Now it suffices to count the contribution of each boundary component, from Corollary 12.5 and Lemma 12.10. Using these contributions and since $\chi(\overline{M}^3)=\frac{1}{2}\chi(\partial\overline{M}^3)$, we get that

$$\dim H^1(\overline{M}^3;\mathfrak{g}_{Ad\rho})=-\chi(\overline{M}^3)\dim G+k\operatorname{rank}G,$$

which concludes the proof of the theorem. □

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