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Stochastic optimal transportation problem and related topics

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First of all we briefly introduce two classes of Stochastic optimal transportation problems and the duality theorems. As applications, we consider marginal problems for Markov processes which is called Markov marginal problems. We also introduce the Knothe-Rosenblatt process as a stochastic optimal control analogue of the Knothe-Rosenblatt rearrangement and show that it can be approximated by a minimizer of a class of Stochastic optimal transportation problems with a small parameter. Finally we also introduce an application of the optimal transportation problem to the characterization of a maximally dependent random variable.

1 Stochastic optimal transportation problems.

Let $W(t)=W(t,\omega)$ be a d-dimensional standard Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$:

(i) $W(0)=0$,

(ii) $W(t)-W(s)$ is independent of $\mathcal{F}_s$, $0\leq s\leq t$,

(iii) The probability law of $W(t)-W(s)$ is $N(0,(t-s)Id)$, $0\leq s\leq t$.

(iv) $W(\cdot)\in C([0,\infty):\mathbf{R}^d)$ a.s..

Let $u(t)=u(t,\omega)$ be d-dimensional, $B([0,1])\times\mathcal{F}$-measurable, and $\mathcal{F}_t$-measurable for $t\in[0,1]$ and $L(t,x;u)\in C([0,1]\times\mathbf{R}^d\times\mathbf{R}^d:[0,\infty))$ be convex in u.

We introduce two classes of Stochastic optimal transportation problems (see [18, 19]).

(Problem I) For any $\{P_0, P_1\}\subset\mathcal{M}_1(\mathbf{R}^d)$(=the space of all Borel probability measures on $\mathbf{R}^d$ with a weak topology),

$$\begin{aligned} V(P_0,P_1) &:= \inf\Big\{E\Big[\int_0^1 L(t,X^u(t);u(t))dt\Big]\Big| \\ &\qquad dX^u(t):=u(t)dt+dW(t), PX(t)^{-1}=P_t, t=0,1\Big\}. \end{aligned} \quad (1.1)$$

(Problem II) For $\mathbf{P} := \{P_t\}_{0\le t\le 1} \subset \mathcal{M}_1(\mathbf{R}^d)$,

$$\begin{aligned}\mathbf{V}(\mathbf{P}) &:= \inf\Big\{E\Big[\int_0^1 L(t, X^u(t); u(t))dt\Big]\Big| \\ &\qquad dX^u(t) := u(t)dt + dW(t), PX^u(t)^{-1} = P_t, 0 \le t \le 1\Big\}. \qquad (1.2)\end{aligned}$$

Remark 1.1 *(i) If we delete $+dW(t)$ in (1.1), then $V(P_0, P_1)$ becomes the Monge-Kantorovich problem (see [3, 7, 8, 10, 13, 16, 17, 22, 23, 26, 27]):*

$$\inf\{E[I(X(0), X(1))] | PX(t)^{-1} = P_t, t = 0, 1\}, \qquad (1.3)$$

where

$$I(x, y) := \inf\Big\{\int_0^1 L\Big(t, X(t); \frac{dX(t)}{dt}\Big)dt\Big]\Big| X(0) = x, X(1) = y\Big\}.$$

(ii) For a fixed $P_0 \in \mathcal{M}_1(\mathbf{R}^d)$, the convex dual $V^*_{P_0}$ of $P \mapsto V(P_0, P)$ is a finite time horizon stochastic optimal control problem (see [9]):

$$\begin{aligned}V^*_{P_0}(f) &:= \sup_{P\in\mathcal{M}_1(\mathbf{R}^d)}\Big\{\int_{\mathbf{R}^d} f(x)P(dx) - V(P_0, P)\Big\} \\ &= \sup\Big\{E\Big[f(X^u(1) - \int_0^1 L(t, X^u(t); u(t))dt\Big]\Big| \\ &\qquad dX^u(t) := u(t)dt + dW(t), PX^u(0)^{-1} = P_0\Big\}. \qquad (1.4)\end{aligned}$$

1.1 Duality Theorems in stochastic optimal transportation

For the sake of simplicity, we set $L = \frac{|u|^2}{2}$. More general results can be found in the references given below.

We state the duality theorems for Problems I and II.

Theorem 1.1 (Duality Theorem) *(Mikami and Thieullen [21]) For any P_0 and $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$,*

$$V(P_0, P_1) = \sup\Big\{\int_{\mathbf{R}^d} \varphi(1, x)P_1(dx) - \int_{\mathbf{R}^d} \varphi(0, x)P_0(dx)\Big\}, \qquad (1.5)$$

where the supremum is taken over all bounded continuous viscosity solutions φ, to the following Hamilton-Jacobi-Bellman (HJB for short) PDE (see e.g.[6]), for which $\varphi(1,\cdot) \in C_b^\infty(\mathbf{R}^d)$:

$$\frac{\partial \varphi(t,x)}{\partial t} + \frac{1}{2}\triangle\varphi(t,x) + \frac{1}{2}|D_x\varphi(t,x)|^2 = 0, \quad (t,x) \in [0,1) \times \mathbf{R}^d. \tag{1.6}$$

Theorem 1.2 (Duality Theorem) *(Mikami [18]). For any* $\mathbf{P} := \{P_t\}_{0\le t\le 1} \subset \mathcal{M}_1(\mathbf{R}^d)$,

$$\begin{aligned} \mathbf{V}(\mathbf{P}) &= \sup\Big\{\int_{[0,1]\times\mathbf{R}^d} f(t,x)dtP_t(dx) - \int_{\mathbf{R}^d}\phi(0,x;f)P_0(dx)\Big| \\ &\qquad f \in C_b^\infty([0,1]\times\mathbf{R}^d)\Big\}, \end{aligned} \tag{1.7}$$

where the supremum is taken over all bounded continuous viscosity solutions ϕ to the following HJB PDE: $\phi(1,x;f) = 0$,

$$\frac{\partial \phi(t,x;f)}{\partial t} + \frac{1}{2}\triangle\varphi(t,x) + \frac{1}{2}|D_x\varphi(t,x)|^2 + f(t,x) = 0, \quad (t,x) \in [0,1) \times \mathbf{R}^d. \tag{1.8}$$

1.2 Marginal problems

We state two classes of marginal problems for which we gave positive answers in (Mikami [18]).

(I) Marginal problem with fixed two end points distributions (see [12, 25]).

For $\{P_0, P_1\} \subset \mathcal{M}_1(\mathbf{R}^d)$, construct a semimartingale $\{X(t)\}_{0\le t\le 1}$ for which there exists a measurable function $b : [0,1] \times \mathbf{R}^d \mapsto \mathbf{R}^d$ and

$$\begin{aligned} dX(t) &= b(t,X(t))dt + dW(t), \\ PX(t)^{-1} &= P_t, \quad t = 0,1. \end{aligned}$$

(II) Marginal problem with fixed marginal distributions for all times.

For $\{P_t\}_{0\le t\le 1} \subset \mathcal{M}_1(\mathbf{R}^d)$, construct a semimartingale $\{X(t)\}_{0\le t\le 1}$ for which there exists a measurable function $b : [0,1] \times \mathbf{R}^d \mapsto \mathbf{R}^d$ and

$$\begin{aligned} dX(t) &= b(t,X(t))dt + dW(t), \\ PX(t)^{-1} &= P_t, \quad t \in [0,1], \end{aligned}$$

Remark 1.2 *Marginal problem (II) is inspired by Nelson's marginal problem in stochastic mechanics: for $\{P_t\}_{0\le t\le 1} \subset \mathcal{M}_1(\mathbf{R}^d)$ and $b \in \mathbf{A}(\{P_t\}_{0\le t\le 1})$, construct a semimartingale $\{X(t)\}_{0\le t\le 1}$ for which the following holds:*

$$\begin{aligned} dX(t) &= b(t, X(t))dt + dW(t), \\ PX(t)^{-1} &= P_t, \quad t \in [0,1], \end{aligned}$$

1.3 Idea of the proof of marginal problems.

We introduce two variational problems to give applications of the duality theorems to the marginal problems for Markov processes.

Definition 1.1 *For $b : [0,1] \times \mathbf{R}^d \mapsto \mathbf{R}^d$ and $\{P_t\}_{0 \le t \le 1} \subset \mathcal{M}_1(\mathbf{R}^d)$, we write $b \in \mathbf{A}(\{P_t\}_{0 \le t \le 1})$ if the Fokker-Planck equation holds: for any $f \in C_b^{1,2}([0,1] \times \mathbf{R}^d)$ and $t \in [0,1]$,*

$$\begin{aligned} &\int_{\mathbf{R}^d} f(t,x)P_t(dx) - \int_{\mathbf{R}^d} f(0,x)P_0(dx) \\ = &\int_0^t ds \int_{\mathbf{R}^d} \Big(\frac{\partial f(s,x)}{\partial s} + \frac{1}{2}\Delta f(s,x) + < b(s,x), D_x f(s,x) > \Big) P_s(dx). \end{aligned}$$

(Problem i) For $\{P_0, P_1\} \subset \mathcal{M}_1(\mathbf{R}^d)$,

$$v(P_0, P_1) := \inf\Big\{ \int_0^1 dt \int_{\mathbf{R}^d} L(t, x; b(t,x)) Q_t(dx) \Big| b \in \mathbf{A}(\{Q_t\}_{0 \le t \le 1}), Q_t = P_t, t = 0, 1 \Big\}. \tag{1.9}$$

(Problem ii) For $\mathbf{P} := \{P_t\}_{0 \le t \le 1} \subset \mathcal{M}_1(\mathbf{R}^d)$,

$$\mathbf{v}(\mathbf{P}) := \inf\Big\{ \int_0^1 dt \int_{\mathbf{R}^d} L(t, x; b(t,x)) P_t(dx) \Big| b \in \mathbf{A}(\mathbf{P}) \Big\}. \tag{1.10}$$

Remark 1.3

$$V(P_0, P_1) \ge v(P_0, P_1),$$

$$\mathbf{V}(\{P_t\}_{0 \le t \le 1}) \ge \mathbf{v}(\{P_t\}_{0 \le t \le 1}).$$

The idea of the proof to marginal problems in section 1.2 is to prove $V = v$ and $\mathbf{V} = \mathbf{v}$ via Duality Theorems.

On the problem (I), for a fixed $P_0 \in \mathcal{M}_1(\mathbf{R}^d)$, consider the convex dual of $P \mapsto v(P_0, P)$:

$$\begin{aligned} v^*_{P_0}(f) &:= \sup_{P\in\mathcal{M}_1(\mathbf{R}^d)}\{\int_{\mathbf{R}^d} f(x)P(dx) - v(P_0,P)\} \\ &= \inf\Big\{\int_{\mathbf{R}^d} f(x)Q_1(dx) - \int_0^1 dt\int_{\mathbf{R}^d} L(t,x;b(t,x))Q_t(dx)\Big| \\ &\qquad b\in \mathbf{A}(\{Q_t\}_{0\le t\le 1}), Q_0 = P_0\Big\}. \end{aligned} \tag{1.11}$$

Prove $P \mapsto V(P_0,P)$ and $v(P_0,P)$ are not identically equal to infinity and lower semicontinuous and convex, which implies the following (see e.g. [5]):

$$V(P_0,P_1) = \sup_{f\in C_b(\mathbf{R}^d)}\{\int_{\mathbf{R}^d} f(x)P_1(dx) - V^*_{P_0}(f)\}, \tag{1.12}$$

$$v(P_0,P_1) = \sup_{f\in C_b(\mathbf{R}^d)}\{\int_{\mathbf{R}^d} f(x)P_1(dx) - v^*_{P_0}(f)\}. \tag{1.13}$$

For $f\in C_b^\infty(\mathbf{R}^d)$, prove

$$V^*_{P_0}(f) = v^*_{P_0}(f) = \int_{\mathbf{R}^d}\varphi(0,x)P_0(dx), \tag{1.14}$$

where φ is a minimal bounded continuous viscosity solution to the following HJB PDE: $\varphi(1,x) = f(x)$,

$$\frac{\partial\varphi(t,x)}{\partial t} + \frac{1}{2}\Delta\varphi(t,x) + H(t,x;D_x\varphi(t,x)) = 0, \quad (t,x)\in[0,1)\times\mathbf{R}^d. \tag{1.15}$$

Here for $(t,x,z)\in[0,1]\times\mathbf{R}^d\times\mathbf{R}^d$,

$$H(t,x;z) := \sup\{< z,u > -L(t,x;u)|u\in\mathbf{R}^d\}.$$

On the problem (II), a similar but more complicated idea can be applied by considering $\mathbf{V}$ and $\mathbf{v}$ as functionals of $dtP_t(dx)$ on $\mathcal{M}_1([0,1]\times\mathbf{R}^d)$ (see [18] for details).

2 Knothe-Rosenblatt processes

The Knothe-Rosenblatt rearrangement plays a crucial role in many fields, e.g., the Brunn-Minkowski inequality and statistics (see [1, 2, 14, 15, 24] and the references therein). We first describe the Knothe-Rosenblatt rearrangement.

Let $d\ge 2$. $\varphi:\mathbf{R}^d\mapsto\mathbf{R}^d$ is called a triangular mapping/transformation if

$$\varphi(x) = (\varphi_1(x_1), \varphi_2(x_1, x_2), \cdots, \varphi_d(x_1, \cdots, x_d)), \quad x = (x_i)_{i=1}^d.$$

A triangular mapping φ is called nondecreasing if $x_i \mapsto \varphi_i(x_1, \cdots, x_i)$ is nondecreasing for all $i = 1, \cdots, d$.

Definition 2.1 *For P_0 and $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$, the Knothe-Rosenblatt rearrangement which maps P_0 to P_1 is a nondecreasing triangular mapping φ such that*

$$P_0 \varphi^{-1} = P_1.$$

For $P_t \in \mathcal{M}_1(\mathbf{R}^d)$,

$$P_{t,i}(A) := P_t(A \times \mathbf{R}^{d-i}), \qquad \text{for Borel } A \subset \mathbf{R}^i.$$

For the sake of simplicity, we only introduce a typical Knothe-Rosenblatt process (see [20] for more general definition).

Definition 2.2 *Let $d > 1$ For P_0, $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$, $\{\mathbf{X}(t) = (X_i(t))_{1 \le i \le d}\}_{0 \le t \le 1}$ is called the* **Knothe-Rosenblatt process** *(for Brownian motion) if there exists $b(t, x) = (b_i(t, x_1, \cdots, x_i))_{i=1}^d$ such that*

$$dX(t) = b(t, X(t))dt + dW(t),$$

and $\{\mathbf{X}_i(t) = (X_j(t))_{1 \le j \le i}\}_{0 \le t \le 1}$ is the unique minimizer of

$$\begin{aligned} &V_i(P_{0,i}, P_{1,i}|\mathbf{X}_{i-1}) \\ :=& \begin{cases} \inf\{E[\int_0^1 \frac{1}{2}|u_1(t)|^2 dt] | d\mathbf{Y}_1^{u_1}(t) = u_1(t)dt + dW(t), \\ \qquad P\mathbf{Y}_1^{u_1}(0)^{-1} = P_{0,1}, P\mathbf{Y}_1^{u_1}(1)^{-1} = P_{1,1}\} =: V_1(P_{0,1}, P_{1,1}) \qquad (i = 1), \\ \inf\{E[\int_0^1 \frac{1}{2}|u_i(t)|^2 dt] | d\mathbf{Y}_i^{\mathbf{u}_i}(t) = \mathbf{u}_i(t)dt + dW(t), \\ \qquad P\mathbf{Y}_i^{\mathbf{u}_i}(0)^{-1} = P_{0,i}, P\mathbf{Y}_i^{\mathbf{u}_i}(1)^{-1} = P_{1,i}, \\ \qquad P(\mathbf{Y}_{i-1})^{-1} = P(\mathbf{X}_{i-1})^{-1}\} \qquad (1 < i \le d). \end{cases} \end{aligned}$$

Remark 2.1 $\mathbf{X}_1$ *is the h-path process for Brownian motion, provided $V_1(P_{0,1}, P_{1,1})$ is finite. In this sense, the Knothe-Rosenblatt process can be considered as a generalization of the h-path process (see [12, 21, 25] and the references therein).*

2.1 Convergence result

We state a typical case of the result [20] which is a stochastic control analogue of [4].

For $i = 2, \cdots, d$ and $\varepsilon > 0$,

$$
\begin{aligned}
& V_i^{\varepsilon}(P_{0,i}, P_{1,i}) \\
:= \ & \inf\Big\{E\Big[\sum_{j=1}^{i} \varepsilon^{j-1} \int_0^1 \frac{1}{2}|u_j(t)|^2 dt\Big]\Big| \\
& d\mathbf{Y}_i^{\mathbf{u}_i}(t) = \mathbf{u}_i(t)dt + dW(t), P\mathbf{Y}_i^{\mathbf{u}_i}(0)^{-1} = P_{0,i}, P\mathbf{Y}_i^{\mathbf{u}_i}(1)^{-1} = P_{1,i}, \Big\}. \quad (2.1)
\end{aligned}
$$

We show that a minimizer of $V_2^{\varepsilon}(P_0, P_1)$ converges to the Knothe-Rosenblatt process as $\varepsilon \to 0$.

Theorem 2.1 *Suppose that $d = 2$. Then for any P_0, $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$ for which the Knothe-Rosenblatt process $\{\mathbf{X}(t)\}_{0 \le t \le 1}$ exists, a minimizer $\{\mathbf{X}^{\varepsilon}(t)\}_{0 \le t \le 1}$ of $V_2^{\varepsilon}(P_0, P_1)$ exists and weakly converges to $\{\mathbf{X}(t)\}_{0 \le t \le 1}$ as $\varepsilon \to 0$.*

$$
\lim_{\varepsilon \to 0} E\Big[\int_0^1 |b_1^{\varepsilon}(t, \mathbf{X}^{\varepsilon}(t))|^2 dt\Big] = V_1(P_{0,1}, P_{1,1}), \quad (2.2)
$$

$$
\lim_{\varepsilon \to 0} E\Big[\int_0^1 |b_2^{\varepsilon}(t, \mathbf{X}^{\varepsilon}(t))|^2 dt\Big] = V_2(P_0, P_1 | \mathbf{X}_1). \quad (2.3)
$$

We consider the case where $d > 2$.

For all $i = 1, \cdots, d$, consider the following HJB PDEs: (HJB)$_i$

$$
\begin{aligned}
& \frac{\partial v_i(t, \mathbf{x}_i)}{\partial t} + \frac{1}{2}\Delta v_i(t, \mathbf{x}_i) + < \nabla_{i-1} v_i(t, \mathbf{x}_i), b_{i-1}(t, \mathbf{x}_{i-1}) > \\
& + \frac{1}{2}\Big|\frac{\partial v_i(t, \mathbf{x}_i)}{\partial x_i}\Big|^2 = 0, \quad (t, \mathbf{x}_i) \in (0,1) \times \mathbf{R}^i, \quad (2.4)
\end{aligned}
$$

where $b_{\mathbf{X}_0} := 0$.

Theorem 2.2 *Suppose that the Knothe-Rosenblatt process $\{\mathbf{X}(t)\}_{0 \le t \le 1}$ exists for P_0, $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$ and that there exists a solution $v_i(t, \mathbf{x}_{d_i}) \in C_b^{1,2}([0,1] \times \mathbf{R}^{d_i})$ to HJB PDE (2.4) such that*

$$
b_i = \frac{\partial v_i(t, \mathbf{x}_i)}{\partial x_i}, \qquad i = 1, \cdots, d.
$$

Then one can construct $P_1^\varepsilon \in \mathcal{M}_1(\mathbf{R}^d)$ such that a minimizer $\{\mathbf{X}^\varepsilon(t)\}_{0\le t\le 1}$ of $V_d^\varepsilon(P_0, P_1^\varepsilon)$ exists and converges to $\{\mathbf{X}(t)\}_{0\le t\le 1}$ in the sense of relative entropy as $\varepsilon \to 0$:

$$H(P(\mathbf{X}^\varepsilon)^{-1}|P(\mathbf{X})^{-1}) \;:=\; \frac{1}{2}E\Big[\int_0^1 |b^\varepsilon(t,\mathbf{X}^\varepsilon(t)) - b(t,\mathbf{X}^\varepsilon(t))|^2 dt\Big] \to 0 \qquad (2.5)$$

($\varepsilon \to 0$). For $i = 1, \cdots, d$, we also have

$$\lim_{\varepsilon\to 0} E\Big[\int_0^1 |b_i^\varepsilon(t,\mathbf{X}^\varepsilon(t))|^2 dt\Big] \;=\; E\Big[\int_0^1 |b_i(t,\mathbf{X}(t))|^2 dt\Big] = V_i(P_{0,i}, P_{1,i}|\mathbf{X}_{i-1}). \quad (2.6)$$

3 Maximally dependent random variable.

Random variables $X_1, \cdots, X_n$ with a common distribution functions F is called maximally dependent if

$$P(\max(X_1, \cdots, X_n) > t) \ge P(\max(Y_1, \cdots, Y_n) > t), \qquad (3.1)$$

for all $t > 0$, for any $Y_1, \cdots, Y_n$ with a common distribution function F.
For a nonnegative random variable X,

$$S_X(t) := P(X > t), \qquad t > 0, \qquad (3.2)$$

is called a survival function of X.
Let $n \ge 2$, $m_i \ge 1$ $(i = 1, \cdots, n)$, and Y_{ij} $(i = 1, \cdots, n, j = 1, \cdots, m_i)$ be real random variables. In this section, we present the maxima and minima of the overall survival functions

$$S_{\min\{\max\{Y_{ij}|1\le j\le m_i\}|1\le i\le n\}}(t), \quad S_{\max\{\min\{Y_{ij}|1\le j\le m_i\}|1\le i\le n\}}(t)$$

in the case where the probability distributions $P(Y_{ij})^{-1}$ are fixed. This can be proved by the optimal transportation problem on $\mathbf{R}$ (see [11]).

For an $\mathbf{R}^n$-valued random variable $\mathbf{Z} = (Z_1, \cdots, Z_n)$ and $k = 2, \cdots, n$, set

$$\begin{aligned}
\phi_{\mathbf{Z},1} &= \Psi_{\mathbf{Z},1} := Z_1,\\
\phi_{\mathbf{Z},2} &= \Psi_{\mathbf{Z},2} := F_{Z_2}^{-1}(1 - F_{Z_1}(Z_1)),\\
\phi_{\mathbf{Z},k} &:= F_{Z_k}^{-1}(1 - F_{\min(\phi_{\mathbf{Z},1},\cdots,\phi_{\mathbf{Z},k-1})}(\min(\phi_{\mathbf{Z},1}, \cdots, \phi_{\mathbf{Z},k-1}))),\\
\Psi_{\mathbf{Z},k} &:= F_{Z_k}^{-1}(1 - F_{\max(\Psi_{\mathbf{Z},1},\cdots,\Psi_{\mathbf{Z},k-1})}(\max(\Psi_{\mathbf{Z},1}, \cdots, \Psi_{\mathbf{Z},k-1}))).
\end{aligned}$$

Then, we have

Theorem 3.1 *(see [11]) Suppose that* $F_{Z_1}(\cdot), \cdots, F_{Z_n}(\cdot)$ *are continuous. Then,*

$$P(\phi_{\mathbf{Z},i})^{-1} = P(\Psi_{\mathbf{Z},i})^{-1} = P(Z_i)^{-1}, \quad \Psi_{\mathbf{Z},i} = -\phi_{-\mathbf{Z},i}, \quad i = 1, \cdots, n), \tag{3.3}$$

$$F_{\min(\phi_{\mathbf{Z},1},\cdots,\phi_{\mathbf{Z},n})}(x) = \min\Big(\sum_{i=1}^{n} F_{Z_i}(x), 1\Big), \tag{3.4}$$

$$F_{\max(\Psi_{\mathbf{Z},1},\cdots,\Psi_{\mathbf{Z},n})}(x) = \max\Big(\sum_{i=1}^{n} F_{Z_i}(x) - n + 1, 0\Big). \tag{3.5}$$

In particular,

$$F_{\min(\phi_{\mathbf{Z},1},\cdots,\phi_{\mathbf{Z},n})}(\min(\phi_{\mathbf{Z},1}, \cdots, \phi_{\mathbf{Z},n})) = \sum_{i=1}^{n} F_{Z_i}(\min(\phi_{\mathbf{Z},1}, \cdots, \phi_{\mathbf{Z},n})), a.s., \tag{3.6}$$

$$F_{\max(\Psi_{\mathbf{Z},1},\cdots,\Psi_{\mathbf{Z},n})}(\max(\Psi_{\mathbf{Z},1}, \cdots, \Psi_{\mathbf{Z},n})) = \sum_{i=1}^{n} F_{Z_i}(\max(\Psi_{\mathbf{Z},1}, \cdots, \Psi_{\mathbf{Z},n})) - n + 1, a.s.. \tag{3.7}$$

We first consider the case in which $m_i = 1$ $(i \geq 2)$, and set $m := m_1$, $Y_j := Y_{1j}$, and $X_i := Y_{i1}$ $(i \geq 2)$ for the sake of simplicity.

Theorem 3.2 *Suppose that* $F_{Y_1}(\cdot), \cdots, F_{Y_m}(\cdot)$ *are continuous. Then,*

$$\begin{aligned} & S_{\max(\min(Y_1,\cdots,Y_m),X_2,\cdots,X_n)}(x) \\ \geq \ & \max\Big(1 - \sum_{i=1}^{m} F_{Y_i}(x), 1 - F_{X_2}(x), \cdots, 1 - F_{X_n}(x)\Big), \quad x \in \mathbf{R}. \end{aligned} \tag{3.8}$$

Here, the equality holds if

$$\begin{aligned} Y_i &= \phi_{\mathbf{Y},i} \quad (i = 2, \cdots, m), \\ X_i &= F_{X_i}^{-1}\Big(\sum_{k=1}^{m} F_{Y_k}(\min(Y_1, \cdots, Y_m))\Big) \quad (i = 2, \cdots, n). \end{aligned} \tag{3.9}$$

Suppose, in addition, that $F_{X_2}(\cdot), \cdots, F_{X_n}(\cdot)$ *are continuous. Then,*

$$S_{\max(\min(Y_1,\cdots,Y_m),X_2,\cdots,X_n)}(x) \le \min\Big(n - \max_{i=1}^{m} F_{Y_i}(x) - \sum_{i=2}^{n} F_{X_i}(x), 1\Big), \quad x \in \mathbf{R}. \quad (3.10)$$

Here, the equality holds if

$$\begin{aligned} Y_i &= F_{Y_i}^{-1}(F_{Y_1}(Y_1)) \quad (i = 2, \cdots, m), \\ X_i &= \Psi_{\mathbf{X},i} \quad (i = 2, \cdots, n), \end{aligned} \quad (3.11)$$

where $X_1 := \min(Y_1, \cdots, Y_m)$.

We can now state the following two corollaries that follow from Theorem 3.2 and from the following:

$$\begin{aligned} & \min(\max(Y_1, \cdots, Y_m), X_2, \cdots, X_n) \\ = & -\max(\min(-Y_1, \cdots, -Y_m), -X_2, \cdots, -X_n). \end{aligned}$$

Corollary 3.1 *Suppose that* $F_{Y_1}(\cdot), \cdots, F_{Y_m}(\cdot)$ *are continuous. Then,*

$$\begin{aligned} & S_{\min(\max(Y_1,\cdots,Y_m),X_2,\cdots,X_n)}(x) \\ \le & \min\Big(m - \sum_{i=1}^{m} F_{Y_i}(x), 1 - F_{X_2}(x), \cdots, 1 - F_{X_n}(x)\Big), \quad x \in \mathbf{R}. \end{aligned} \quad (3.12)$$

Here, the equality holds if

$$\begin{aligned} Y_i &= \Psi_{\mathbf{Y},i} \quad (i = 2, \cdots, m), \\ X_i &= F_{X_i}^{-1}\Big(\sum_{k=1}^{m} F_{Y_k}(\max(Y_1, \cdots, Y_m)) - m + 1\Big), \quad i = 2, \cdots, n. \end{aligned} \quad (3.13)$$

Suppose, in addition, that $F_{X_2}(\cdot), \cdots, F_{X_n}(\cdot)$ *are continuous. Then,*

$$S_{\min(\max(Y_1,\cdots,Y_m),X_2,\cdots,X_n)}(x) \ge \max\Big(1 - \min_{i=1}^{m} F_{Y_i}(x) - \sum_{i=2}^{n} F_{X_i}(x), 0\Big), \quad x \in \mathbf{R}. \quad (3.14)$$

Here, the equality holds if

$$\begin{aligned} Y_i &= F_{Y_i}^{-1}(F_{Y_1}(Y_1)) \quad (i = 2, \cdots, m), \\ X_i &= \phi_{\mathbf{X},i} \quad (i = 2, \cdots, n), \end{aligned} \quad (3.15)$$

where $X_1 := \max(Y_1, \cdots, Y_m)$.

Corollary 3.2 *Suppose that there exist $m > 1$ and $i_0 \geq 1$ such that $m_i = m$ ($i = 1, \cdots, i_0$) and $m_i = 1$ ($i = i_0 + 1, \cdots, n$) and that for each $i = 1, \cdots, n$, $\{Y_{ij}\}_{j=1}^{m_i}$ are identically distributed with a continuous distribution function. Then,*

$$S_{\max\{\min\{Y_{ij}|1\leq j\leq m_i\}|1\leq i\leq n\}}(x) \geq \max\Big(\max\{1 - m_i F_{Y_{i1}}(x)|1 \leq i \leq n\}, 0\Big), x \in \mathbf{R}, \tag{3.16}$$

where the equality holds if

$$\begin{aligned} Y_{ij} &= \phi_{\mathbf{Y}_i,j} \quad (j = 2, \cdots, m_i), \\ Y_{i1} &= \begin{cases} F_{Y_{i1}}^{-1}(F_{Y_{11}}(Y_{11})) & i = 1, \cdots, i_0, \\ F_{Y_{i1}}^{-1}(mF_{Y_{11}}(\min(Y_{11}, \cdots, Y_{1m}))) & i = i_0 + 1, \cdots, n, \end{cases} \end{aligned} \tag{3.17}$$

where $\mathbf{Y}_i := (Y_{i1}, \cdots, Y_{im_i})$.

$$S_{\min\{\max\{Y_{ij}|1\leq j\leq m_i\}|1\leq i\leq n\}}(x) \leq \min(\min\{m_i - m_i F_{Y_{i1}}(x)|1 \leq i \leq n\}, 1), x \in \mathbf{R}, \tag{3.18}$$

where the equality holds if

$$\begin{aligned} Y_{ij} &= \Psi_{\mathbf{Y}_i,j} \quad (j = 2, \cdots, m_i), \\ Y_{i1} &= \begin{cases} F_{Y_{i1}}^{-1}(F_{Y_{11}}(Y_{11})) & i = 1, \cdots, i_0, \\ F_{Y_{i1}}^{-1}(mF_{Y_{11}}(\max(Y_{11}, \cdots, Y_{1m})) - m + 1) & i = i_0 + 1, \cdots, n. \end{cases} \end{aligned} \tag{3.19}$$

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