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ON NON-LINEAR SPECTRAL GAP FOR SYMMETRIC MARKOV CHAINS WITH COARSE RICCI CURVATURES

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ABSTRACT. In this note, we report the summary of [12] for the case that the target space is a complete separable CAT(0)-space. We prove an upper estimate of spectral radius for (non-linear) transition operator P over L^p-maps in the framework of symmetric Markov chains on a Polish space with positive lower bound of n-step coarse Ricci curvatures without its proof. As consequences, strong L^p-Liouville property for P-harmonic maps, a global Poincaré inequality (spectral gaps) for energy functional over L^2-maps (or functions), and spectral bounds of L^2-generator of Markov chains are presented.

1. COARSE RICCI CURVATURE

Throughout this note, let (E,d) be a Polish space with complete distance d and $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. Denote by $\mathcal{P}^p(E)$, the family of probability measures on (E,d) with finite p-th moment. We consider a conservative Markov chain $\mathbf{X}= (\Omega, X_k, \theta_k, \mathcal{F}_k, \mathcal{F}_\infty, \boldsymbol{P}_x)_{x\in E}$ with state space (E,d). Then the transition kernel $P(x,dy)$ (or $P_x(dy)$ in short) of $\mathbf{X}$ defined by $P(x,dy) := \boldsymbol{P}_x(X_1\in dy)$, $x\in E$ satisfies

(P1) for each $x\in E$, $\mathcal{B}(E)\ni A\mapsto P(x,A)$ is a probability measure on $(E,\mathcal{B}(E))$.

(P2) for each $A\in\mathcal{B}(E)$, $E\ni x\mapsto P(x,A)$ is $\mathcal{B}(E)$-measurable.

Conversely, for $P(x,dy)$ satisfying (P1) and (P2), we can construct a conservative Markov chain $\mathbf{X}$ such that $P(x,dy) = \boldsymbol{P}_x(X_1\in dy)$, $x\in E$. We set $Pf(x) := \int_X f(y)P(x,dy) = \boldsymbol{E}_x[f(X_1)]$ for any non-negative or bounded $\mathcal{B}(E)$-measurable function f on E. For $n\in\mathbb{N}$, if we set $P^n f(x) := P(P^{n-1}f)(x)$ inductively, then $P^n f(x) = \boldsymbol{E}_x[f(X_n)]$

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and $P^n(x,A) := (P^n\mathbf{1}_A)(x) = \boldsymbol{P}_x(X_n \in A)$. For any non-negative measure ν on $(E,\mathcal{B}(E))$ and $n \in \mathbb{N}$, we define a measure νP^n by $\nu P^n(A) := \langle \nu, P^n\mathbf{1}_A\rangle := \int_E P^n(x,A)\nu(dx) = \boldsymbol{P}_\nu(X_n \in A)$, $A \in \mathcal{B}(E)$. Note that $\delta_x P^n = P_x^n$, $x \in E$.

We further assume the following condition to $\mathbf{X}$:

(P3) for each $x \in E$, $P_x \in \mathcal{P}^1(E)$.

For the given Markov chain $\mathbf{X}$ as above and a fixed $n \in \mathbb{N}$, a Markov chain $\mathbf{X}^n = (\Omega, X_k^n, \theta_k^n, \mathcal{F}_k^n, \mathcal{F}_\infty^n, \boldsymbol{P}_x^n)_{x\in E}$ with state space (E,d) defined by the transition kernel $P^n(x,dy)$ is called an *n-step Markov chain*. Note that if $\mathbf{X}$ satisfies (P3), then $\mathbf{X}^n$ does so.

For $\mu, \nu \in \mathcal{P}^1(E)$, the L^1-Wasserstein/Kantorovich-Rubinstein distance $d_{W_1}(\mu,\nu)$ is defined by

$$d_{W_1}(\mu,\nu) := \inf\left\{ \int_{E\times E} d(x,y)\pi(dxdy) \;\middle|\; \pi \in \Pi(\mu,\nu)\right\},$$

where $\Pi(\mu,\nu) := \{\pi \in \mathcal{P}(E\times E) \mid \pi(A\times E) = \mu(A), \pi(E\times B) = \nu(B) \text{ for any } A, B \in \mathcal{B}(E)\}$.

Definition 1.1 (Coarse Ricci Curvature, [22]). For a pair of distinct points $x, y \in E$, the *coarse Ricci curvature* $\kappa(x,y)$ of $\mathbf{X}$ along (xy) is defined to be

$$\kappa(x,y) := 1 - \frac{d_{W_1}(P_x,P_y)}{d(x,y)}(> -\infty), \quad (x,y) \in E\times E \setminus \mathsf{diag}$$

and $\kappa := \inf\{\kappa(x,y) \mid (x,y) \in E\times E\setminus \mathsf{diag}\} \in [-\infty, 1]$ is said to be the *lower bound of the coarse Ricci curvature.* The *n-step coarse Ricci curvature* $\kappa_n(x,y)$ of $\mathbf{X}$ along (xy) is defined to be

$$\kappa_n(x,y) := 1 - \frac{d_{W_1}(P_x^n,P_y^n)}{d(x,y)}(\geq -\infty), \quad (x,y) \in E\times E \setminus \mathsf{diag}$$

and $\kappa_n := \inf\{\kappa_n(x,y) \mid (x,y) \in E\times E\setminus \mathsf{diag}\} \in [-\infty, 1]$ is said to be the *lower bound of the n-step coarse Ricci curvature.*

Remark 1.2. We denote the family of Lipschitz functions on E by $\mathrm{Lip}(E)$.

(1) If $\kappa \in \mathbb{R}$, then $P^n f \in \mathrm{Lip}(E)$ for any $f \in \mathrm{Lip}(E)$ and $\mathrm{Lip}(P^n f) \leq (1-\kappa)^n \mathrm{Lip}(f)$ by [22, Proposition 20], which implies that (P3) holds for all $\mathbf{X}^n$ provided (P3) holds for $\mathbf{X}$ and $\kappa \in \mathbb{R}$, in particular, $\kappa_n(x,y) > -\infty$ for all $n \in \mathbb{N}$ and $x \neq y$ under $\kappa \in \mathbb{R}$.

(2) The n-step coarse Ricci curvature $\kappa_n(x,y)$ is nothing but the coarse Ricci curvature for $\mathbf{X}^n$ and $\kappa_1(x,y) = \kappa(x,y)$ for $(x,y) \in E\times E\setminus \mathsf{diag}$. In general, we have

$$(1-\kappa_{k+\ell}) \leq (1-\kappa_k)(1-\kappa_\ell), \quad k, \ell \in \mathbb{N},$$

which implies $\lim_{\ell\to\infty}(1-\kappa_\ell)^{1/\ell} = \inf_{n\in\mathbb{N}}(1-\kappa_n)^{1/n} \in [0,+\infty]$. In particular, $\kappa \in \mathbb{R}$ implies $\kappa_n \in \mathbb{R}$ for all $n \in \mathbb{N}$.

The following example due to [4] shows that the lower bound 0 for the coarse Ricci curvature does not necessarily mean the same bound for the n-step coarse Ricci curvature.

Example 1.3 (Simple Random Walk on Cycle Graph, see [4]). Let $G=(V,E)$ be a cycle graph of size N, that is, G is an unweighted finite graph with vertices $V:=\{x_i\}_{i=1}^N$ and edges $E:=\{x_ix_{i+1}\}_{i=1}^N$ by regarding $x_{N+i}=x_i$ ($i\in\mathbb{N}$). The degree $d_x(G)$ for $x\in V$ is the number of edges starting from x is given by $d_x(G)=2$ for this cycle graph G. The weight w_{xy} for $xy\in E$ is given by $w_{x_ix_{i+1}}=1$ for $i=1,2,\cdots,N$. Consider a symmetric Markov chain $\mathbf{X}$ defined by the transition kernel $P_{x_i}(dy):=\frac{1}{2}\delta_{x_{i-1}}(dy)+\frac{1}{2}\delta_{x_{i+1}}(dy)$. As for the simple random walk on $\mathbb{Z}^1$, the coarse Ricci curvature $\kappa(x,y)$ on $\mathbf{X}$ satisfies $\kappa(x,y)=0$ for $(x,y)\in V\times V\setminus\mathsf{diag}$ (by [8, Theorems 2,3,4 and 5], [4, Theorems 6 and 7]), hence the n-step coarse Ricci curvature $\kappa_n(x,y)$ satisfies $\kappa_n(x,y)\geq 0$ for $(x,y)\in V\times V\setminus\mathsf{diag}$ by [22, Proposition 25]. $\mathbf{X}$ (hence the n-step Markov chain $\mathbf{X}^n$) is m-symmetric with respect to $m(dy):=\frac{1}{N}\sum_{i=1}^N\delta_{x_i}(dy)$. For simplicity, hereafter, we assume $N=5$. 3-step Markov chain $\mathbf{X}^3$ is associated with the Cayley graph $G^3:=(V^3,E^3)$ defined by $V^3:=V$ and $E^3:=\{x_ix_j\mid i,j=1,2,3,4,5 \text{ with } i\neq j\}$. G^3 is a weighted complete graph. The transition kernel $P_x^3(dy)$ of $\mathbf{X}^3$ is given by $P_{x_i}^3(dy)=\frac{1}{8}\delta_{x_{i-2}}(dy)+\frac{3}{8}\delta_{x_{i-1}}(dy)+\frac{3}{8}\delta_{x_{i+1}}(dy)+\frac{1}{8}\delta_{x_{i+2}}(dy)$. G^3 is a weighted graph with no loop and the degree $d_x(G^3)$ for $x\in V$ is given by $d_x(G^3)=4$. The weight w_{xy} for $xy\in E^3$ is given by $w_{x_ix_i}=\frac{3}{2}$, $w_{x_{i-2}x_i}=w_{x_ix_{i+2}}=\frac{3}{4}$. Note here that our degree $d_x(G^3)=4$ and the way for weighting on edges are different from those used in Section 6 of [4], but the conclusion is the same as we calculate below. The 3-step coarse Ricci curvature $\kappa_3(x,y)$ for $xy\in E^3$ can be estimated by use of [4, Theorems 6 and 7]:

$$\kappa_3(x_i,x_{i+1})=\frac{3}{8},\quad \frac{5}{8}\leq\kappa_3(x_i,x_{i+2})\leq\frac{7}{8}.$$

Therefore, $\kappa_3(x,y)\geq\frac{3}{8}$ for all $(x,y)\in V\times V\setminus\mathsf{diag}$.

Remark 1.4. For a continuous time parameter Markov process $\mathbf{M}$, the notion of *coarse Ricci curvature* $\kappa(x,y)$ for $\mathbf{M}$ is discussed in [22], [28]:

$$\kappa(x,y):=\varlimsup_{t\to 0}\frac{1}{t}\left(1-\frac{d_{W_1}(P_t(x,\cdot),P_t(y,\cdot))}{d(x,y)}\right)\quad\text{for } (x,y)\in E\times E\setminus\mathsf{diag}.$$

We can also define the n-step coarse Ricci curvature $\kappa_n(x,y)$:

$$\kappa_n(x,y):=\varlimsup_{t\to 0}\frac{1}{t}\left(1-\frac{d_{W_1}(P_{nt}(x,\cdot),P_{nt}(y,\cdot))}{d(x,y)}\right)\quad\text{for } (x,y)\in E\times E\setminus\mathsf{diag},$$

which is nothing but the coarse Ricci curvature for the time changed process $\mathbf{M}^n=(\Omega,X_{nt},\boldsymbol{P}_x)$. Then, we easily see $\kappa_n(x,y)=n\kappa(x,y)$ for $(x,y)\in E\times E\setminus\mathsf{diag}$, in particular, the positivity of lower bound for curvature is equivalent to each other between both coarse Ricci curvatures. In our discrete setting, we have no such a relation.

Example 1.5 (Riemannian manifold). Let (M,g) be a d-dimensional complete smooth Riemannian manifold whose Ricci curvature is bound below by $\kappa > 0$. In view of Bonnet-Myers theorem, M is compact. Consider a Brownian motion $\mathbf{M}=(\Omega, X_t, \boldsymbol{P}_x)$ on M associated with the following Dirichlet energy form on $L^2(M;m)$;

$$\begin{cases} D(\mathcal{E}) &:= \{u \in L^2(M;m) \mid \int_M g(\nabla f, \nabla f)dm < \infty\} \\ \mathcal{E}(f,g) &:= \int_M g(\nabla f, \nabla g)dm, \quad f,g \in D(\mathcal{E}) \end{cases}$$

where m is the volume element of (M,g). Let $P_t(x,dy)$ be the transition kernel of $\mathbf{M}$. Under the Ricci curvature lower bound, $\mathbf{M}$ is a conservative process, that is, $P_t(x,\cdot) \in \mathcal{P}(M)$ for any $t > 0$. Moreover, we see $P_t(x,\cdot) \in \mathcal{P}^1(M)$ for any $t > 0$. We set $P(x,dy) := P_1(x,dy)$ and consider an m-symmetric Markov chain $\mathbf{X}$ associated with $P(x,dy)$. It is proved in [30] that

$$d_{W_1}(P_t(x,\cdot), P_t(y,\cdot)) \leq e^{-\kappa t} d(x,y), \quad x,y \in M.$$

So the coarse Ricci curvature $\kappa_{\mathbf{M}}(x,y)$ of $\mathbf{M}$ has the lower estimate

$$\begin{aligned} \kappa_{\mathbf{M}}(x,y) &:= \varliminf_{t\to 0} \frac{1}{t}\left(1 - \frac{d_{W_1}(P_t(x,\cdot), P_t(y,\cdot))}{d(x,y)}\right) \\ &\geq \frac{d}{dt}(1-e^{-\kappa t})\Big|_{t=0} = \kappa > 0, \quad (x,y) \in M \times M \setminus \mathsf{diag}. \end{aligned}$$

On the other hand, the n-step coarse Ricci curvature $\kappa_n(x,y)$ of $\mathbf{X}$ has the lower estimate

$$\kappa_n(x,y) = 1 - \frac{d_{W_1}(P_x^n, P_y^n)}{d(x,y)} \geq 1 - e^{-\kappa n} > 0, \quad (x,y) \in M \times M \setminus \mathsf{diag}.$$

Note that the same conclusion also holds for a Markov process whose coarse Ricci curvature is bounded below by $\kappa > 0$.

2. CAT(0)-SPACES

In this section, we summarize the notions of CAT(0)-space and its properties.

Definition 2.1 (CAT(0)-space). A metric space (Y,d) is called the *CAT(0)-space* (*Hadamard space, or global NPC space*) if for any pair of points $\gamma_0, \gamma_1 \in Y$ and any $t \in [0,1]$ there exists a point $\gamma_t \in Y$ such that for any $z \in Y$

$$(2.1) \quad d_Y^2(z,\gamma_t) \leq (1-t)d_Y^2(z,\gamma_0) + t d_Y^2(z,\gamma_1) - t(1-t)d_Y^2(\gamma_0,\gamma_1).$$

By definition, $\gamma := (\gamma_t)_{t\in[0,1]}$ is the minimal geodesic joining γ_0 and γ_1. Any CAT(0)-space is simply connected. Hadamard manifolds, Euclidean Bruhat-Tits buildings (e.g. metric tree), spiders, booklets and Hilbert spaces are typical examples of CAT(0)-spaces (cf. [25]).

Let (Y, d_Y) be a CAT(0)-space. Then the distance function $d_Y : Y \times Y \to [0, \infty[$ is convex (Corollary 2.5 in [25]) and Jensen's inequality (Theorem 6.3 in [25]) can be applied to the convex function $Y \ni w \mapsto d_Y(w, z)$ for each $z \in Y$.

The inequality (2.1) yields the (strict) convexity of $Y \ni x \mapsto d_Y^2(z, x)$ for a fixed $z \in Y$. Any closed convex subset of a CAT(0)-space is again a CAT(0)-space.

The unique existence of projection (or foot-point) to closed convex set of CAT(0)-space is proved in [14] in more general setting.

Lemma 2.2 (Projection Map to Convex Set, see [25]). *Let (Y, d_Y) be a complete CAT(0)-space. The following hold:*

(1) *Let F be a closed convex subset of (Y, d_Y). Then, for each $x \in Y$, there exists a unique element $\pi_F(x) \in F$ such that $d_Y(x, F) = d_Y(\pi_F(x), x)$ holds. We call $\pi_F : Y \to F$ the projection map to F.*

(2) *Let F be as above. Then π_F satisfies*

$$d_Y^2(z, \pi_F(z)) + d_Y^2(\pi_F(z), w) \leq d_Y^2(z, w), \quad \text{for } z \in Y, w \in F, \tag{2.2}$$

in particular, $d_Y(\pi_F(z), w) \leq d_Y(z, w)$ for $z \in Y, w \in F$.

Let (Y, d_Y) be a metric space and $\mathcal{P}(Y)$ a family of Borel probability measures on Y. For $p \geq 1$, we set

$$\mathcal{P}^p(Y) := \left\{ \mu \in \mathcal{P}(Y) \;\Big|\; \int_Y d_Y^p(x, y)\mu(dy) < \infty \text{ for any/some } x \in Y \right\}.$$

Each element $\mu \in \mathcal{P}^p(Y)$ is called a *probability measure with p-th moment.*

Definition 2.3 (Barycenter or Center of Mass, see [25]). For $\mu \in \mathcal{P}^2(Y)$, if $z \mapsto \int_Y d_Y^2(z, x)\mu(dx)$ has a minimizer $b(\mu) \in Y$, then we call $b(\mu)$ the *barycenter, or center of mass* of $\mu \in \mathcal{P}^2(Y)$. For $\mu \in \mathcal{P}^1(Y)$ and $w \in Y$, we consider the following function F_w:

$$F_w(z) := \int_Y (d_Y^2(z, x) - d_Y^2(w, x))\mu(dx). \tag{2.3}$$

We easily see

$$|F_w(z)| \leq 2d_Y(z, w) \int_Y (d_Y(z, x) + d_Y(w, x))\mu(dx) < \infty.$$

If $Y \ni z \mapsto F_w(z)$ admits a minimizer $b(\mu)$ independent of w in the sense that $F_w(z) \geq F_w(b(\mu))$ if and only if $F_v(z) \geq F_v(b(\mu))$ for all $z, w, v \in Y$, we call it *barycenter, or center of mass* of $\mu \in \mathcal{P}^1(Y)$. If the barycenter of $\mu \in \mathcal{P}^2(Y)$ exists, then it is a barycenter of $\mu \in \mathcal{P}^1(Y)$.

Assume that (Y, d_Y) is a geodesic space. For a subset F of Y, denote by $C(F)$ the closed convex hull of F. That is, $C(F)$ is the smallest closed convex subset of Y containing F.

If (Y, d_Y) is a complete CAT(0)-space, we can obtain the unique existence of barycenter of $\mu \in \mathcal{P}^1(Y)$ proved in [25].

Lemma 2.4 ([25], cf. [16],[21]). *Let (Y, d_Y) be a complete $CAT(0)$-space. Then $\mu \in \mathcal{P}^1(Y)$ admits a unique barycenter.*

For any metric space (Y, d_Y), we easily see $b(\delta_x) = x$ for $x \in Y$.
The following proposition is proved in Proposition 5.5 in [25].

Theorem 2.5 (Jensen's Inequality, see [25, Theorem 6.3]). *Let (Y, d_Y) be a complete $CAT(0)$-space. Let φ be a lower semi-continuous convex function on Y and $\mu \in \mathcal{P}^1(Y)$. Suppose $\varphi \in L^1(Y;\mu)$. Then we have*

$$\varphi(b(\mu)) \leq \int_Y \varphi(x)\mu(dx). \tag{2.4}$$

Corollary 2.6 (Fundamental Contraction Property, see [25]). *Let (Y, d_Y) be a complete $CAT(0)$-space. Let $\mu, \nu \in \mathcal{P}^1(Y)$. Then*

$$d_Y(b(\mu), b(\nu)) \leq d_{W_1}(\mu, \nu),$$

where $d_{W_1}(\mu,\nu)$ is the L^1-Wasserstein distance on $\mathcal{P}^1(Y)$ defined by

$$d_{W_1}(\mu,\nu) := \inf_{\pi \in \Pi(\mu,\nu)} \int_{Y \times Y} d_Y(x,y)\pi(dxdy).$$

Here $\Pi(\mu,\nu) := \{\pi \in \mathcal{P}(Y \times Y) \mid \pi(A \times Y) = \mu(A), \pi(Y \times B) = \nu(B)$ for $A, B \in \mathcal{B}(Y)\}$.

3. L^p-MAPS

Let $(E, \mathcal{E}, \mu)$ be a σ-finite measure space and $\mathcal{E}^\mu$ a completion of $\mathcal{E}$ with respect to μ. In what follows, we say *measurable* (resp. *μ-measurable*) for $\mathcal{E}$-measurable (resp. $\mathcal{E}^\mu$-measurable). For function $f : E \to [-\infty, \infty]$, we set $\|f\|_p := \left(\int_E |f(x)|^p \mu(dx)\right)^{1/p}$, $\|f\|_\infty := \inf\{\lambda > 0 \mid |f(x)| \leq \lambda$ μ-a.e. $x \in E\}$. For two $\overline{\mathbb{R}}$-valued functions f, g, they are said to be *μ-equivalent* if $f = g$ μ-a.e.

For $p \in]0, \infty]$, $L^p(E;\mu)$ denotes the family of μ-equivalence class of functions with finite $\|\cdot\|_p$-norm. Also $L^0(E;\mu)$ denotes the family of μ-equivalence class of functions having finite value μ-a.e. Fix a metric space (Y, d_Y). For $p \in]0, \infty]$ and measurable maps $u, v : E \to Y$, the pseudo-distance $d_{L^p}(u,v)$ is defined by $d_{L^p}(u,v) := \|d_Y(u,v)\|_p$. More precisely, for $p \in]0, \infty[$ we set

$$d_{L^p}(u,v) := \left(\int_E d_Y^p(u(x), v(x))\mu(dx)\right)^{1/p},$$

and for $p = \infty$, $d_\infty(u,v)$ is the μ-essentially supremum of $x \mapsto d_Y(u(x), v(x))$. We say that u and v are μ-equivalent ($u \overset{\mu}{\sim} v$ in short) if

$$u(x) = v(x) \ \ \mu\text{-a.e. } x \in E.$$

For a fixed measurable map $h : E \to Y$, we set

$$L^p_h(E, Y; \mu) := \{f \in \mathcal{E}/\mathcal{B}(Y) \mid d_Y(f, h) \in L^p(E; \mu)\}/ \overset{\mu}{\sim}.$$

Such a map $h : E \to Y$ is called a *base map* of $L^p_h(E, Y; \mu)$. If $\mu(E) < \infty$ and the image of $h : E \to Y$ is bounded, $L^p_h(E, Y; \mu)$ is independent of the choice of such a base map h. In this case, we can assume $h \equiv o$ for some fixed point $o \in Y$.

Proposition 3.1 ([23, Proposition 3.3]). *Let (Y, d_Y) be a metric space and $h : E \to Y$ a measurable map. Take $p \in [1, \infty]$. Then we have the following:*

(1) *If (Y, d_Y) is complete, then $(L^p_h(E, Y; \mu), d_{L^p})$ is so.*
(2) *If (Y, d_Y) is a geodesic space and any point γ_t of the constant speed geodesic $\gamma : [0, 1] \to Y$ joining γ_0 to γ_1 is a continuous map with respect to (γ_0, γ_1), then $(L^p_h(E, Y; \mu), d_{L^p})$ is also a geodesic space.*

In what follows, we assume $m(E) < \infty$. Let $L^p(E, Y; m)$ be the space of L^p-maps with bounded base maps, that is,

$$L^p(E, Y; m) := \left\{ u : E \to Y \;\middle|\; \begin{array}{l} u \text{ is } m\text{-measurable}, \\ \displaystyle\int_E d^p_Y(u, o)dm < \infty \text{ for some } o \in Y \end{array} \right\} \Big/ \overset{m}{\sim}.$$

Definition 3.2 (Lipschitz Maps). Let (Y, d_Y) be a geodesic space and (E, d) a metric space. For a map $u : E \to Y$, we set $\mathrm{Lip}(u) := \sup_{x \neq y} \frac{d_Y(u(x), u(y))}{d(x,y)}$ and

$$\mathrm{Lip}(E, Y) := \{u : E \to Y \mid \mathrm{Lip}(u) < \infty\}.$$

Lemma 3.3. *Let (Y, d_Y) be a geodesic space and (E, d) a metric space. Suppose that m has a p-th moment, that is, $\int_E d^p(x, x_0)m(dx) < \infty$ for some/any point $x_0 \in E$. Then $\mathrm{Lip}(E, Y) \subset L^p(E, Y; m)$.*

Let $S(E, Y)$ be a space of finite valued maps from E to Y. Any element of $S(E, Y)$ is called a *step map* or a *simple map*. Since $m(E) < \infty$, $S(E, Y)$ (more precisely $S(E, Y)/ \overset{m}{\sim}$) is a subset of $L^p(E, Y; m)$.

Theorem 3.4. *Suppose that (E, d) is a Polish space and (Y, d_Y) is a separable geodesic space. Take $p \in [1, \infty[$. Then any element of $L^p(E, Y; m)$ can be L^p-approximated by elements in $S(E, Y)$. In particular, if $E = \mathrm{supp}[m]$, then $(L^p(E, Y; m), d_{L^p})$ is a separable metric space. Moreover, if m has a finite p-th moment, then $L^p(E, Y; m)$ can be L^p-approximated by elements in $\mathrm{Lip}(E, Y)$, if further $E = \mathrm{supp}[m]$, then $\mathrm{Lip}(E, Y)$ is a dense subset of $L^p(E, Y; m)$.*

In what follows, (E, d) denotes a Polish space with complete distance d.

Definition 3.5 ($P^\ell u$ for Borel Map u)**.** Let $\mathbf{X}$ be a conservative Markov chain on (E,d). Suppose that (Y,d_Y) is a complete CAT(0)-space and a $\mathcal{B}(E)/\mathcal{B}(Y)$-measurable map $u : E \to Y$ satisfies $u_\sharp P_x^\ell \in \mathcal{P}^1(Y)$ for $\ell \in \mathbb{N}$. Then we set

$$P^\ell u(x) := b(u_\sharp P_x^\ell).$$

Here $u_\sharp P_x^\ell$ is a push-forward measure of $P(x,\cdot)$ by u; $u_\sharp P_x^\ell(A) := P^\ell(x, u^{-1}(A))$, $A \in \mathcal{B}(Y)$.

Remark 3.6. Note that any $u \in S(E,Y) \cup \mathrm{Lip}(E,Y)$ satisfies $u_\sharp P_x \in \mathcal{P}^1(Y)$. Indeed, for $u \in S(E,Y)$, u is a constant on each Borel set A_i, where $\{A_i\}_{i=1}^l$ is a finite family of disjoint Borel sets satisfying $E = \bigcup_{i=1}^l A_i$, hence $\int_E d_Y(z_0,z)u_\sharp P_x(dz) = \sum_{i=1}^l \|d_Y(z_0,u)\|_{\infty,A_i} P_x(A_i) < \infty$. For $u \in \mathrm{Lip}(E,Y)$, we have

$$\begin{aligned}\int_E d_Y(z_0,z)u_\sharp P_x(dz) &= \int_E d_Y(z_0,u(y))P_x(dy)\\ &\le d_Y(z_0,u(y_0)) + \mathrm{Lip}(u)\int_E d(y_0,y)P_x(dy) < \infty.\end{aligned}$$

Lemma 3.7 (Lemma 6.4 in [23])**.** *Let $\mathbf{X}$ be a conservative Markov chain on (E,d). Suppose that (Y,d_Y) is a complete separable CAT(0)-space. Then, for any Borel map $u : E \to Y$ satisfying $u_\sharp P_x \in \mathcal{P}^1(Y)$ for all $x \in E$, $Pu : E \to Y$ is $\mathcal{B}(E)/\mathcal{B}(Y)$-measurable.*

Definition 3.8 (Pu for L^p-map u)**.** Fix $p \ge 1$. Let $\mathbf{X}$ be an m-symmetric conservative Markov chain on (E,d). Suppose that (Y,d_Y) is a complete CAT(0)-space and $u \in L^p(E,Y;m)$, we can define $Pu \in L^p(E,Y;m)$ in the following way: Let $\{u_k\} \subset S(E,Y)$ be an L^p-approximating sequence to u. Applying the Jensen's inequality to the convex function d_Y^p on $Y \times Y$ and the m-symmetry, we have the following inequality for any maps $v,w \in S(E,Y)$.

$$\begin{aligned}d_{L^p}^p(Pv,Pw) &= \int_E d_Y^p(Pv(x),Pw(x))m(dx)\\ &\le \int_E Pd_Y^p(v,w)dm \le d_{L^p}^p(v,w).\end{aligned} \tag{3.1}$$

These mean that $\{Pu_k\}$ forms an L^p-Cauchy sequence. We set $Pu := \lim_k Pu_k \in L^p(E,Y;m)$. The well-definedness of Pu is clear from (3.1) and this is valid for any $v,w \in L^p(E,Y;m)$.

Definition 3.9 (P-harmonic Map, [16],[15])**.** A (lower or upper) bounded Borel function $f : E \to \mathbb{R}$ is said to be *P-subharmonic* if $f \le Pf$ on E and it is said to be *P-harmonic* if both f and $-f$ are P-subharmonic. A Borel map $u : E \to Y$ is said to be *P-harmonic* if $u = Pu$ on E holds under that $u_\sharp P_x \in \mathcal{P}^1(Y)$ for all $x \in E$.

Lemma 3.10. *Let* $\mathbf{X}$ *be a Markov chain on* (E,d). *Fix* $n \in \mathbb{N}$ *and assume* $\kappa \in \mathbb{R}$. *Suppose that* (Y,d_Y) *is a complete CAT(0)-space. Then for* $u \in \mathrm{Lip}(E,Y)$ *and* $\ell \in \mathbb{N}$, *we have* $P^{n\ell}u \in \mathrm{Lip}(E,Y)$ *and*

$$\mathrm{Lip}(P^{n\ell}u) \le (1-\kappa_n)^{\ell}\mathrm{Lip}(u),$$

in particular,

$$\mathrm{Lip}(P^{\ell}u) \le (1-\kappa)^{\ell}\mathrm{Lip}(u).$$

Corollary 3.11 (Strong Liouville Property for Lipschitz Maps). *Let* $\mathbf{X}$ *be a Markov chain on* (E,d). *Assume that* $\kappa \in \mathbb{R}$ *and there exists* $n \in \mathbb{N}$ *such that* $\kappa_n > 0$. *Suppose that* (Y,d_Y) *is a complete CAT(0)-space. Then any* P^n*-harmonic map* $u \in \mathrm{Lip}(E,Y)$ *is a constant map.*

Definition 3.12 (Variance). Fix $p \ge 1$, $\mu \in \mathcal{P}(E)$, a metric space (Y,d_Y) and $u \in L^p(E,Y;\mu)$. The *p-variance* $\mathrm{Var}^p_\mu(u)$ of u is defined by

$$\mathrm{Var}^p_\mu(u) := \inf_{y\in Y}\int_E d^p_Y(u(x),y)\mu(dx)(<\infty).$$

The *quasi p-variance* $\overline{\mathrm{Var}}^p_\mu(u)$ is defined by

$$\overline{\mathrm{Var}}^p_\mu(u) := \frac{1}{2}\int_E\int_E d^p_Y(u(y),u(x))\mu(dx)\mu(dy)(<\infty).$$

We easily see $\mathrm{Var}^p_\mu(u) \le 2\overline{\mathrm{Var}}^p_\mu(u)$. When $p=2$, we write $\mathrm{Var}_\mu(u) := \mathrm{Var}^2_\mu(u)$ and $\overline{\mathrm{Var}}_\mu(u) := \overline{\mathrm{Var}}^2_\mu(u)$, and call them simply *variance*, *quasi variance*, respectively. Let (Y,d_Y) be a complete CAT(0)-space. If $u \in L^2(E,Y;\mu)$, then $\mathrm{Var}_\mu(u) = \int_E d^2_Y(u(x),b(u_\sharp\mu)))\mu(dx)$ holds. For $u \in L^2(E,Y;\mu)$, we have $\mathrm{Var}_\mu(u) \le \overline{\mathrm{Var}}_\mu(u)$. If (Y,d_Y) is a Hilbert space H, then we have $\mathrm{Var}_\mu(u) = \overline{\mathrm{Var}}_\mu(u)$. In this case we can define the *covariance* $\mathrm{Cov}_\mu(f,g)$ for $f,g \in L^2(E,H;\mu)$ by

$$\begin{aligned}\mathrm{Cov}_\mu(f,g) &:= \int_E \langle f(x)-\langle\mu,f\rangle, g(x)-\langle\mu,g\rangle\rangle_H\mu(dx)\\ &= \langle\mu,\langle f,g\rangle_H\rangle - \langle\langle\mu,f\rangle,\langle\mu,g\rangle\rangle_H\\ &= \frac{1}{2}\int_E\int_E\langle f(y)-f(x),g(y)-g(x)\rangle_H\mu(dx)\mu(dy),\end{aligned}$$

where $\langle\mu,f\rangle := \int_H f(x)\mu(dx) \in H$ is the barycenter of $f_\sharp\mu \in \mathcal{P}^2(Y)$.

Definition 3.13 (Energy of Maps). Take $m \in \mathcal{P}(E)$ and let $\mathbf{X}$ be an m-symmetric Markov chain and (Y,d_Y) is a metric space. For $u \in L^p(E,Y;m)$,

$$E^p(u) := \frac{1}{2}\int_E\int_E d^p_Y(u(y),u(x))P(x,dy)m(dx)$$

is said to be *p-energy* of u with respect to $\mathbf{X}$ and

$$E^p_*(u) := \frac{1}{2}\int_E d^p_Y(Pu(x),u(x))m(dx) = \frac{1}{2}d^p_{L^p}(Pu,u)$$

is said to be *quasi p-energy* of u with respect to $\mathbf{X}$ for $p \geq 1$ when (Y, d_Y) is a complete separable CAT(0)-space.

When $p = 2$, we simply say *energy* (resp. *quasi 2-energy*) and write $E(u) := E^2(u)$ (resp. $E_*(u) := E_*^2(u)$). Since

$$\mathrm{Var}^p_{P_x}(u) \leq \int_E d_Y^p(u(y), u(x))P(x, dy),$$

we see

$$\frac{1}{2}\int_E \mathrm{Var}^p_{P_x}(u)m(dx) \leq E^p(u). \tag{3.2}$$

We use

$$\begin{cases} D(E^p) &:= \{u \in L^p(E, Y; m) \mid E^p(u) < \infty\} \\ E^p(u) &:= \frac{1}{2}\int_E\int_E d_Y^p(u(y), u(x))P(x, dy)m(dx), \quad u \in D(E^p). \end{cases}$$

When (Y, d_Y) is a Hilbert space H, we use the symbol $\mathcal{E}$ instead of E for the (2-)energy on $L^2(E, H; m)$ and set

$$\mathcal{E}(f, g) := \frac{1}{2}\int_{E\times E}\langle f(y) - f(x), g(y) - g(x)\rangle_H P_x(dy)m(dx)$$

for $f, g \in D(\mathcal{E})$. We see $\mathcal{E}(f) = \mathcal{E}(f, f)$ for $f \in L^2(E, H; m)$.

Proposition 3.14. *Let $\mathbf{X}$ be an m-symmetric Markov chain on (E, d) and (Y, d_Y) is a metric space. Fix $p \in [1, \infty[$. For measurable maps $u, v : E \to Y$, the following inequalities hold:*

$$E^p(u)^{\frac{1}{p}} \leq E^p(v)^{\frac{1}{p}} + 2^{1-\frac{1}{p}} d_{L^p}(u, v), \tag{3.3}$$

$$\mathrm{Var}^p_m(u)^{\frac{1}{p}} \leq \mathrm{Var}^p_m(v)^{\frac{1}{p}} + d_{L^p}(u, v), \tag{3.4}$$

$$\overline{\mathrm{Var}}^p_m(u)^{\frac{1}{p}} \leq \overline{\mathrm{Var}}^p_m(v)^{\frac{1}{p}} + 2^{1-\frac{1}{p}} d_{L^p}(u, v). \tag{3.5}$$

Corollary 3.15. *Let $\mathbf{X}$ be an m-symmetric Markov chain on (E, d). Suppose that (Y, d_Y) is a complete separable CAT(0)-space. For $p \geq 1$ and $u \in L^p(E, Y; m)$, the following inequalities hold:*

$$\mathrm{Var}^p_m(u)^{\frac{1}{p}} \leq \mathrm{Var}^p_m(Pu)^{\frac{1}{p}} + 2^{\frac{1}{p}} E^p_*(u)^{\frac{1}{p}}, \tag{3.6}$$

$$\overline{\mathrm{Var}}^p_m(u)^{\frac{1}{p}} \leq \overline{\mathrm{Var}}^p_m(Pu)^{\frac{1}{p}} + 2E^p_*(u)^{\frac{1}{p}}. \tag{3.7}$$

If $u \in L^2(E, Y; m)$, we have

$$E_*^2(u) \leq 4E^2(u). \tag{3.8}$$

Corollary 3.16 (Lower Semi Continuity of Energy)**.** *Let $\mathbf{X}$ be an m-symmetric Markov chain with $m \in \mathcal{P}(E)$ and (Y, d_Y) a metric space. Take $p \geq 1$ and let $(E^p, D(E^p))$ be the p-energy on $L^p(E, Y; m)$ associated with $\mathbf{X}$. We set $E^p(u) := \infty$ for $u \in L^p(E, Y; m) \setminus D(E^p)$. Then E^p is a $[0, \infty]$-valued lower semi continuous functional on $L^p(E, Y; m)$.*

Remark 3.17. When $p=2$, $Y=\mathbb{R}$, the lower semi continuity of energy is equivalent to the completeness of $D(\mathcal{E})$ with respect to the norm $\|\cdot\|_{\mathcal{E}_1}$ defined by $\|f\|_{\mathcal{E}_1}:=\sqrt{\mathcal{E}_1(f,f)}$. Here $\mathcal{E}_1(f,g):=\mathcal{E}(f,g)+(f,g)_m$, $f,g\in D(\mathcal{E})$.

Lemma 3.18 (Contraction Property). *Let* $\mathbf{X}$ *be an* m*-symmetric Markov chain on* (E,d) *with* $m\in\mathcal{P}(E)$. *Fix* $p\geq 1$. *Let* (Y,d_Y) *be a complete separable* $CAT(0)$*-space. Then, for any* $u\in L^p(E,Y;m)$, *we have*

$$\mathrm{Var}_m^p(Pu)\leq \mathrm{Var}_m^p(u), \tag{3.9}$$

$$\overline{\mathrm{Var}}_m^p(Pu)\leq \overline{\mathrm{Var}}_m^p(u). \tag{3.10}$$

4. Main results

In this section, we fix $p\geq 1$ and assume $m\in\mathcal{P}(E)$ and $\mathrm{supp}[m]=E$.

Theorem 4.1 (Non-linear Spectral Radius of P on $L^p(E,Y;m)/\{\mathrm{const}\}$). *Let* $\mathbf{X}$ *be an* m*-symmetric Markov chain on* (E,d) *with* $m\in\mathcal{P}^p(E)$ *and assume* $\kappa\in\mathbb{R}$. *Let* (Y,d_Y) *be a complete separable* $CAT(0)$*-space. Then, we have*

$$\varlimsup_{\ell\to\infty}\left(\sup_{u\in L^p(E,Y;m)}\frac{\mathrm{Var}_m^p(P^\ell u)}{\mathrm{Var}_m^p(u)}\right)^{\frac{1}{p\ell}}\leq \inf_{n\in\mathbb{N}}(1-\kappa_n)^{\frac{1}{n}}\wedge 1, \tag{4.1}$$

$$\varlimsup_{\ell\to\infty}\left(\sup_{u\in L^p(E,Y;m)}\frac{\overline{\mathrm{Var}}_m^p(P^\ell u)}{\overline{\mathrm{Var}}_m^p(u)}\right)^{\frac{1}{p\ell}}\leq \inf_{n\in\mathbb{N}}(1-\kappa_n)^{\frac{1}{n}}\wedge 1. \tag{4.2}$$

Corollary 4.2 (Linear Spectral Radius of P on $L^2(E,H;m)/\{\mathrm{const}\}$). *Let* $\mathbf{X}$ *be an* m*-symmetric Markov chain on* (E,d) *and* H *a real separable Hilbert space. Assume* $m\in\mathcal{P}^2(E)$ *and* $\kappa\in\mathbb{R}$. *Then, we have*

$$\lim_{\ell\to\infty}\left(\sup_{f\in L^2(E,H;m)}\frac{\mathrm{Var}_m(P^\ell f)}{\mathrm{Var}_m(f)}\right)^{\frac{1}{2\ell}}\leq \inf_{n\in\mathbb{N}}(1-\kappa_n)^{\frac{1}{n}}\wedge 1. \tag{4.3}$$

Consequently, P *is an* $\inf_{n\in\mathbb{N}}(1-\kappa_n)^{\frac{1}{n}}\wedge 1$*-contraction operator on* $L^2(E,H;m)/\{\mathrm{const}\}$. *In particular, for* $f\in L^2(E,H;m)/\{\mathrm{const}\}$, *the following hold:*

$$\mathrm{Var}_m(Pf)\leq (\inf_{n\in\mathbb{N}}(1-\kappa_n)^{\frac{2}{n}}\wedge 1)\mathrm{Var}_m(f), \tag{4.4}$$

$$|\mathrm{Cov}_m(Pf,f)|\leq (\inf_{n\in\mathbb{N}}(1-\kappa_n)^{\frac{1}{n}}\wedge 1)\mathrm{Var}_m(f). \tag{4.5}$$

The main part of the following theorem is a slight generalization of [22, Corollary 31], and its proof is similar as in [22] based on Corollary 4.2 above.

Theorem 4.3 (Poincaré Inequality, cf. Corollary 31 in [22]). *Assume $m \in \mathcal{P}^2(E)$ and $\kappa \in \mathbb{R}$. Let $\mathbf{X}$ be an m-symmetric Markov chain on (E,d) and H a real separable Hilbert space. Then, for $f \in L^2(E,H;m)$*

$$(4.6)\qquad (1-\inf_{n\in\mathbb{N}}(1-\kappa_n)^{\frac{2}{n}}\wedge 1)\mathrm{Var}_m(f) \le \int_E \mathrm{Var}_{P_x}(f)m(dx),$$

$$(4.7)\qquad 1-\inf_{n\in\mathbb{N}}(1-\kappa_n)^{\frac{1}{n}}\wedge 1 \le \frac{\mathcal{E}(f)}{\mathrm{Var}_m(f)} \le 1+\inf_{n\in\mathbb{N}}(1-\kappa_n)^{\frac{1}{n}}\wedge 1.$$

In particular, if $\kappa_n > 0$ for some $n \in \mathbb{N}$, we have a global Poincaré inequality:

$$\begin{aligned} 0 < 1-\inf_{n\in\mathbb{N}}(1-\kappa_n)^{\frac{1}{n}} &\le \inf_{f\in L^2(E,H;m)} \frac{\mathcal{E}(f)}{\mathrm{Var}_m(f)} \\ &\le \sup_{f\in L^2(E,H;m)} \frac{\mathcal{E}(f)}{\mathrm{Var}_m(f)} \le 1+\inf_{n\in\mathbb{N}}(1-\kappa_n)^{\frac{1}{n}} < 2. \end{aligned}$$

Moreover, if $\mathbf{X}$ is an even step Markov chain obtained from an m-symmetric Markov chain, then

$$(4.8)\qquad \sup_{f\in L^2(E,H;m)} \frac{\mathcal{E}(f)}{\mathrm{Var}_m(f)} \le 1.$$

Corollary 4.4 (Estimates of Eigenvalues). *Let $\mathbf{X}$ be an m-symmetric Markov chain on (E,d) with $m \in \mathcal{P}^2(E)$ and assume that $\kappa \in \mathbb{R}$ and the embedding $D(\mathcal{E}) \subset L^2(E;m)$ is compact. Then any non-zero eigenvalue λ of the L^2-operator $-\Delta = I - P$ on $L^2(E;m)$ satisfies*

$$(4.9)\qquad 1-\inf_{n\in\mathbb{N}}(1-\kappa_n)^{\frac{1}{n}}\wedge 1 \le \lambda \le 1+\inf_{n\in\mathbb{N}}(1-\kappa_n)^{\frac{1}{n}}\wedge 1.$$

Moreover, if $\mathbf{X}$ is an even step Markov chain obtained from an m-symmetric Markov chain, then any eigenvalue λ satisfies $\lambda \le 1$.

Corollary 4.5 (Recurrence). *Assume $m \in \mathcal{P}^2(E)$ and $\kappa \in \mathbb{R}$. Let $\mathbf{X}$ be an m-symmetric Markov chain on (E,d). Suppose that there exists $n \in \mathbb{N}$ such that $\kappa_n > 0$. Then $\mathbf{X}$ is recurrent, that is, for any non-trivial $f \in L^1_+(E;m)$, we have $Gf = \infty$ m-a.e. Here $Gf := \sum_{i=0}^{\infty} P^i f$.*

Remark 4.6.
(1) When $\mathbf{X}$ is an m-symmetric random walk on a finite undirected weighted connected graph $G = (V,E)$ with $m(\{x\}) := d_x(G)$, the degree of G at $x \in V$, Bauer-Jost-Liu [4] proved (4.9) for any $n \in \mathbb{N}$.
(2) If we assume the existence of non-constant Lipschitz eigenfunction of $-\Delta := I - P$, then we can directly prove the estimate for the associated real eigenvalue λ;

$$(4.10)\qquad 1-(1-\kappa_n)^{\frac{1}{n}} \le \lambda \le 1+(1-\kappa_n)^{\frac{1}{n}}$$

under $\kappa_n(x,y) \ge \kappa_n(\in \mathbb{R})$ for $(x,y) \in E\times E\setminus \mathsf{diag}$ without assuming the m-symmetricity of $\mathbf{X}$. If $\kappa_n > 0$, (4.10) is equivalent to (4.9). We show (4.10) as mentioned above. Let f

be a non-constant Lipschitz eigenfunction and assume that λ is a real eigenvalue of f with respect to $-\Delta$. Then, we have $(I-P)f = \lambda f$, equivalently, $P^k f = (1-\lambda)^k f$ for any $k \in \mathbb{N}$. By scaling, we may assume that the Lipschitz constant of f is 1. Kantorovich-Rubinstein duality formula yields

$$\begin{aligned} d(x,y)(1-\kappa_n) \geq d_{W_1}(P_x^n, P_y^n) &\geq P^n f(x) - P^n f(y) \\ &= (1-\lambda)^n (f(x) - f(y)) \end{aligned}$$

for $(x,y) \in E \times E$, which implies $(1-\kappa_n) \geq |1-\lambda|^n$, that is, we obtain (4.10).

Theorem 4.7 (Strong L^p-Liouville Property). *Assume $m \in \mathcal{P}^p(E)$. Let $\mathbf{X}$ be an m-symmetric Markov chain on (E,d). Suppose that $\kappa \in \mathbb{R}$ and there exists $n \in \mathbb{N}$ such that $\kappa_n > 0$. Let (Y, d_Y) be a complete separable $CAT(0)$-space. Suppose that $u \in L^p(E, Y; m)$ satisfies $Pu = u$ m-a.e. on E. Then u is a constant map m-a.e. In particular, if $u \in \mathrm{Lip}(E,Y)$ is P-harmonic, then u is a constant map.*

Corollary 4.8 (Ergodicity). *Let $\mathbf{X}$ be an m-symmetric Markov chain on (E,d). Suppose that $\kappa \in \mathbb{R}$ and there exists $n \in \mathbb{N}$ such that $\kappa_n > 0$. Then $\mathbf{X}$ is ergodic, that is, for any P-invariant Borel set A, $m(A) = 0$ or $m(A^c) = 0$.*

Theorem 4.9 (Poincaré Inequality). *Assume $m \in \mathcal{P}^2(E)$ and $\kappa \in \mathbb{R}$. Let $\mathbf{X}$ be an m-symmetric Markov chain on (E,d). Suppose that there exists $n \in \mathbb{N}$ such that $\kappa_n > 0$. Let (Y, d_Y) be a complete separable $CAT(0)$-space. Then for any $\varepsilon \in]0, 1-(1-\kappa_n)^{\frac{1}{n}}[$, there exists $\ell_0 \in \mathbb{N}$ depending on $\varepsilon, \kappa_n, (E, d, m, \mathbf{X})$ and (Y, d_Y) such that*

$$\inf_{u \in L^2(E,Y;m)} \frac{E(u)}{\mathrm{Var}_m(u)} \geq \frac{(1-(1-\kappa_n)^{\frac{1}{n}} \wedge 1 - \varepsilon)^2}{8\ell_0^2} > 0.$$

Remark 4.10. (1) For the random walk on an undirected weighted finite graph $G = (V, E)$ with $N := |V|$, Bauer-Jost-Liu [4] proved the equivalence among the following:

(i) G is non-bipartite.

(ii) $\lambda_{N-1} < 2$.

(iii) There exists $n \in \mathbb{N}$ such that $\kappa_n > 0$.

Since G is connected, we have $\lambda_1 > 0$. Here λ_1 (resp. λ_{N-1}) is the smallest non-zero (resp. maximum) eigenvalue of the Laplace operator on G. Under the equivalent conditions (i)–(iii), we have a positivity of non-linear spectral gap as in Theorem 4.9. Remark that the positivity of non-linear spectral gap on the finite connected weighted graph G (having no loop and no multi-edges) with graph distance is already proved by Izeki-Kondo-Nayatani [7] for CAT(0)-space target.

(2) Our Theorem 4.9 covers the case for the random walk derived from the Brownian motion on Riemannian manifolds with positive Ricci curvature as in Example 1.5.

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