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Logic characterized by Boolean algebras with conjugate

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1 Introduction

In [1], Jarvinen and Kortelainen considered properties of lower (upper) approximation operators in rough set theory by use of the algebras with conjugate pair of maps. Let B be any Boolean algebra. A pair (f, g) of maps $f, g : B \to B$ is called *conjugate* ([2]) if, for all $x, y \in B$, the following condition is satisfied:

$$x \wedge f(y) = 0 \iff y \wedge g(x) = 0$$

Moreover if a pair (f, f) is conjugate, then f is called *self-conjugate.* If a Boolean algebra has a pair of conjugate maps, then we say simply a Boolean algebra with conjugate.

By **B** we mean the class of all Boolean algebras with conjugate. In this short note we show that **B** characterizes a ceratin kind of *tense logic* K_t^*, that is, for the class Φ of all formulas of K_t^*,

For any $B \in \mathbf{B}$ and a map $\xi : \Phi \to B$, we have $\xi(A) = 1 \iff \vdash^*_{K_t} A$

2 tense logic K_t^*

We define a certain kind of tense logic named K_t^* here. The logic is obtained from the minimal tense logic K_t by removing the axioms $(sym) : A \to GPA,\ A \to HFA$ and $(cl) : GA \to GGA,\ HA \to HHA$.

Let Φ_0 be a countable set $p_o, p_1, p_2, \cdots$ of propositional variables and $\wedge, \vee, \to, \neg, G, H$ are logical symbols. A formula of K_t^* is defined as follows:

(1) Every propositional variable is a formula;

(2) If A and B are formulas, then so are $A \wedge B, A \vee B, A \to B, \neg A, GA, HA$.

Let Φ be the set of all formulas of K_t^*. We define symbols F and P respectively by

$$FA \equiv \neg G \neg A,\ PA \equiv \neg H \neg A.$$

A logical system K_t^* has the following axioms and rules of inference ([3]):
Axioms :

(1) $A \to (B \to A)$

(2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(3) $(\neg A \to \neg B) \to (B \to A)$

(4) $G(A \to B) \to (GA \to GB),\ H(A \to B) \to (HA \to HB)$

Rule of Inference :

(MP) Deduce B from A and $A \to B$;

(Nec) Deduce GA and HA from A.

We list typical axioms which chracterize some properties of conjugate:

(ext) : $GA \to A, HA \to A$
(sym) : $A \to GPA,\ A \to HFA$
(cl) : $GA \to GGA,\ HA \to HHA$

A well-known tense logic K_t is an axiomatic extension of K_t^*, which has extra axioms (*sym*) and (*cl*), that is,

$$K_t = K_t^* + (sym) + (cl)$$

A formula A is called ***provable*** when there is a finite sequence $A_1, A_2, \cdots, A_n (= A)$ $(n \geq 1)$ of formulas such that, for every i $(1 \leq i \leq n)$,

(1) A_i is an axiom;

(2) A_i is deduced from A_j, A_k $(j, k < i)$ by (MP);

(3) A_i is done from A_j $(j < i)$ by (Nec).

We denote that A is provable by

$$\vdash_{K_t^*} A \quad (\text{or simply } \vdash A).$$

A relational structure (W, R) is called a *Kripke frame*, where W is a non-empty set and R is a binary relation on it. A valuation v is a map from Φ_0 to $\mathcal{P}(W)$, that is, $v : \Phi_0 \to \mathcal{P}(W)$. It is easy to show that a valuation v can be extended uniquely to the set Φ of all formulas:

(1) $v(A \wedge B) = v(A) \cap v(B)$

(2) $v(A \vee B) = v(A) \cup v(B)$

(3) $v(A \to B) = v(A)^c \cup v(B)$

(4) $v(\neg A) = v(A)^c$

(5) $v(GA) = \{x \in W \mid \forall y((x, y) \in R \Longrightarrow y \in v(A))\}$

(6) $v(HA) = \{x \in W \mid \forall y((y, x) \in R \Longrightarrow y \in v(A))\}$

Thus we call the extended valuation above simply a valuation and denote it by the same symbol v.

Since, for all formulas A and B

$$\vdash_{K_t^*} A \wedge \neg A \to B \wedge \neg B, \quad \vdash_{K_t^*} A \vee \neg A \to B \vee \neg B,$$

We define symbols $\bot$ and $\top$ respectively by

$$\bot \equiv A \wedge \neg A, \quad \top \equiv A \vee \neg A.$$

Then for every formula $A \in \Phi$, we have

$$\vdash_{K_t^*} \bot \to A, \quad \vdash_{K_t^*} A \to \top.$$

A structure $\mathcal{M} = (W, R, v)$ is called a *Kripke model*, where (W, R) is a Kripke frame and v is a valuation on it. Given a Kripke model $\mathcal{M} = (W, R, v)$, we can interpret the formulas on it as follows: For $x \in W$, a formula A is said to be *true* at x on the Kripke model $\mathcal{M}$ if

$$x \in v(A),$$

and denoted by

$$\mathcal{M} \models_x A.$$

If $v(A) = W$, that is, A is true at ever $x \in W$ on the Kripke model $\mathcal{M}$, then A is called *true* on $\mathcal{M}$ and denoted by

$$\mathcal{M} \models A.$$

Moreover A is called ***valid*** if A is true on every Kripke model $\mathcal{M}$ and denoted by

$$\models A.$$

It is easy to show the next result ([3]):

Theorem 1. *(Completeness Theorem) For every formula A, we have*

$$\vdash_{K_t^*} A \iff A : \text{valid}$$

We can get the next result by use of *filtration* method ([3]):

Theorem 2. *For every formula A, we have*

$$\vdash_{K_t^*} A \iff A : \textit{true} \text{ for any finite Kripke model } \mathcal{M}.$$

3 Boolean algebra with conjugate pair

Let $\mathcal{B} = (B, \wedge, \vee, ', 0, 1)$ be a *Boolean algebra.* A pair (φ, ψ) of maps $\varphi, \psi : B \to B$ is called a *conjugate pair* if , for all $x, y \in B$,

$$x \wedge \varphi(y) = 0 \iff y \wedge \psi(x) = 0.$$

We define some properties about a map $\varphi : B \to B$ as follows:

$$\begin{aligned} \varphi : \text{extensive} &\iff x \leq \varphi(x) \quad (\forall x \in B) \\ \varphi : \text{symmetric} &\iff x \leq \varphi(y) \text{ implies } y \leq \varphi(x) \quad (\forall x, y \in B) \\ \varphi : \text{closed} &\iff y \leq \varphi(x) \text{ implies } \varphi(y) \leq \varphi(x) \quad (\forall x, y \in B) \end{aligned}$$

It is clear that the following holds for a conjugate pair (φ, ψ) ([1]):

$$\begin{aligned} \varphi : \text{extensive} &\iff \psi : \text{extensive} \\ \varphi : \text{symmetric} &\iff \varphi : \text{self} - \text{conjugate} \\ \varphi : \text{closed} &\iff \psi : \text{closed} \end{aligned}$$

We introduce two operators $\varphi^\partial, \psi^\partial$ for the sake of simplicity

$$\varphi^\partial(x) = (\varphi(x'))', \ \psi^\partial(x) = (\psi(x'))' \quad (x \in B).$$

A conjugate pair (φ, ψ) can be represented by

$$\varphi(x) \leq y \iff x \leq \psi^\partial(y) \quad (x, y \in B).$$

It is obvious from definition that

Proposition 1. *For every $x \in B$ we have*

$$\begin{aligned} \varphi : \text{extensive} &\iff \varphi^\partial(x) \leq x \\ \varphi : \text{symmetric} &\iff x \leq \varphi^\partial(\varphi(x)) \\ \varphi : \text{closed} &\iff \varphi^\partial(x) \leq \varphi^\partial(\varphi^\partial(x)) \end{aligned}$$

Let **B** be a Boolean algebra with conjugate and $\xi : \Phi \to B$ be a map. Each formula of K_t^* is interpreted on the algebra as follows:

(1) $\xi(A \wedge B) = \xi(A) \wedge \xi(B)$
(2) $\xi(A \vee B) = \xi(A) \vee \xi(B)$
(3) $\xi(A \to B) = (\xi(A))' \vee \xi(B)$
(4) $\xi(\neg A) = (\xi(A))'$
(5) $\xi(GA) = (\varphi((\xi(A))'))' = \varphi^\partial(\xi(A))$
(6) $\xi(HA) = (\psi(\xi(A)'))' = \psi^\partial(\xi(A))$

Lemma 1. *For every formula A, we have*

$$\vdash_{K_t^*} A \implies \xi(A) = 1 \text{ for all } \xi : \Phi \to B$$

Proof. It is sufficient to verify that each axiom α of K_t^* has a value $\xi(\alpha) = 1$ and each rule of inference is preserved, that is, for the case of (MP),

$$\xi(A) = \xi(A \to B) = 1 \text{ imply } \xi(B) = 1$$

and for the case of (Nec)

$$\xi(A) = 1 \text{ implies } \xi(GA) = \xi(HA) = 1.$$

We omit their proof. □

We can show the converse direction of the above. In order to do that we prepare some lemmas. At first we define a relation $\equiv$ on the set Φ of formulas of K_t^* : For $A, B \in \Phi$,

$$A \equiv B \iff \vdash_{K_t^*} A \to B \text{ and } \vdash_{K_t^*} B \to A$$

As to the relation $\equiv$ we can prove that

Lemma 2. $\equiv$ *is a congruence on* Φ, *that is, it is an equivalence relation and satisfies the compatible property : If* $A \equiv B$ *and* $C \equiv D$, *then*

$$\begin{aligned} A \wedge C \equiv B \wedge D&, \ A \vee C \equiv B \vee D, \\ A \to D &\equiv B \to D, \\ \neg A &\equiv \neg B, \\ GA \equiv GB&, \ HA \equiv HB \end{aligned}$$

Proof. We only prove that if $A \equiv B$ then $GA \equiv GB$. It follows from assumption that $\vdash A \to B$. From (Nec) we get

$$\vdash G(A \to B).$$

On the other hand, since $\vdash G(A \to B) \to (GA \to GB)$, we have from (MP)

$$\vdash GA \to GB.$$

Similarly, by $\vdash B \to A$, we get

$$\vdash GB \to GA.$$

This means that

$$GA \equiv GB.$$

□

Since $\equiv$ is the congruence, we can define operations on $\Phi/\equiv$: For $A, B \in \Phi$, we define

$$\begin{aligned} [A] \sqcap [B] &= [A \wedge B], \\ [A] \sqcup [B] &= [A \vee B], \\ [A]^* &= [\neg A], \\ \varphi([A]) &= [\neg G \neg A] = [FA], \\ \psi([A]) &= [\neg H \neg A] = [PA], \\ \mathbf{0} = [\bot]&, \ \mathbf{1} = [\top]. \end{aligned}$$

Lemma 3. $(\Phi/\equiv, \sqcap, \sqcup, ^*, \mathbf{0}, \mathbf{1})$ *is a Boolean algebra with* (φ, ψ) *as a conjugate pair.*

Proof. We show that (φ, ψ) is the conjugate pair. Let $[A], [B] \in \Phi/\equiv$. We have to prove

$$[A] \sqcap \varphi([B]) = \mathbf{0} \iff [B] \sqcap \psi([A]) = \mathbf{0},$$

that is,

$$[A \wedge FB] = \mathbf{0} \iff [B \wedge PA] = \mathbf{0}.$$

Suppose that $[A \wedge FB] = \mathbf{0}$. Since $\vdash A \wedge FB \to \bot$, we have $\vdash FB \to \neg A$. From (Nec) we get $\vdash HFB \to H\neg A$. Since $\vdash B \to HFB$, we also have $\vdash B \to H\neg A$. Thus we obtain

$$\vdash \neg(B \wedge PA),$$

that is,

$$[B \wedge PA] = \mathbf{0}.$$

It is similar the converse. □

Lemma 4. *For any formula $A \in \Phi$,*

$$\vdash_{K_t^*} A \iff [A] = \mathbf{1} \textit{ in } \Phi/\equiv$$

Proof.

$$\begin{aligned} \vdash_{K_t} A &\iff \vdash_{K_t} A \to \top \text{ and } \vdash_{K_t} \top \to A \\ &\iff [A] = [\top] = \mathbf{1} \end{aligned}$$

□

From the above, we can prove the next theorem.

Theorem 3. *Let $A \in \Phi$.*

For any Boolean algebra B with conjugate and a map $\xi : \Phi \to B$, we have $\xi(A) = 1$ $\iff \vdash_{K_t^*} A$

Proof. We have already proved if part. To show the only if part, we assume that $\nvdash_{K_t^*} A$. Since $\Phi/\equiv$ is the Boolean algebra with conjugate, if we take a map

$$\xi : \Phi \to \Phi/\equiv,\ \xi(A) = [A],$$

then on $\Phi/\equiv$ we get

$$\xi(A) \neq \mathbf{1}$$

by $\nvdash_{K_t^*} A$. □

We can characterize some logics by Boolean algebras with conjugate.

Theorem 4. *Logical systems $K_t^* + (ext)$, $K_t^* + (sym)$, $K_t^* + (cl)$ are characterized respectively by the Boolean algebras with extensive, symmetric, closed conjugate, that is, for any formula $A \in \Phi$*

(1) for any Boolean algebra B with extensive conjugate and a map $\xi : \Phi \to B$, we have $\xi(A) = 1 \iff \vdash_{K_t^*+(ext)} A$

(2) for any Boolean algebra B with symmetric conjugate and a map $\xi : \Phi \to B$, we have $\xi(A) = 1 \iff \vdash_{K_t^*+(sym)} A$

(3) for any Boolean algebra B with closed conjugate and a map $\xi : \Phi \to B$, we have $\xi(A) = 1$ $\iff \vdash_{K_t^*+(cl)} A$

Proof. We only show that, in any Boolean algebra with typical property, $\xi(A) = 1$ for the correspondent typical axioms A in respective cases. Suppose that $\xi(A) = x \in B$.

(1) For an extensive conjugate (φ, ψ), we have to prove that $\xi(GA \to A) = 1$. Since

$$\begin{aligned}\xi(GA \to A) = 1 &\Longleftrightarrow \xi(GA) \leq \xi(A) \\ &\Longleftrightarrow (\varphi(x'))' \leq x \\ &\Longleftrightarrow x' \leq \varphi(x')\end{aligned}$$

and φ is extensive, we have $\xi(GA \to A) = 1$.

(2) Let (φ, ψ) be a symmetric conjugate. Since $\varphi = \psi$ by assumption, we have

$$\begin{aligned}\xi(A \to GPA) = 1 &\Longleftrightarrow \xi(A) \leq \xi(GPA) \\ &\Longleftrightarrow x \leq \varphi^\partial(\psi(x)) \\ &\Longleftrightarrow x \leq \psi^\partial(\psi(x)) \\ &\Longleftrightarrow \varphi(x) \leq \psi(x) \\ &\Longleftrightarrow \varphi(x) \leq \varphi(x).\end{aligned}$$

Thus, $\xi(A \to GPA) = 1$.

(3) Suppose that (φ, ψ) is a closed conjugate. It follows from the assumption that $\varphi^\partial(x) \leq \varphi^\partial(\varphi^\partial(x))$ $(x \in B)$ and hence that

$$\begin{aligned}\xi(GA \to GGA) = 1 &\Longleftrightarrow \xi(GA) \leq \xi(GGA) \\ &\Longleftrightarrow \varphi^\partial(x) \leq \varphi^\partial(\varphi^\partial(x)).\end{aligned}$$

This means that $\xi(A \to GPA) = 1$. □

4 Decidability

It is well-known that the minimal tense logic K_t can be characterized by the class of *finite* Kripke models. Similarly we can show that K_t^* is characterized by the class $\mathbf{B}^*$ of *finite* Boolean algebras with conjugate.

Suppose that $\not\vdash_{K_t^*} A$. There is a finite Kripke model $\mathcal{M}^* = (W, R, v)$ such that $x \notin v(A)$ for some $x \in W$, that is, $v(A) \neq W$. We construct a finite Boolean algebra B^* with conjugate from the finite Kripke model $\mathcal{M}^*$ as follows:

$B^* = \mathcal{P}(W)$
$\varphi, \psi : B \to B$ are defined respectively by

$$\begin{aligned}\varphi(X) &= \{x \in B \mid R(x) \cap X \neq \emptyset\} \\ \psi(X) &= \{x \in B \mid R^{-1}(x) \cap X \neq \emptyset\},\end{aligned}$$

where $R(x),\ R^{-1}(x)$ are defined by

$$R(x) = \{y \in B \mid (x, y) \in R\},\ R^{-1}(x) = \{y \in B \mid (y, x) \in R\}$$

We can prove the fundamental result.

Lemma 5. *B^* is a finite Boolean algebra with a conjugate pair $\varphi, \psi : B^* \to B^*$.*

Proof. It is sufficient to prove that $\varphi, \psi : B^* \to B^*$ are conjugate. That is, we have to prove that for $X, Y \subseteq W$ (*i.e.*, $X, Y \in B^*$),

$$X \cap \varphi(Y) = \emptyset \iff Y \cap \psi(X) = \emptyset.$$

Suppose that $Y \cap \psi(X) \neq \emptyset$. Since $y \in \psi(X)$ for some $y \in Y$, it follows from definition of $\psi(X)$ that

$$\exists x \in X \ s.t.\ (x, y) \in R.$$

We also have $(x, y) \in R$ and $y \in Y$. This implies that

$$R(x) \cap Y \neq \emptyset$$

and $x \in \varphi(Y)$. The fact that $x \in X$ means

$$x \in X \cap \varphi(Y), \text{ that is, } X \cap \varphi(Y) \neq \emptyset.$$

The converse can be proved similarly. Thus B^* is the finite Boolean algebra with the conjugate pair $\varphi, \psi : B^* \to B^*$. □

Moreover if we take $\xi^* : \Phi \to B^*$ as

$$\xi^*(A) = v(A),$$

then we have $\xi^*(A) \neq 1$ from $v(A) \neq W$. This means that $\not\vdash_{K_t^*} A$ implies $\xi^*(A) \neq 1$ for some finite Boolean algebra with conjugate and $\xi^* : B^* \to B^*$. It is obvious the converse statement. We thus obtain the next result.

Theorem 5. *The logic K_t^* can be characterized by the finite Boolean algebras with conjugate.*

We can show the following similarly.

Theorem 6. *The logics $K_t^* + (ext)$, $K_t^* + (sym)$, $K_t^* + (cl)$ are characterized by the class of all finite Boolean algebras with extensive, symmetric, closed conjugate pair, respectively.*

Thus we can conclude that our logical systems $K_t^*(+(ext), +(sym), +(cl))$ are decidable, that is, we can determine whether a given formula is provable or not by finite steps.

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