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Heat equation with absorption and non-decaying initial data

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1 Introduction

This paper is a joint work with Professor Kazuhiro Ishige and a part of [10]. In this paper we are concerned with the large time behavior of the solution of the Cauchy problem for the heat equation with absorption,

$$\begin{cases} \partial_t u = \Delta u - u^\beta & \text{in} \quad \mathbf{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0 & \text{in} \quad \mathbf{R}^N, \end{cases} \tag{1.1}$$

where $N \geq 1$, $\partial_t = \partial/\partial t$, and $\beta > 1$. The large time behavior of the solution of (1.1) depends on β and the behavior of the initial function u_0 at the space infinity, and has been studied intensively by many mathematicians (see for example [1]–[9], [11], [13]–[16]). If the initial function u_0 decays slowly at the space infinity, then the solution u of (1.1) behaves like the function

$$\left(\frac{1}{\beta-1}\right)^{\frac{1}{\beta-1}} t^{-\frac{1}{\beta-1}}$$

as $t \to \infty$, which is the solution of the ordinary differential equation

$$\zeta' = -\zeta^\beta. \tag{1.2}$$

More precisely, if

$$u_0 \in L^p(\mathbf{R}^N) \text{ for some } p \in [1,\infty), \qquad \lim_{|x|\to\infty} \operatorname{ess} |x|^{\frac{2}{\beta-1}} u_0(x) = \infty, \tag{1.3}$$

then

$$\lim_{t\to\infty} t^{\frac{1}{\beta-1}} u(x,t) = \left(\frac{1}{\beta-1}\right)^{\frac{1}{\beta-1}} \tag{1.4}$$

uniformly on the set $E_c := \{x \in \mathbf{R}^N : |x| \leq ct^{1/2}\}$ for any $c > 0$ (see [3, Theorem 2.1]). We remark that the behavior of the solution u at the space infinity depends on the behavior of the initial function u_0 at the space infinity (see [6, Theorem 1]).

In this paper we consider problem (1.1) with the non-decaying initial data

$$u_0(x) = \lambda + \varphi(x) \geq 0, \tag{1.5}$$

where $\lambda > 0$ and φ is a bounded continuous function in $\mathbf{R}^N$ such that $\varphi \in L^p(\mathbf{R}^N)$ for some $1 \le p < \infty$, and study the large time behavior of the solution u of (1.1). Let ζ_λ be the solution of (1.2) with $\zeta(0) = \lambda$, that is,

$$\zeta_\lambda(t) := (\lambda^{-(\beta-1)} + (\beta-1)t)^{-\frac{1}{\beta-1}}. \tag{1.6}$$

Then

$$\zeta_\lambda(t) = \left(\frac{1}{\beta-1}\right)^{\frac{1}{\beta-1}} t^{-\frac{1}{\beta-1}}(1+O(t^{-1})) \quad \text{as} \quad t \to \infty, \tag{1.7}$$

and it follows from the local L^∞- estimates for parabolic equations and the comparison principle that

$$u(x,t) = \left(\frac{1}{\beta-1}\right)^{\frac{1}{\beta-1}} t^{-\frac{1}{\beta-1}}(1+O(t^{-1})) \quad \text{as} \quad t \to \infty \tag{1.8}$$

uniformly on $\mathbf{R}^N$ (see Proposition 3.1). Here we put

$$v(x,t) := \frac{\lambda^\beta[u(x,t) - \zeta_\lambda(t)]}{\zeta_\lambda(t)^\beta}. \tag{1.9}$$

Then, by (1.7) and (1.9) we see that

$$\|v(t)\|_{L^\infty(\mathbf{R}^N)} = O(1) \quad \text{as} \quad t \to \infty.$$

In this paper we are especially interested in the precise description of the large time behavior of the function v, and prove that the function v behaves like a solution of the heat equation. Furthermore, we give decay estimates on the difference between the function v and its asymptotic profile (see Theorems 1.1 and 1.2), and prove that the solution u behaves like

$$\left((\lambda + h(x,t))^{-(\beta-1)} + (\beta-1)t\right)^{-\frac{1}{\beta-1}}$$

as $t \to \infty$ uniformly on $\mathbf{R}^N$, where h is a solution of the heat equation (see Corollary 1.1). As far as we know, there are no results giving the precise descriptions of the large time behavior of the solution of (1.1) with non-decaying initial data.

We introduce some notation. Let $B(x,R) := \{y \in \mathbf{R}^N \,:\, |y-x| < R\}$ for $x \in \mathbf{R}^N$ and $R > 0$. For any $k \in \mathbf{R}$, let $[k]$ be an integer such that $k-1 < [k] \le k$. Let $\mathbf{M} := (\mathbf{N} \cup \{0\})^N$. For any $\nu = (\nu_1, \ldots, \nu_N) \in \mathbf{M}$, we put

$$|\nu| := \sum_{i=1}^N \nu_i, \qquad x^\nu := x_1^{\nu_1} \cdots x_N^{\nu_N}, \qquad \nu! := \nu_1! \cdots \nu_N!, \qquad \partial_x^\nu := \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1} \cdots \partial x_N^{\nu_N}},$$
$$J(\nu) := \{\mu = (\mu_1, \ldots, \mu_N) \in \mathbf{M} \,:\, \mu_i \le \nu_i \, (i = 1, \ldots, N),\ \mu \ne \nu\}.$$

Let $BC(\mathbf{R}^N) := C(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. For any $r \in [1,\infty]$, we denote by $\|\cdot\|_r$ the usual norm of $L^r(\mathbf{R}^N)$. For any $k \ge 0$, we denote by $|||\cdot|||_k$ the norm of $L^1(\mathbf{R}^N, (1+|x|^k)dx)$, that is,

$$|||f|||_k := \int_{\mathbf{R}^N} |f(x)|(1+|x|^k)dx \qquad \text{for} \qquad f \in L^1(\mathbf{R}^N, (1+|x|^k)dx).$$

Let G be the fundamental solution of the heat equation on $\mathbf{R}^N$, that is,

$$G(x,t) := (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right), \tag{1.10}$$

and put

$$g_\nu(x,t) := \frac{(-1)^{|\nu|}}{\nu!}(\partial_x^\nu G)(x,t+1) \qquad \text{for any} \quad \nu \in \mathbf{M}.$$

In particular, we write $g(x,t) := g_0(x,t)$ for simplicity. For any $\phi \in L^\infty(\mathbf{R}^N)$, let $e^{t\Delta}\phi$ be a unique bounded solution of the heat equation $\partial_t z = \Delta z$ in $\mathbf{R}^N \times (0,\infty)$ with $z(0) = \phi$, that is,

$$(e^{t\Delta}\phi)(x) := \int_{\mathbf{R}^N} G(x-y,t)\phi(y)dy. \tag{1.11}$$

For any sets Λ and Σ, let $f = f(\lambda,\sigma)$ and $h = h(\lambda,\sigma)$ be maps from $\Lambda \times \Sigma$ to $(0,\infty)$. Then we say

$$f(\lambda,\sigma) \preceq h(\lambda,\sigma) \quad \text{for all} \quad \lambda \in \Lambda$$

if, for any $\sigma \in \Sigma$, there exists a positive constant C such that $f(\lambda,\sigma) \le Ch(\lambda,\sigma)$ for all $\lambda \in \Lambda$. In addition, we say

$$f(\lambda,\sigma) \asymp h(\lambda,\sigma) \quad \text{for all} \quad \lambda \in \Lambda$$

if $f(\lambda,\sigma) \preceq h(\lambda,\sigma)$ and $f(\lambda,\sigma) \succeq h(\lambda,\sigma)$ for all $\lambda \in \Lambda$.

We are ready to state the main results of this paper. In Theorems 1.1 and 1.2 we give the large time behavior of the functions v by using the heat equation. Furthermore, we give decay estimates of the difference between the functions v and its asymptotic profile. See also [10, Theorem 1.1, Theorem 1.2].

Theorem 1.1 *Assume* (1.5) *and*

$$\varphi \in BC(\mathbf{R}^N) \cap L^p(\mathbf{R}^N) \tag{1.12}$$

for some $p \in [1,\infty)$. Let u be a solution of (1.1). *Then, for any $q \in [p,\infty]$,*

$$\sup_{t>0} t^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}\|v(t)\|_q < \infty. \tag{1.13}$$

Furthermore, if $p > 1$, then

$$t^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}\|v(t) - e^{t\Delta}\varphi\|_q = O(t^{-\frac{N}{2p}}) + O(t^{-\frac{N}{2}(1-\frac{1}{p})}) \tag{1.14}$$

as $t \to \infty$, for any $q \in [p,\infty]$.

Theorem 1.2 *Assume* (1.5) *and*

$$\varphi \in BC(\mathbf{R}^N) \cap L^1(\mathbf{R}^N, (1+|x|^K)dx) \tag{1.15}$$

for some $K \ge 0$ with

$$K < N + 2. \tag{1.16}$$

Let u be a solution of (1.1). *Then the following hold:*
(i) *For any $l \in [0, K]$,*

$$\sup_{t>0}(1+t)^{-\frac{l}{2}}|||v(t)|||_l < \infty;$$

(ii) *For any $\nu \in \mathbf{M}$ with $|\nu| \le K$, put*

$$\begin{aligned} M_\nu(t) &:= \int_{\mathbf{R}^N} x^\nu v(x,t)dx \quad \text{if} \quad |\nu| \le 1, \\ M_\nu(t) &:= \int_{\mathbf{R}^N} x^\nu v(x,t)dx - \sum_{\mu \in J(\nu)} M_\mu(t) \int_{\mathbf{R}^N} x^\nu g_\mu(x,t)dx \quad \text{if} \quad |\nu| \ge 2. \end{aligned} \tag{1.17}$$

Then there exists a constant M_ν such that

$$M_\nu = \lim_{t\to\infty} M_\nu(t);$$

(iii) *Put*

$$W(x,t) := \sum_{|\nu|\le K} M_\nu g_\nu(x,t).$$

Then

$$t^{\frac{N}{2}(1-\frac{1}{q})}\|v(t) - W(t)\|_q = o(t^{-\frac{K}{2}})$$

as $t \to \infty$, for any $q \in [1, \infty]$.

As a corollary of Theorems 1.1 and 1.2 with $q = \infty$, we have the following result. See also [10, Corollary 1.1].

Corollary 1.1 *Let u be a solution of* (1.1) *and assume* (1.5). *Then the following hold:*
(i) *Assume $\varphi \in BC(\mathbf{R}^N) \cap L^p(\mathbf{R}^N)$ for some $1 < p < \infty$. Then*

$$\begin{aligned} u(x,t) &= \zeta_\lambda(t) + \lambda^{-\beta}\zeta_\lambda(t)^\beta (e^{t\Delta}\varphi)(x) + O(t^{-\frac{N}{p'}}\zeta_\lambda(t)^\beta) \\ &= \left[(e^{t\Delta}u_0)(x)^{-(\beta-1)} + (\beta-1)t\right]^{-\frac{1}{\beta-1}} + O(t^{-\frac{N}{p'}}\zeta_\lambda(t)^\beta) \end{aligned}$$

in $L^\infty(\mathbf{R}^N)$ as $t \to \infty$, where $p' = \max\{p, 2\}$;
(ii) *Assume $\varphi \in BC(\mathbf{R}^N) \cap L^1(\mathbf{R}^N, (1+|x|^K)dx)$ for some $K \ge 0$ with* (1.16). *Let W be the function given in assertion* (iii) *of Theorem* 1.2. *Then*

$$\begin{aligned} u(x,t) &= \zeta_\lambda(t) + \lambda^{-\beta}\zeta_\lambda(t)^\beta W(x,t) + o(t^{-\frac{N}{2}-\frac{K}{2}}) \\ &= \left[(e^{t\Delta}(\lambda + W(0)))(x)^{-(\beta-1)} + (\beta-1)t\right]^{-\frac{1}{\beta-1}} + o(t^{-\frac{N}{2}-\frac{K}{2}}) \end{aligned}$$

in $L^\infty(\mathbf{R}^N)$ as $t \to \infty$.

Corollary 1.1 gives more precise description of the large time behavior of u than that of (1.8) and shows that the heat equation plays an important role of determining the large time behavior of the solution u of nonlinear parabolic problem (1.1).

Our analysis in this paper is based on the arguments in [8] and [9], where Ishige and Kawakami studied the large time behavior of the solutions of nonlinear parabolic equations and obtained higher order asymptotic expansions of the solutions behaving like the heat kernel. In order to prove Theorems 1.1 and 1.2, we first prove that the function v is a solution of the Cauchy problem,

$$\begin{cases} \partial_t v = \Delta v + F(x,t) & \text{in} \quad \mathbf{R}^N \times (0,\infty), \\ v(x,0) = \varphi(x) & \text{in} \quad \mathbf{R}^N \end{cases} \tag{1.18}$$

(see (3.3) and Proposition 3.1). Here the inhomogeneous term F satisfies

$$|F(x,t)| \preceq (1+t)^{-2}|v(x,t)|^2 \tag{1.19}$$

for all $(x,t) \in \mathbf{R}^N \times (0,\infty)$. Then, under the condition that $\varphi \in BC(\mathbf{R}^N) \cap L^p(\mathbf{R}^N)$ for some $1 \le p < \infty$, we can give

$$\|v(t)\|_q + t^{1/2}\|\nabla v(t)\|_q \preceq t^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}, \qquad t > 0, \tag{1.20}$$

for any $p \le q \le \infty$ (see (3.14)). Since it follows from (1.18) that

$$v(t) = e^{t\Delta}\varphi + \int_0^t e^{(t-s)\Delta}F(s)ds, \qquad t > 0, \tag{1.21}$$

we apply (1.19) and (1.20) to (1.21), and prove Theorem 1.1. Furthermore, under the conditions (1.15) and (1.16), we apply the results in [9, Section 4] with the aid of (1.19) and (1.20), and prove Theorem 1.2.

The rest of this paper is organized as follows. In Section 2 we recall some properties of $e^{t\Delta}\varphi$, and give some results on the large time behavior of solutions of nonlinear parabolic equations. In Section 3 we give some decay estimates of v, and prove Theorems 1.1 and 1.2 and Corollary 1.1.

2 Preliminaries

In this section we recall some fundamental properties of $e^{t\Delta}\varphi$, and give some results on the large time behavior of solutions of nonlinear parabolic equations.

Let G be the fundamental solution of the heat equation on $\mathbf{R}^N$ (see (1.10)). Then, for any $\nu \in \mathbf{M}$ and $j = 0,1,2,\dots$, there exists a constant C_1 such that

$$|\partial_t^j \partial_x^\nu G(x,t)| \le C_1 t^{-\frac{N+|\nu|+2j}{2}} \left[1 + \left(\frac{|x|}{t^{1/2}}\right)^{|\nu|+2j}\right] \exp\left(-\frac{|x|^2}{4t}\right) \tag{2.1}$$

for all $(x,t) \in \mathbf{R}^N \times (0,\infty)$. This inequality yields the inequalities

$$\|g_\nu(t)\|_q \preceq (1+t)^{-\frac{N}{2}(1-\frac{1}{q})-\frac{|\nu|}{2}}, \quad \int_{\mathbf{R}^N} |x|^l |g_\nu(x,t)| dx \preceq (1+t)^{\frac{l-|\nu|}{2}}, \quad t>0, \tag{2.2}$$

for any $q \in [1,\infty]$ and $l \geq 0$. Furthermore, by (1.11) and (2.1) we have:

(G) For any $\nu \in \mathbf{M}$ and $1 \leq p \leq q \leq \infty$, there exists a constant $c_{|\nu|}$, independent of p and q, such that

$$\|\partial_x^\nu e^{t\Delta}\varphi\|_q \leq c_{|\nu|} t^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})-\frac{|\nu|}{2}} \|\varphi\|_p, \quad t>0.$$

In particular, there holds $\|e^{t\Delta}\varphi\|_q \leq \|\varphi\|_q$ for all $t>0$.

Let z be a solution of the Cauchy problem for the nonlinear parabolic equation,

$$\begin{cases} \partial_t z = \Delta z + H(x,t,z,\nabla z) & \text{in} \quad \mathbf{R}^N \times (0,\infty), \\ z(x,0) = \phi(x) & \text{in} \quad \mathbf{R}^N, \end{cases} \tag{2.3}$$

where $H \in C(\mathbf{R}^N \times (0,\infty) \times \mathbf{R} \times \mathbf{R}^N)$. Assume that

$$|H(x,t,z(x,t),\nabla z(x,t))| \preceq (1+t)^{-A}(|z(x,t)| + (1+t)^{\frac{1}{2}}|\nabla z(x,t)|) \tag{2.4}$$

holds in $\mathbf{R}^N \times (0,\infty)$ for some $A \in \mathbf{R}$. Then, by the same argument as in the proof of [9, Lemma 3.1], we have

$$\sup_{0<t\leq T} t^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})} \left[\|z(t)\|_q + t^{\frac{1}{2}}\|\nabla z(t)\|_q\right] < \infty \tag{2.5}$$

for any $T>0$ and $q \in [p,\infty]$. Furthermore, we have the following lemma.

Lemma 2.1 *Let z be a solution of* (2.3) *and assume* (2.4) *for some $A>1$. Then the following hold:*
(i) *If $\phi \in BC(\mathbf{R}^N) \cap L^p(\mathbf{R}^N)$ for some $1 \leq p < \infty$, then*

$$\sup_{t>0} t^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})} \left[\|z(t)\|_q + t^{\frac{1}{2}}\|\nabla z(t)\|_q\right] < \infty$$

for any $q \in [p,\infty]$;
(ii) *If $\phi \in BC(\mathbf{R}^N) \cap L^1(\mathbf{R}^N, (1+|x|^k)dx)$ for some $k \geq 0$, then*

$$\sup_{t>0} (1+t)^{-\frac{l}{2}} \left[|||z(t)|||_l + t^{\frac{1}{2}}|||\nabla z(t)|||_l\right] < \infty$$

for any $l \in [0,k]$.

Proof. Assertion (ii) is given in [9, Theorem 3.1]. Assertion (i) is also proved by the modification of the proof of [9, Theorem 3.1]. See [10, Lemma 2.1]. □

Next we assume that $\phi \in BC(\mathbf{R}^N) \cap L^1(\mathbf{R}^N, (1+|x|^k)dx)$ for some $k \geq 0$, and recall some results on the asymptotics of the solution z of (2.3). By assertion (ii) of Lemma 2.1, for any $\nu \in \mathbf{M}$ with $|\nu| \leq k$, we can define

$$
\begin{aligned}
m_\nu(z(t),t) &:= \int_{\mathbf{R}^N} x^\nu z(x,t)dx \quad \text{if} \quad |\nu| \leq 1, \\
m_\nu(z(t),t) &:= \int_{\mathbf{R}^N} x^\nu z(x,t)dx \\
&\qquad - \sum_{\mu \in J(\nu)} m_\mu(z(t),t) \int_{\mathbf{R}^N} x^\nu g_\mu(x,t)dx \quad \text{if} \quad |\nu| \geq 2,
\end{aligned} \tag{2.6}
$$

inductively. Put

$$[P_k(t)z(t)](x) := z(x,t) - \sum_{|\nu|\leq k} m_\nu(z(t),t) g_\nu(x,t).$$

Then we have:

(Z_1) Let $\nu \in \mathbf{M}$ with $|\nu| \leq k$. If $A > 1 + |\nu|/2$, then there exists a constant m_ν such that

$$|m_\nu(z(t),t) - m_\nu| \preceq (1+t)^{-(A-1)+|\nu|/2}, \qquad t > 0.$$

Furthermore, there holds

$$m_\nu(z(t),t) = \begin{cases} O(t^{-(A-1)+|\nu|/2}) & \text{if} \quad A < 1 + |\nu|/2, \\ O(\log t) & \text{if} \quad A = 1 + |\nu|/2, \end{cases}$$

as $t \to \infty$;

(Z_2) For any $\mu \in \mathbf{M}$ with $|\mu| \leq k$,

$$\int_{\mathbf{R}^N} x^\mu [P_k(t)z(t)](x)dx = 0, \qquad t > 0;$$

(Z_3) The function $P_k(t)z(t)$ satisfies

$$P_k(t)z(t) = e^{(t-\tau)\Delta} P_k(\tau)z(\tau) + \int_\tau^t e^{(t-\tau)\Delta} P_k(s)H(s)ds$$

for all $t \geq \tau \geq 0$.

(See [7, Section 2], [8], and [9, Section 4].) Furthermore, by properties (Z_1)–(Z_3) we have the following lemma. (See [9, Theorem 4.1].)

Lemma 2.2 *Assume the same conditions as in Lemma* 2.1. *Let*

$$\phi \in BC(\mathbf{R}^N) \cap L^1(\mathbf{R}^N, (1+|x|^k)dx)$$

for some $k \geq 0$. *Then, for any* $q \in [1, \infty]$,

$$t^{\frac{N}{2}(1-\frac{1}{q})} \|P_k(t)z(t)\|_q = \begin{cases} O(t^{-(A-1)}) & \text{if} \quad 2(A-1) < k, \\ O(t^{-\frac{k}{2}} \log t) & \text{if} \quad 2(A-1) = k, \\ o(t^{-\frac{k}{2}}) & \text{if} \quad 2(A-1) > k, \end{cases}$$

as $t \to \infty$.

3 Proof of Theorems 1.1 and 1.2

Let u be a solution of Cauchy problem (1.1), and assume the same conditions as in Theorem 1.1. By the comparison principle and the maximum principle we have

$$0 < u(x,t) \le \zeta_{\|u_0\|_\infty}(t), \qquad (x,t) \in \mathbf{R}^N \times (0,\infty). \tag{3.1}$$

By (1.9) we have

$$1 + \lambda^{-\beta}\zeta_\lambda(t)^{\beta-1}v = \frac{u(x,t)}{\zeta_\lambda(t)} > 0, \qquad (x,t) \in \mathbf{R}^N \times (0,\infty), \tag{3.2}$$

and see that v is a solution of the Cauchy problem,

$$\begin{cases} \partial_t v = \Delta v + F(x,t) & \text{in} \quad \mathbf{R}^N \times (0,\infty), \\ v(x,0) = \varphi(x) & \text{in} \quad \mathbf{R}^N, \end{cases} \tag{3.3}$$

where $F(x,t) := -a(x,t)v$ and

$$a(x,t) := \frac{\lambda^\beta(1 + \lambda^{-\beta}\zeta_\lambda(t)^{\beta-1}v)^\beta - \lambda^\beta - \beta\zeta_\lambda(t)^{\beta-1}v}{v}. \tag{3.4}$$

We first prove that the function v is bounded in $\mathbf{R}^N \times (0,\infty)$.

Proposition 3.1 *Assume* (1.5) *and* $\varphi \in BC(\mathbf{R}^N) \cap L^p(\mathbf{R}^N)$ *for some* $p \in [1,\infty)$. *Then*

$$\sup_{0<t<\infty} \|v(t)\|_\infty < \infty.$$

Proof. By (1.9) and (3.1) we have only to prove

$$\|v(t)\|_\infty = O(1) \tag{3.5}$$

as $t \to \infty$. Let $T > 1$. By (3.1)–(3.4) we apply (2.5) to obtain

$$\sup_{0<t\le T} t^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})} \left[\|v(t)\|_q + t^{\frac{1}{2}}\|\nabla v(t)\|_q\right] < \infty \tag{3.6}$$

for any $q \in [p,\infty]$. This together with (3.4) implies that

$$\sup_{T-1<t<T} \|a(t)\|_\infty < \infty. \tag{3.7}$$

By (3.7) we apply the local L^∞-estimates for parabolic equations (see [12, Chapter III, Theorem 8.1]) to (3.3), and by the Hölder inequality we obtain

$$|v(x,T)| \le C\left(\int_{T-1}^T \int_{B(x,1)} |v(x,t)|^2 dxdt\right)^{1/2} \le C'\left(\int_{T-1}^T \int_{B(x,1)} |v(x,t)|^{p'} dxdt\right)^{1/p'} \tag{3.8}$$

for all $x \in \mathbf{R}^N$, where C and C' are positive constants and $p' = \max\{p, 2\}$. On the other hand, it follows from (3.6) that

$$\int_{T-1}^{T} \int_{\mathbf{R}^N} |v(x,t)|^{p'} dxdt < \infty,$$

and for any $\epsilon > 0$, we can find a constant R such that

$$\int_{T-1}^{T} \int_{|x| \geq R} |v(x,t)|^{p'} dxdt < \epsilon. \tag{3.9}$$

Therefore, by (3.8) and (3.9) we have

$$|v(x,T)| \leq C' \epsilon^{1/p'} \tag{3.10}$$

for all $x \in \mathbf{R}^N$ with $|x| > R+1$. Taking a sufficiently small ϵ if necessary, by (1.9) and (3.10) we have

$$u(x,T) \geq \zeta_\lambda(T) - C'\lambda^{-\beta}\zeta_\lambda(T)^\beta \epsilon^{1/p'} \geq \zeta_\lambda(T)/2 > 0$$

for all $x \in \mathbf{R}^N$ with $|x| > R+1$. This together with (3.1) implies that

$$m := \inf_{x \in \mathbf{R}^N} u(x,T) > 0. \tag{3.11}$$

Therefore, by (3.11) we apply the comparison principle to obtain

$$u(x,t) \geq \zeta_m(t-T), \qquad (x,t) \in \mathbf{R}^N \times [T, \infty). \tag{3.12}$$

On the other hand, it follows from (1.6) that

$$\zeta_\mu(t) = \left(\frac{1}{\beta-1}\right)^{\frac{1}{\beta-1}} t^{-\frac{1}{\beta-1}}(1+O(t^{-1})) = \zeta_\lambda(t)(1+O(t^{-1})) \quad \text{as} \quad t \to \infty$$

for any $\mu > 0$. Then, by (3.1) and (3.12) we obtain

$$u(x,t) = \zeta_\lambda(t)(1+O(t^{-1})) = \zeta_\lambda(t) + \zeta_\lambda(t)^\beta O(1) \quad \text{in} \quad L^\infty(\mathbf{R}^N)$$

as $t \to \infty$. This implies (3.5), and the proof of Proposition 3.1 is complete. □

By Proposition 3.1 we apply the Taylor theorem to (3.4), and obtain

$$a(x,t) = \frac{\beta(\beta-1)}{2\lambda^\beta}\zeta_\lambda(t)^{2(\beta-1)}(1+\lambda^{-\beta}\theta(x,t)\zeta_\lambda(t)^{\beta-1}v)^{\beta-2}v$$

for all $x \in \mathbf{R}^N$ and all sufficiently large t, where $0 < \theta(x,t) < 1$. These together with (1.6) and Proposition 3.1 imply that

$$|a(x,t)| \preceq \zeta_\lambda(t)^{2(\beta-1)}|v(x,t)| \asymp (1+t)^{-2}|v(x,t)| \preceq (1+t)^{-2} \tag{3.13}$$

for all $(x,t) \in \mathbf{R}^N \times (0, \infty)$, and we have

$$|F(x,t)| \preceq (1+t)^{-2}|v(x,t)|$$

in $\mathbf{R}^N \times (0,\infty)$. Then, by Lemma 2.1 we have

$$\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}\left[\|v(t)\|_q + t^{\frac{1}{2}}\|\nabla v(t)\|_q\right] < \infty \quad \text{for any } q \in [p,\infty]. \tag{3.14}$$

This together with (3.13) implies that

$$|F(x,t)| \preceq (1+t)^{-2}|v(x,t)|[|v(x,t)| + (1+t)^{\frac{1}{2}}|\nabla v(x,t)|] \tag{3.15}$$

$$\preceq (1+t)^{-2-\frac{N}{2p}}[|v(x,t)| + (1+t)^{\frac{1}{2}}|\nabla v(x,t)|] \tag{3.16}$$

for all $(x,t) \in \mathbf{R}^N \times (0,\infty)$.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By (1.9) and (3.14) we have (1.13), and it suffices to prove (1.14). Let $\varphi \in BC(\mathbf{R}^N) \cap L^p(\mathbf{R}^N)$ for some $p \in (1,\infty)$ and $1 < p \le q \le \infty$. By (3.3) (see also (??)) we have

$$\|v(t) - e^{t\Delta}\varphi\|_q \le \left\|\int_{t/2}^t e^{(t-s)\Delta}F(s)ds\right\|_q + \left\|\int_0^{t/2} e^{(t-s)\Delta}F(s)ds\right\|_q \tag{3.17}$$

for all $t > 0$. By (G), (3.14), and (3.16) we have

$$\left\|\int_{t/2}^t e^{(t-s)\Delta}F(s)ds\right\|_q \le \int_{t/2}^t \|F(s)\|_q ds$$

$$\preceq t^{-2-\frac{N}{2p}}\int_{t/2}^t \left[\|v(s)\|_q + s^{\frac{1}{2}}\|\nabla v(s)\|_q\right] ds \preceq t^{-1-\frac{N}{2p}} t^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})} \tag{3.18}$$

for all $t \ge 1$. Similarly, by (G), (3.14), (3.15), and the Hölder inequality we have the following:

(i) If $p \ge 2$, then

$$\left\|\int_0^{t/2} e^{(t-s)\Delta}F(s)ds\right\|_q \preceq \int_0^{t/2} (t-s)^{-\frac{N}{2}(\frac{2}{p}-\frac{1}{q})}\|F(s)\|_{p/2}ds$$

$$\preceq t^{-\frac{N}{2}(\frac{2}{p}-\frac{1}{q})}\int_0^{t/2} (1+s)^{-2}\|v(s)\|_p\left[\|v(s)\|_p + (1+s)^{\frac{1}{2}}\|\nabla v(s)\|_p\right] ds$$

$$\preceq t^{-\frac{N}{2}(\frac{2}{p}-\frac{1}{q})} \quad \text{for all } t > 1; \tag{3.19}$$

(ii) If $1 < p < 2$, then

$$\left\|\int_0^{t/2} e^{(t-s)\Delta}F(s)ds\right\|_q \preceq \int_0^{t/2} (t-s)^{-\frac{N}{2}(1-\frac{1}{q})}\|F(s)\|_1 ds$$

$$\preceq t^{-\frac{N}{2}(1-\frac{1}{q})}\int_0^{t/2} (1+s)^{-2}\|v(s)\|_{\frac{p}{p-1}}\left[\|v(s)\|_p + (1+s)^{\frac{1}{2}}\|\nabla v(s)\|_p\right] ds$$

$$\preceq t^{-\frac{N}{2}(1-\frac{1}{q})} \quad \text{for all } t > 1. \tag{3.20}$$

Therefore, by (3.17)–(3.20) we have

$$t^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}\|v(t) - e^{t\Delta}\varphi\|_q \preceq \begin{cases} t^{-1-\frac{N}{2p}} + t^{-\frac{N}{2p}} & \text{if } \ p \ge 2, \\ t^{-1-\frac{N}{2p}} + t^{-\frac{N}{2}(1-\frac{1}{p})} & \text{if } \ 1 < p < 2, \end{cases}$$

for all $t > 1$, and obtain

$$t^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}\|v(t) - e^{t\Delta}\varphi\|_q = O(t^{-\frac{N}{2p}}) + O(t^{-\frac{N}{2}(1-\frac{1}{p})}) \tag{3.21}$$

as $t \to \infty$. This together with (1.9) implies (1.14), and Theorem 1.1 follows. □

Next we apply Lemmas 2.1 and 2.2 to prove Theorem 1.2.

Proof of Theorem 1.2. Assume $\varphi \in BC(\mathbf{R}^N) \cap L^1(\mathbf{R}^N, (1+|x|^K)dx)$ for some $K \geq 0$ with (1.16). Then it follows from (3.16) that

$$|F(x,t)| \preceq (1+t)^{-2-\frac{N}{2}}|v(x,t)|$$

for all $(x,t) \in \mathbf{R}^N \times (0,\infty)$. Then Lemma 2.1 gives assertion (i). On the other hand, by (1.16) we have

$$2\left(2+\frac{N}{2}-1\right) > K. \tag{3.22}$$

Then, since it follows from (1.17) and (2.6) that

$$m_\nu(v(t),t) = M_\nu(t), \qquad t > 0, \tag{3.23}$$

applying property (Z_1) and Lemma 2.2 with $A = 2 + N/2$, by (3.22) and (3.23) we have:

- For any $\nu \in \mathbf{M}$ with $|\nu| \leq K$, there exists a constant M_ν such that

$$|M_\nu(t) - M_\nu| \preceq (1+t)^{-(1+N/2)+|\nu|/2} = o(1) \tag{3.24}$$

  as $t \to \infty$;

- For any $q \in [1,\infty]$,

$$t^{\frac{N}{2}(1-\frac{1}{q})}\left\|v(t) - \sum_{|\nu|\leq K} M_\nu(t)g_\nu(t)\right\|_q = o(t^{-\frac{K}{2}}) \tag{3.25}$$

  as $t \to \infty$.

Assertion (ii) follows from (3.24). Furthermore, by (2.2), (3.22), (3.24), and (3.25) we have

$$\begin{aligned} &t^{\frac{N}{2}(1-\frac{1}{q})}\|v(t) - W(t)\|_q \\ &\leq t^{\frac{N}{2}(1-\frac{1}{q})}\left\|v(t) - \sum_{|\nu|\leq K} M_\nu(t)g_\nu(t)\right\|_q + t^{\frac{N}{2}(1-\frac{1}{q})}\sum_{|\nu|\leq K}|M_\nu(t) - M_\nu|\|g_\nu(t)\|_q \\ &= o(t^{-\frac{K}{2}}) \end{aligned}$$

as $t \to \infty$. This together with (1.9) implies assertion (iii), and the proof of Theorem 1.2 is complete. □

Proof of Corollary 1.1. Assume $\varphi \in BC(\mathbf{R}^N) \cap L^p(\mathbf{R}^N)$ for some $p \in (1,\infty)$. By Theorem 1.1 and (1.9) we have

$$\begin{aligned} u(x,t) &= \zeta_\lambda(t) + \lambda^{-\beta}\zeta_\lambda(t)^\beta v(x,t) \\ &= \zeta_\lambda(t) + \lambda^{-\beta}\zeta_\lambda(t)^\beta[(e^{t\Delta}\varphi)(x) + O(t^{-\frac{N}{p}}) + O(t^{-\frac{N}{2}})] \\ &= \zeta_\lambda(t) + \lambda^{-\beta}\zeta_\lambda(t)^\beta[(e^{t\Delta}\varphi)(x) + O(t^{-\frac{N}{p'}})] \end{aligned} \tag{3.26}$$

in $L^\infty(\mathbf{R}^N)$ as $t \to \infty$. On the other hand, we have

$$\begin{aligned} &\left[(e^{t\Delta}u_0)(x)^{-(\beta-1)} + (\beta-1)t\right]^{-\frac{1}{\beta-1}} = \left[(\lambda + (e^{t\Delta}\varphi)(x))^{-(\beta-1)} + (\beta-1)t\right]^{-\frac{1}{\beta-1}} \\ &= \left[\lambda^{-(\beta-1)}\left(1 - (\beta-1)\lambda^{-1}(e^{t\Delta}\varphi)(x) + O(t^{-\frac{N}{p}})\right) + (\beta-1)t\right]^{-\frac{1}{\beta-1}} \\ &= \left(\lambda^{-(\beta-1)} + (\beta-1)t\right)^{-\frac{1}{\beta-1}}\left[1 + \frac{-(\beta-1)\lambda^{-\beta}(e^{t\Delta}\varphi)(x) + O(t^{-\frac{N}{p}})}{\lambda^{-(\beta-1)} + (\beta-1)t}\right]^{-\frac{1}{\beta-1}} \\ &= \zeta_\lambda(t)\left[1 + \frac{\lambda^{-\beta}(e^{t\Delta}\varphi)(x) + O(t^{-\frac{N}{p}})}{\lambda^{-(\beta-1)} + (\beta-1)t} + O(t^{-\frac{N}{p}-2})\right] \\ &= \zeta_\lambda(t)\left[1 + \lambda^{-\beta}\zeta_\lambda(t)^{\beta-1}\left((e^{t\Delta}\varphi)(x) + O(t^{-\frac{N}{p}})\right)\right] \\ &= \zeta_\lambda(t) + \lambda^{-\beta}\zeta_\lambda(t)^\beta\left((e^{t\Delta}\varphi)(x) + O(t^{-\frac{N}{p}})\right) \end{aligned}$$

in $L^\infty(\mathbf{R}^N)$ as $t \to \infty$. This together with (3.26) yields assertion (i). Similarly, by Theorem 1.2 we can prove assertion (ii), and Corollary 1.1 follows. □

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