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On the convergence of an area minimizing scheme for the anisotropic mean curvature flow

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1 Introduction

In this article we present the convergence of an area minimizing scheme for for the anisotropic mean curvature flow (AMCF for short) and its application to an approximation of the crystalline curvature flow (CCF) in the plane.

A family $\{\Gamma(t)\}_{t\geq 0}$ of hypersurfaces in $\mathbb{R}^N$ is called an AMCF provided that $\Gamma(t)$ evolves by the equation of the form

$$V=-\mathrm{div}\,\xi(\mathbf{n}) \quad \text{on } \Gamma(t), t>0. \tag{1.1}$$

Here $\mathbf{n}$ is the Euclidean outer unit normal vector field of $\Gamma(t)$, the function $\gamma=\gamma(p)$ is the *surface energy density*, $\xi=\nabla_p\gamma:=(\gamma_{p_1},\cdots,\gamma_{p_N})$ is called the *Cahn-Hoffman vector*. The function γ is assumed to be convex. In particular, if $\gamma(p)=|p|$, then (1.1) is the usual mean curvature flow (MCF) equation:

$$V=-\mathrm{div}\,\mathbf{n} \quad \text{on } \Gamma(t),\ t>0. \tag{1.2}$$

These equations arise in geometry, interface dynamics, crystal growth and image processing etc. Many people have been studying MCF, AMCF and CCF from various viewpoints. With relation to the applications mentioned above, numerical schemes have also been studied.

Among them, Chambolle [4] proposed an algorithm for MCF. His algorithm is described as follows: Let $E_0\subset\mathbb{R}^N$ be a compact set and fix a time step $h>0$. We choose a bounded domain $\Omega\subset\mathbb{R}^N$ including E_0 and take a function $w_0\in L^2(\Omega)\cap BV(\Omega)$ as a unique minimizer of the functional $J_h(\cdot,E_0)$ defined by

$$J_h(v,E_0):=\begin{cases}\displaystyle\int_\Omega|Dv|+\frac{1}{2h}\|v-d_{E_0}\|^2_{L^2(\Omega)} & \text{if } v\in L^2(\Omega)\cap BV(\Omega),\\ +\infty & \text{if } v\in L^2(\Omega)\backslash BV(\Omega).\end{cases} \tag{1.3}$$

Here $\int_\Omega|Dv|$ is the total variation of v, Dv is the gradient of v in the sense of distribution, and $d(E_0)=d(\cdot,E_0)$ denotes the Euclidean signed distance function to ∂E_0, namely,

$$d(x,E_0):=\mathrm{dist}(x,E_0)-\mathrm{dist}(x,\mathbb{R}^N\backslash E_0) \quad \text{for } x\in\mathbb{R}^N. \tag{1.4}$$

Then we set

$$E_1:=\{w_0\leq 0\}. \tag{1.5}$$

Throughout this paper we use the notations $\{f \geq \mu\} := \{x \in \mathbb{R}^N \mid f(x) \geq \mu\}$, $\{f \leq \mu\} := \{x \in \mathbb{R}^N \mid f(x) \leq \mu\}$ etc. Next we take a function $w_1 \in L^2(\Omega) \cap BV(\Omega)$ as a unique minimizer of the functional $J_h(\cdot, E_1)$ and define E_2 as the set in (1.5) with w_1 replacing w_0. Repeating this process, we have a sequence $\{E_k\}_{k=0,1,\dots}$ of compact sets. We then set

$$E^h(t) := E_k \quad \text{for } t \in [kh, (k+1)h) \text{ and } k = 0, 1, \dots \tag{1.6}$$

Sending $h \to 0$, we obtain a limit $\{E(t)\}_{t \geq 0}$ of $\{E^h(t)\}_{t \geq 0, h > 0}$ and formally observe that $\{\Gamma(t) = \partial E(t)\}_{t \geq 0}$ is an MCF starting from $\Gamma(0)(= \partial E_0)$.

In this paper we extend Chambolle's algorithm to the AMCF by use of the elliptic differential inclusion:

$$\frac{w - d(E)}{h} \in \operatorname{div} \partial_p \gamma(\nabla w) w \quad \text{in } \mathbb{R}^N. \tag{1.7}$$

(See section 3 below for the precise description of our algorithm.) Note that this is the Euler - Lagrange equation for such a variational problem as (1.3). This idea is essentially given by Caselles - Chambolle [3]. Also, the (1.7) is a time discretization of the parabolic initial-value problem:

$$\begin{cases} v_t \in \operatorname{div} \partial_p \gamma(\nabla v) & \text{in } (0, +\infty) \times \mathbb{R}^N, \\ v(0, x) = d(x, E) & \text{for } x \in \mathbb{R}^N. \end{cases} \tag{1.8}$$

There are some papers studying anisotropic extensions of Chambolle's algorithm. See Bellettini - Caselles - Chambolle - Novaga [2], Caselles - Chambolle [3], Chambolle - Novaga [5], [6] and Eto - Giga - Ishii [8]. In these papers the convergences are proved in the sense of the Hausdorff distance and are locally uniform with respect to the time variable (except for [5]). As for the proofs of the convergences, the authors of [3], [2], [6] and [5] used some variational techniques. In [8] the authors applied some ideas from mathematical morphology, level set method and the theory of viscosity solutions.

The main purpose of this paper is to provide a different proof of the convergence of an anisotropic Chambolle's algorithm from those given in [2], [3], [5], [6] and [8]. Moreover, we apply our results to an approximation of the noncompact and nonconvex CCF.

The main idea is to employ the signed distance functions and the eikonal equations. This is motivated by Soner [18] and Goto - Ishii - Ogawa [12], in which they discussed, respectively, the convergence of Allen - Cahn equations and that of the Bence - Merriman - Osher algorithm for MCF. Consequently, under the nonfattening condition, we are able to show that the approximate flow by (1.7) converges to an AMCF in the sense of the Hausdorff distance and that it is locally uniform with respect to the time variable. Also we are able to apply our results to an approximation to CCF in the plane.

This is a brief report of [14].

2 Preliminaries

2.1 Anisotropies and an elliptic differential inclusion

We make the following assumptions on γ.

(A1) $\gamma : \mathbb{R}^N \longrightarrow [0, +\infty)$: convex.

(A2) $\gamma(-p) = \gamma(p)$ and $\gamma(ap) = a\gamma(p)$ for all $p \in \mathbb{R}^N$ and $a > 0$.

(A3) $\Lambda^{-1}|p| \leq \gamma(p) \leq \Lambda|p|$ for all $p \in \mathbb{R}^N$ and some $\Lambda > 0$.

We easily see by (A1) - (A3) that γ Lipschitz continuous in $\mathbb{R}^N$. Let $\partial_p\gamma(p)$ be the subdifferential of ζ at $p \in \mathbb{R}^N$:

$$\partial_p\gamma(p) := \{\xi \in \mathbb{R}^N \mid \langle \xi, q - p\rangle \leq \gamma(q) - \gamma(p) \text{ for all } q \in \mathbb{R}^N\}.$$

If γ is differentiable at p, then we write $\nabla_p\gamma(p)$ in place of $\partial_p\gamma(p)$. It follows from [8, Lemma 2.1] that $\partial_p\gamma(p) \subset \partial_p\gamma(0) \subset \operatorname{cl} B(0, \Lambda)$ for all $p \in \mathbb{R}^N$. Here and in the sequel, $B(x, r) := \{y \in \mathbb{R}^N \mid |y - x| < r\}$ for $x \in \mathbb{R}^N$ and $r > 0$ and $\operatorname{cl} A$ is the closure of $A \subset \mathbb{R}^N$.

We define the support function γ° of the convex set $\{\gamma \leq 1\}$ (often called Frank diagram for γ) by

$$\gamma^\circ(p) := \sup_{\gamma(q)\leq 1} \langle p, q\rangle.$$

We observe that γ° also satisfies (A1) - (A3) and Lipschitz continuity in $\mathbb{R}^N$.

In addition to (A1) - (A3), we assume some regularity on γ.

(A4) $\gamma \in C^2(\mathbb{R}^N\backslash\{0\})$, $\nabla_p^2\gamma^2 > O$ in $\mathbb{R}^N\backslash\{0\}$.

Remark 2.1. (1) The second condition of (A4) is equivalent to the strict convexity of $\{\gamma \leq 1\}$ and that if ζ satisfies (A4), then γ° does so (cf. [16, Section 2.5] and [10, Remark 1.7.5]).

(2) In fact, we are able to derive (A3) from (A1), (A2) and (A4) (cf. [15]).

Assume that (A1), (A2) and (A4) hold and that ∂E is smooth. The anisotropic mean curvature is defined as follows.

Definition 2.1. *Let E be an open set in $\mathbb{R}^N$ with the smooth boundary ∂E. Then the anisotropic mean curvature $\kappa_{\gamma^\circ}(x, E)$ of ∂E is defined by*

$$\kappa_{\gamma^\circ}(x, E) := -\operatorname{div} \nabla_p\gamma(\mathbf{n})(= -\operatorname{div} \xi(\mathbf{n}(x))) \quad \textit{for } x \in \partial E.$$

Next we introduce the anisotropic total variation. Let $\Omega \subset \mathbb{R}^N$ be an open set with Lipschitz boundary. Denote by $BV(\Omega)$ the space of all functions of bounded variation and by $BV_{loc}(\Omega)$ the class of all functions of locally bounded variation.

We define the anisotropic total variation of $u \in BV(\Omega)$ with respect to γ in Ω as

$$\int_\Omega \gamma(Du) := \sup\left\{ \int_\Omega u \operatorname{div}\varphi\, dx \;\middle|\; \varphi \in C_0^1(\Omega; \mathbb{R}^N),\ \gamma^\circ(\varphi) \leq 1 \text{ in } \Omega \right\}.$$

Set $X(\Omega) := \{z \in L^\infty(\Omega; \mathbb{R}^N) \mid \operatorname{div} z \in L^2(\Omega)\}$. For $w \in L^2(\Omega) \cap BV(\Omega)$ and $z \in X(\Omega)$, we define a functional on $C_0^1(\Omega)$ as

$$\text{(2.1)} \qquad \int_\Omega (z, Dw)\psi := -\int_\Omega w\psi \operatorname{div} z\, dx - \int_\Omega w\langle z, \nabla\psi\rangle dx \quad \text{for } \psi \in C_0^1(\Omega).$$

We can extend this functional to a linear one on $C_0(\Omega)$. Hence (z, Dw) is a Radon measure. We recall Green's formula for $w \in L^2(\Omega) \cap BV(\Omega)$ and $z \in X(\Omega)$.

Theorem 2.1. ([1]) *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. Let $w \in L^2(\Omega) \cap BV(\Omega)$ and $z \in X(\Omega)$. Then there exists $[z \cdot \mathbf{n}] \in L^\infty(\partial\Omega)$ such that $\|[z \cdot \mathbf{n}]\|_{L^\infty(\partial\Omega)} \leq \|z\|_{L^\infty(\Omega)}$ and*

$$\int_\Omega w \operatorname{div} z dx + \int_\Omega (z, Dw) = \int_{\partial\Omega} [z \cdot \mathbf{n}] w \, d\mathcal{H}^{N-1},$$

where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure. In the case $\Omega = \mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N} w \operatorname{div} z dx + \int_{\mathbb{R}^N} (z, Dw) = 0$$

for all $w \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$ and $z \in X(\mathbb{R}^N)$.

We briefly review some results on solutions of an elliptic differential inclusion:

$$\frac{w-g}{h} \in \operatorname{div} \partial_p \gamma(\nabla w) \ni g \quad \text{in } \mathbb{R}^N, \tag{2.2}$$

where $g \in L^2_{loc}(\mathbb{R}^N)$ and $h > 0$.

We give the definition of weak solutions of (2.2).

Definition 2.2. *We say that $w \in L^2_{loc}(\mathbb{R}^N) \cap BV_{loc}(\mathbb{R}^N)$ is a weak solution of (2.2) provided that there exists $z \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$, $\operatorname{div} z \in L^2_{loc}(\mathbb{R}^N)$ such that*

(1) *$z \in \partial\gamma(\nabla w)$ a.e. in $\mathbb{R}^N$,*

(2) *$(z, Dw) = \gamma(Dw)$ locally as measures in $\mathbb{R}^N$,*

(3) *$\dfrac{w-g}{h} = \operatorname{div} z$ in $\mathcal{D}'(\mathbb{R}^N)$.*

The existence, uniqueness and regularity of solutions of (2.2) are stated as follows.

Theorem 2.2. (cf. [3] and [8]) *Assume (A1) - (A3). For any $g \in L^2_{loc}(\mathbb{R}^N)$, (2.2) admits a unique weak solution. Moreover, a weak solution w of (2.2) is Lipschitz continuous in $\mathbb{R}^N$ and $|\nabla w| \leq 1$ for a.e. in $\mathbb{R}^N$ and all $h > 0$.*

2.2 Generalized AMCF

Assume that γ satisfies (A1), (A2) and (A4). The level set equation for (1.1) is the following:

$$u_t - |\nabla u| \operatorname{div} \xi(\nabla u) = 0 \quad \text{in } (0,T) \times \mathbb{R}^N. \tag{2.3}$$

Notice that $\operatorname{div} \xi(\nabla u) = \operatorname{tr}(\nabla_p^2 \gamma(\nabla u) \nabla^2 u)$ if $\nabla u \neq 0$.

We give the definition of viscosity solutions of (2.3). Let U be a subset of a metric space (X, ρ) and let f be a function on U. The upper (resp., lower) semicontinuous envelope f^* (resp., f_*) is defined as follows: For each $x \in \overline{U}$,

$$f^*(x) := \limsup_{y \in U, \rho(y,x) \to 0} f(y), \quad f_*(x) := \liminf_{y \in U, \rho(y,x) \to 0} f(y). \tag{2.4}$$

Definition 2.3. *Let* $u : [0,T) \times \mathbb{R}^N \longrightarrow \mathbb{R}$.

(1) *We say that u is a viscosity subsolution (resp., supersolution) of (2.3) provided that* $u^*(t,x) < +\infty$ *(resp.,* $u_*(t,x) > -\infty$*) for all* $(t,x) \in [0,T) \times \mathbb{R}^N$ *and for any* $\phi \in C^\infty((0,T) \times \mathbb{R}^N)$, *if* $u^* - \phi$ *takes a local maximum (resp., minimum) at* $(\hat{t},\hat{x})$, *then*

$$\phi_t(\hat{t},\hat{x}) - |\nabla\phi(\hat{t},\hat{x})| \operatorname{div}\xi(\nabla\varphi(\hat{t},\hat{x})) \leq 0 \ (resp., \geq 0) \ \text{if} \ \nabla\varphi(\hat{t},\hat{x}) \neq 0,$$
$$\phi_t(\hat{t},\hat{x}) \leq 0 \ (resp., \geq 0) \ \text{if} \ \nabla\varphi(\hat{t},\hat{x}) = 0 \ \text{and} \ \nabla^2\varphi(\hat{t},\hat{x}) = O.$$

(2) *We say that u is a viscosity solution of (2.3) if u is a viscosity sub- and super-solution of (2.3).*

A family $\{\Gamma(t)\}_{t\geq 0}$ of hypersurfaces in $\mathbb{R}^N$ is called a generalized AMCF (or a generalized motion by (1.1)) if $\Gamma(t) = \{u(t,\cdot) = 0\}$, where u is a viscosity solution of (2.3). We refer to [10] for the theory of generalized motion of surface evolution equations including (1.1).

In sections 4 and 5 we use the notion of distance solutions for AMCF developed by [17]. Let $\{\Gamma(t)\}_{t\geq 0}$ be a family of hypersurfaces and $E(t)$ a closed set such that $\Gamma(t) = \partial E(t)$. Let $d = d(t,\cdot)$ be the signed distance function to $\Gamma(t)$ given by (1.4) with $E_0 = E(t)$.

Definition 2.4. *We say that* $\{\Gamma(t)\}_{t\geq 0}$ *is a distance solution of (1.1) provided that* $d \wedge 0$ *and* $d \vee 0$ *are, respectively, a viscosity subsolution and a viscosity supersolution of (2.3).*

Remark 2.2. In section 5 we will discuss an approximation of CCF and not assume (A4). Then $-\operatorname{div}\xi(\mathbf{n})$ in (1.1) is not defined in the classical sense. However, in two dimensional case it can be regarded as the crystalline curvature due to [19], [13] etc., more generally as the nonlocal curvature due to [9]. In [9] the authors develop the theory of the generalized motion by nonlocal curvature including CCF.

3 An anisotropic version of Chambolle's algorithm

An anisotropic version of Chambolle's algorithm is stated in the following way.

Fix $E_0 \in \mathcal{C}(\mathbb{R}^N)$. Let $w(E_0) := w(\cdot, E_0)$ be a weak solution of (1.7) with $E = E_0$. We then define a new set E_1 by

$$E_1 := \{w(\cdot, E_0) \leq 0\}.$$

Notice by Theorem 2.2 that $E_1 \in \mathcal{C}(\mathbb{R}^N)$. Let $w(E_1)$ be a weak solution of (1.7) with $E = E_1$. Again we define a new set E_2 by

$$E_2 := \{w(\cdot, E_1) \leq 0\}.$$

Repeating this process, we have a sequence $\{E_k\}_{k=0}^{[T/h]}$ of closed subsets of $\mathbb{R}^N$. Set

$$E^h(t) := E_{[t/h]} \quad \text{for } t \geq 0. \tag{3.1}$$

Letting $h \to 0$, we obtain a limit flow $\{E(t)\}_{t\geq 0}$ of $\{E^h(t)\}_{t\geq 0, h>0}$ and formally observe that $\partial E(t)$ is an AMCF starting from ∂E_0.

4 Convergence

In this section we assume (A1), (A2) and (A4) and formally show the convergences of $\{d^h\}_{h>0}$, $\{w^h)\}_{h>0}$ and $\{E^h(t)\}_{t\geq 0,h>0}$. We also establish that $\{\Gamma(t) = \partial E(t)\}_{t\geq 0}$ is a distance solution of (1.1).

For $E_0 \in \mathcal{C}(\mathbb{R}^N)$ let $\{E^h(t)\}_{t\geq 0,h>0}$, $\{d(E^h(t))\}_{t\geq 0,h>0}$, and $\{w(E^h(t))\}_{t\geq 0,h>0}$ be defined in the previous section. Set

$$d^h(t,x) := d(x, E^h(t)),\ w^h(t,x) := w(x, E^h(t)) \quad \text{for } t \in [0,T) \text{ and } x \in \mathbb{R}^N.$$

We mention our strategy to prove the convergence of our scheme. Since $w^h(t,\cdot)$ satisfies

$$w^h(t,\cdot) - h \operatorname{div} \partial_p \gamma(\nabla w^h(t,\cdot)) \ni d^h(t,\cdot) \quad \text{in } \mathbb{R}^N,$$

in a weak sense, letting $h \to 0$, we get $\lim_{h\to 0} w^h(t,x) = \lim_{h\to 0} d^h(t,x)$ at least formally. By this observation we compute the limit of $\{d^h\}_{h>0}$ as $h \to 0$ to obtain that of $\{w^h\}_{h>0}$.

The formula $\lim_{h\to 0} w^h(t,x) = \lim_{h\to 0} d^h(t,x)$ can be obtained as follows. First, we remark that any weak solution of (2.2) is a minimizer of the associated variational problem.

Proposition 4.1. ([3, Proposition 3.1]) *Let $g \in L^2_{loc}(\mathbb{R}^N)$ and $w \in L^2_{loc}(\mathbb{R}^N) \cap BV_{loc}(\mathbb{R}^N)$. The following assertions are equivalent.*

(1) *w is a weak solution of (2.2).*

(2) *For each $r > 0$, w satisfies*

$$\begin{aligned}\int_{B(0,r)} \gamma(Dw) + \frac{1}{2h}\|w-g\|^2_{L^2(B(0,r))} \ \leq\ & \int_{B(0,r)} \gamma(Dv) + \frac{1}{2h}\|v-g\|^2_{L^2(B(0,r))} \\ & + \int_{\partial B(0,r)} \gamma(\mathbf{n}(B(0,r)))|v-w|\,d\mathcal{H}^{N-1}\end{aligned}$$

for all $v \in L^2(B(0,r)) \cap BV(B(0,r))$. where $\mathcal{H}^{N-1}$ denotes the $(N-1)$ dimensional Hausdorff measure.

Applying this proposition with $g = v = d^h(t,\cdot)$ and $w = w^h(t,\cdot)$, we get

$$\frac{1}{2h}\|w^h(t,\cdot) - d^h(t,\cdot)\|^2_{L^2(B(0,r))} \leq \int_{B(0,r)} \gamma(\nabla d^h(t,\cdot)) + \int_{\partial B(0,r)} \gamma(\mathbf{n})|d^h(t,\cdot) - w^h(t,\cdot)|\,d\mathcal{H}^{N-1}$$

It is seen by (A3) and the fact $|\nabla d^h(t,\cdot)| = 1$ for a.e. in $\mathbb{R}^N$ that the first term of the right-hand side of this inequality is uniformly bounded for $h > 0$. Since we can observe that the second term is also uniformly bounded for $h > 0$, we have

$$\sup_{t\in[0,T)} \|w^h(t,\cdot) - d^h(t,\cdot)\|_{L^2(B(0,r))} \leq C\sqrt{h},$$

where $C > 0$ is independent of $h > 0$. Moreover, note that $\{d^h(t,\cdot)\}_{t\geq 0,h>0}$ and $\{w^h(t,\cdot)\}_{t\geq 0,h>0}$ are equi-Lipschitz continuous in $\mathbb{R}^N$ (cf. Theorem 2.2). Hence combining these facts, we obtain $\overline{w} = \overline{d}$ and $\underline{w} = \underline{d}$ in $[0,T) \times \mathbb{R}^N$. Here $\overline{w}$, $\underline{w}$ is defined by (4.2) with w^h replacing d^h. These formulae and Theorem 4.1 yield $\lim_{h\to 0} w^h(t,x) = \lim_{h\to 0} d^h(t,x)$. Therefore it is sufficient to consider the limit of $\{d^h\}_{h>0}$ instead of that of $\{d^h\}_{h>0}$.

We observe that for each $t \in [0,T)$, $d^h(t,\cdot)$ satisfies

$$(4.1) \qquad |\nabla d^h| - 1 = 0 \quad \text{in } \{d^h(t,\cdot) > 0\}, -|\nabla d^h| + 1 = 0 \quad \text{in } \{d^h(t,\cdot) < 0\},$$

in the sense of viscosity solutions. Then setting

$$(4.2) \qquad \overline{d}(t,x) := \limsup_{(h,s,y)\to(0,t,x)} d^h(s,y), \ \underline{d}(t,x) := \liminf_{(h,s,y)\to(0,t,x)} d^h(s,y),$$

we can verify by the stability of viscosity solutions that $\rho(= \overline{d}, \underline{d})$ is a viscosity solution of

$$|\nabla \rho| - 1 = 0 \quad \text{in } \{\rho(t,\cdot) > 0\}, -|\nabla \rho| + 1 = 0 \quad \text{in } \{\rho(t,\cdot) < 0\},$$

Besides, it is seen from the barrier construction argument that

$$\overline{d}(0,\cdot) = \underline{d}(0,\cdot) = d(\cdot, E_0) \quad \text{in } \mathbb{R}^N.$$

We impose an important assumption: Set $\Gamma(t) := \{\underline{d}(t,\cdot) \le 0 \le \overline{d}(t,\cdot)\}$.

$$(4.3) \qquad \Gamma(t) \neq \emptyset, \ \Gamma(t) = \partial\{\overline{d}(t,\cdot) < 0\} = \partial\{\underline{d}(t,\cdot) > 0\} \quad \text{for all } t \in [0,T).$$

Then we observe that the map $t \mapsto \Gamma(t)$ is continuous in $[0,T)$ in the sense that

$$(4.4) \qquad \lim_{s\to t} d_H(\Gamma(s), \Gamma(t)) = 0 \quad \text{for each } t \in [0,T),$$

where d_H is the Hausdorff distance defined by

$$d_H(A,B) := \max\left\{\sup_{x\in A} \operatorname{dist}(x,B), \sup_{x\in B} \operatorname{dist}(x,A)\right\} \quad \text{for } A, B \subset \mathbb{R}^N.$$

Hence we have the convergence of $\{d^h\}_{h>0}$. Let $d = d(t,\cdot)$ be the signed distance function to $\Gamma(t)$ given by (1.4) with $E_0 = E(t)$. Note that d is continuous in $[0,T) \times \mathbb{R}^N$ under the assumption (4.3) because of (4.4) and Lipschitz continuity of $d(t,\cdot)$ for all $t \in [0,T)$.

Theorem 4.1. *Assume (A1), (A2), (A4) and (4.3). Then $\overline{d} = \underline{d} = d$ in $[0,T) \times \mathbb{R}^N$. Thus $\{d^h\}_{h>0}$ converges to d as $h \to 0$ locally uniformly in $[0,T) \times \mathbb{R}^N$. Moreover, $\partial E^h(t)$ converges to $\Gamma(t)$ as $h \to 0$ in the sense of the Hausdorff distance, locally uniformly in $[0,T)$.*

Theorem 4.2. *Assume (A1), (A2), (A4) and (4.3). Then $\{w^h\}_{h>0}$ converges to d as $h \to 0$ locally uniformly in $[0,T) \times \mathbb{R}^N$.*

Now we show that $\Gamma(t)$ is an AMCF. For simplicity we assume that $\lim_{h\to 0} w^h = d^h$ in the $C^{1,2}$ sense. We get from (1.7) with $w = w^h(t,\cdot)$ and $E_0 = E^h(t)$

$$(4.5) \qquad \frac{w^h(t,\cdot) - d(\cdot, E^h(t))}{h} \in \operatorname{div} \partial_p \gamma(\nabla w^h(t,\cdot)) \quad \text{on } \partial E^h(t).$$

Recall that $E^h(t)$ is given by

$$E^h(t) = E_{[t/h]} = \{w(\cdot, E_{[t/h]-1}) \le 0\} = \{w^h(t-h,\cdot) \le 0\}.$$

Hence $w^h(t-h,\cdot) = 0$ on $\partial E^h(t)$. Since $d(\cdot, E^h(t)) = 0$ and $|\nabla d(\cdot, E^h(t))| = 1$ on $\partial E^h(t)$, we obtain from (4.5)

$$\frac{w^h(t,\cdot) - w^h(t-h,\cdot)}{h} = \operatorname{div}\xi(\nabla w^h(t,\cdot)) \quad \text{on } \partial E^h(t).$$

Sending $h \to 0$, we have

$$d_t = \operatorname{div}\xi(\nabla d) \quad \text{on } \Gamma(t),\ t > 0.$$

This equation is nothing but (1.1) because $d_t = -V$ and $\nabla d = \mathbf{n}$.

The above arguments are justified in the sense of a distance solution, mentioned at the end of subsection 2.2.

Theorem 4.3. *Assume (A1), (A2), (A4) and (4.3). Then $\{\Gamma(t)\}_{t\ge 0}$ is a distance solution of (1.1).*

5 An application to CCF

The purpose of this section is to apply the results in section 4 to an approximation for CCF.

Fix $n(\ge 2) \in \mathbb{N}$. Let $\theta_i := i\pi/n$ and let $q_i := (\cos\theta_i, \sin\theta_i)$. Define $\gamma(p) := \max_{1\le i\le 2n}\langle q_i, p\rangle$ for $p \in \mathbb{R}^2$. Then this γ satisfies (A1) - (A3), but not (A4). In this case $\operatorname{div}\xi(\mathbf{n})$ cannot be defined in the classical sense, as mentioned in Remark 2.2. Hence we rewrite (1.1) as follows:

$$V = -\text{“}\operatorname{div}\xi(\mathbf{n})\text{''} = 0 \quad \text{on } \Gamma(t),\ t > 0. \tag{5.1}$$

Here $\Gamma(t)$ is a simple and closed curve in $\mathbb{R}^2$ and “$\operatorname{div}\xi(\mathbf{n})$'' is interpreted as the crystalline curvature (cf. [13], [20]). The family $\{\Gamma(t)\}_{t\ge 0}$ evolving by (5.1) is is often called a crystalline curvature flow (CCF).

The level set equation for (5.1) is given by

$$u_t - |\nabla u|\, \text{“}\operatorname{div}\xi(\nabla u)\text{''} \quad \text{in } (0,T) \times \mathbb{R}^2. \tag{5.2}$$

The generalized CCF $\{\Gamma(t)\}_{t\ge 0}$ (or generalized motion by (5.1)) is defined by $\Gamma(t) := \{u(t,\cdot) = 0\}$ for each $t \in [0,T)$. Here u is a viscosity solution of (5.2). We use the results in [9] to show the convergence of our scheme to a generalized CCF, although we omit the detail.

For our purpose we approximate γ by smooth functions. By [11, Lemma 2.5] there is a sequence $\{\gamma_\tau\}_{\tau>0}$ satisfying (A1), (A2), (A4) and

$$\gamma_\tau \longrightarrow \gamma \quad \text{as } \tau \to 0 \text{ locally uniformly in } \mathbb{R}^2, \tag{5.3}$$

$$\frac{1}{2\Lambda}|p| \le \gamma_\tau(p) \le 2\Lambda|p| \quad \text{for } p \in \mathbb{R}^2 \text{ and } \tau > 0. \tag{5.4}$$

We use $\{\gamma_\tau\}_{\tau>0}$ to construct approximate sequences: Fix a compact set $E_0 \subset \mathbb{R}^N$ and set $E_0^\tau := E_0$. Let $w^\tau(E_0)$ be a weak solution of (1.7) with $\gamma = \gamma_\tau$ and $E_0 := E_0^\tau$. Then we define a new set $E_1^\tau := \{w^\tau(E_0) \leq 0\}$. Next take $w^\tau(E_1)$ as a weak solution of (1.7) with $\gamma = \gamma_\tau$ and $E_0 := E_1^\tau$. Define a new set $E_2^\tau := \{w^\tau(E_1) \leq 0\}$. Repeating the process, we have sequences $\{E_k^\tau\}_{k=0,1,\ldots}$, $\{d(E_k^\tau)\}_{k=0,1,\ldots}$ and $\{w^\tau(E_k^\tau)\}_{k=0,1,\ldots}$.

For $t \geq 0$ and $x \in \mathbb{R}^N$, set

$$E^{\tau,h}(t) := E^\tau_{[t/h]},\ d^{\tau,h}(t,x) := d(x, E^{\tau,h}(t)),\ w^{\tau,h}(t,x) := w^\tau(x, E^{\tau,h}(t)).$$

Define

$$\overline{\rho}(t,x) := \limsup_{(\tau,h,s,y)\to(0,0,t,x)} d^{\tau,h}(s,y),\ \underline{\rho}(t,x) := \liminf_{(\tau,h,s,y)\to(0,0,t,x)} d^{\tau,h}(s,y), \tag{5.5}$$

and $\Gamma(t) := \{\underline{\rho}(t,\cdot) \leq 0 \leq \overline{\rho}(t,\cdot)\}$. Similar arguments to those before Theorem 4.1 yield the following theorem. Let $d = d(t,\cdot)$ be the signed distance function to $\Gamma(t)$ given by (1.4) with $E_0 = E(t)$

Theorem 5.1. *Assume (A1) - (A3) and*

$$\Gamma(t) \neq \emptyset \text{ and } \Gamma(t) = \partial\{\overline{\rho}(t,\cdot) < 0\} = \partial\{\underline{\rho}(t,\cdot) > 0\} \quad \text{for all } t \in [0,T). \tag{5.6}$$

Then $\overline{d} = \underline{d} = d$ in $[0,T) \times \mathbb{R}^N$. Thus $\{d^{\tau,h}\}_{\tau,h>0}$ converges to d as τ, $h \to 0$ locally uniformly in $[0,T) \times \mathbb{R}^N$. Moreover, $\partial E^{\tau,h}(t)$ converges to $\Gamma(t)$ as τ, $h \to 0$ in the sense of the Hausdorff distance, locally uniformly in $[0,T)$.

Thanks to (5.4), we directly apply Proposition 4.1 with $\gamma = \gamma_\tau$ to get

$$\sup_{t\in[0,T),\tau>0} \|w^{\tau,h}(t,\cdot) - d^{\tau,h}(t,\cdot)\|_{L^2(B(0,r))} \leq C\sqrt{h},$$

where $C > 0$ is independent of τ, $h > 0$. Since we observe that $|\nabla d^{\tau,h}(t,\cdot)| = 1$ and $|w^{\tau,h}(t,\cdot)| \leq 1$ a.e. in $\mathbb{R}^N$ for all $t \in [0,T)$, combining these facts, we have the convergence of $\{w^{\tau,h}\}_{\tau,h>0}$.

Theorem 5.2. *Assume (A1) - (A3) and (5.6). Then $\{w^{\tau,h}\}_{\tau,h>0}$ converges to d as τ, $h \to 0$ locally uniformly in $[0,T) \times \mathbb{R}^N$.*

The characterization of $\{\Gamma(t)\}_{t\geq 0}$ is shown by using the results due to [17] and [9].

Theorem 5.3. *Assume (A1) - (A4) and (5.6). Then $\{\Gamma(t)\}_{t\geq 0}$ is a distance solution of (5.1). In other words, $d \wedge 0$ and $d \vee 0$ are, respectively, a viscosity subsolution and a viscosity supersolution of (5.2).*

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