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Borel sums of Voros coefficients of hypergeometric differential equations with a large parameter

By

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§ 1. Introduction

The notion of Voros coefficients was introduced by Voros [11] for some Schrödinger equations with irregular singularities. It plays a role in the analysis of Stokes phenomena for WKB solutions with respect to parameters which are contained in the potentials. For Weber equations and for Whittaker equations, concrete forms of the Voros coefficient were obtaind by Shen-Silverstone [8], Takei [9] and by Koike-Takei [7].

Voros coefficients can be defined also for equations with regular singularities. In [2], the authors give a definition of them and a concrete form of a Voros coefficient for hypergeometric differential equations with a large parameter for a special case. As in the case of irregular singularities, we want to analyze the Stokes phenomena for WKB solutions in parameters by using Voros coefficients of hypergeometric equations. For this purpose, we must compute the Borel sums of them.

In this report, we give a concrete form of the Voros coefficient for each regular singular point and the Borel sums of it for hypergeometric equations. Detailed discussions and proofs will be given in our article in preparation.

§ 2. Voros coefficients

We consider the following Schrödinger-type equation with a large parameter η:

$$\left(-\frac{d^2}{dx^2}+\eta^2 Q\right)\psi=0 \tag{2.1}$$

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with $Q = Q_0 + \eta^{-2}Q_1$, where we set

$$Q_0 = \frac{(\alpha-\beta)^2x^2 + 2(2\alpha\beta - \alpha\gamma - \beta\gamma)x + \gamma^2}{4x^2(x-1)^2} \tag{2.2}$$

and

$$Q_1 = -\frac{x^2 - x + 1}{4x^2(x-1)^2}. \tag{2.3}$$

Then α, β and γ are complex parameters. Equation (2.1) is obtained from the hypergeometric differential equation:

$$x(1-x)\frac{d^2w}{dx^2} + (c - (a+b+1)x)\frac{dw}{dx} - abw = 0, \tag{2.4}$$

that is, we introduce a large parameter η by setting $a = 1/2 + \eta\alpha$, $b = 1/2 + \eta\beta$, $c = 1 + \eta\gamma$ with complex parameters α, β and γ and eliminate the first-order term by taking

$$\psi = x^{\frac{1}{2}+\frac{\eta\gamma}{2}}(1-x)^{\frac{1}{2}+\frac{\eta(\alpha+\beta-\gamma)}{2}}w$$

as unknown function. Then we have equation (2.1). Let

$$\psi_\pm = \frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp(\pm\int_{a_k}^{x} S_{\mathrm{odd}}\,dx), \tag{2.5}$$

be WKB solutions of (2.1) (cf. [7]). Here $a_k (k = 0, 1)$ is a turning points of (2.1), that is, zeros of Q_0 and S_{odd} denotes the odd-order part of the formal solution $S = \sum_{h=-1}^{\infty}\eta^{-h}S_h$ in η^{-1} of the Riccati equation

$$\frac{dS}{dx} + S^2 = \eta^2 Q \tag{2.6}$$

associated with (2.1). We consider the following integrals which are called Voros coefficients:

$$V_0 = V_0(\alpha,\beta,\gamma) := \int_0^{a_k}(S_{\mathrm{odd}} - \eta S_{-1})dx,$$

$$V_1 = V_1(\alpha,\beta,\gamma) := \int_1^{a_k}(S_{\mathrm{odd}} - \eta S_{-1})dx$$

and

$$V_2 = V_2(\alpha,\beta,\gamma) := \int_\infty^{a_k}(S_{\mathrm{odd}} - \eta S_{-1})dx$$

of equation (2.1). Since the residues of S_{odd} and ηS_{-1} at the singular points coincide (See [6] for the computation of residues of S_{odd}.), these integrals are well-defined for every homotopy class of the path of integration and we have a formal series V_j (α, β, γ)

$(j=0,1,2)$ in η^{-1}. Note that, there are two turning points a_0, a_1 in general, however, V_0, V_1 and V_2 are independent of the choice of a_k $(k=0,1)$.

For $j=0,1$ and 2, $V_j(\alpha,\beta,\gamma)$ describes the discrepancy between WKB solutions normalized at a_k and those normalized at singular points $b_0=0, b_1=1$ and $b_2=\infty$, respectively, that is, when we set

$$\psi_\pm = \frac{1}{\sqrt{S_{\rm odd}}}\exp\left(\pm\int_{a_k}^x S_{\rm odd}dx\right) \tag{2.7}$$

and

$$\psi_\pm^{(b_j)} = \frac{1}{\sqrt{S_{\rm odd}}}\exp\left(\pm\int_{b_j}^x (S_{\rm odd}-\eta S_{-1})dx \pm \eta\int_{a_k}^x S_{-1}dx\right), \tag{2.8}$$

we have

$$\psi_\pm^{(b_j)} = \exp(\pm V)\psi_\pm. \tag{2.9}$$

Here the paths of integration should be chosen suitably. Voros coefficient V_j satisfies a system difference equations with respect to parameters α, β and γ. Solving the system we have the following Theorem.

Theorem 2.1. *Voros coefficients V_j have the following forms:*

$$V_0 = \frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n\eta^{1-n}}{n(n-1)}\left\{(1-2^{1-n})\left(\frac{1}{\alpha^{n-1}}+\frac{1}{\beta^{n-1}}+\frac{1}{(\gamma-\alpha)^{n-1}}+\frac{1}{(\gamma-\beta)^{n-1}}\right)+\frac{2}{\gamma^{n-1}}\right\},$$

$$V_1 = \frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n\eta^{1-n}}{n(n-1)}\left\{(1-2^{1-n})\left(\frac{1}{\alpha^{n-1}}+\frac{1}{\beta^{n-1}}-\frac{1}{(\gamma-\alpha)^{n-1}}-\frac{1}{(\gamma-\beta)^{n-1}}\right)+\frac{2}{(\alpha+\beta-\gamma)^{n-1}}\right\}$$

and

$$V_2 = \frac{1}{2}\sum_{n=2}^{\infty}\frac{B_n\eta^{1-n}}{n(n-1)}\left\{(1-2^{1-n})\left(\frac{1}{\alpha^{n-1}}-\frac{1}{\beta^{n-1}}-\frac{1}{(\gamma-\alpha)^{n-1}}+\frac{1}{(\gamma-\beta)^{n-1}}\right)-\frac{2}{(\beta-\alpha)^{n-1}}\right\}.$$

Here B_n are Bernoulli numbers defined by

$$\frac{te^t}{e^t-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}t^n.$$

§ 3. Stokes graphs

A characterization of Stokes graphs in term of parameters of (2.1) is given in [2]. A Stokes curve emanating from the turning point a_k $(k = 0, 1)$ is a curve defined by

$$\mathrm{Im} \int_{a_k}^{x} \sqrt{Q_0}\, dx = 0.$$

A Stokes curve flows into a singular point or a turning point. The Stokes graph ([1]) of (2.1) is, by definition, a two-colored sphere graph consisting of all Stokes curves (emanating from a_0 and a_1) as edges, $\{a_0, a_1\}$ as vertices of the first color and $\{b_0, b_1, b_2\}$ as vertices of the second color. The Stokes graph of (2.4) is, by definition, that of (2.1). We define that the sets H_j $(j = 0, 1, 2)$ of the parameters α, β, γ as follows:

$$H_0 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \alpha \cdot \beta \cdot \gamma \cdot (\alpha - \beta) \cdot (\alpha - \gamma) \cdot (\beta - \gamma) \cdot (\alpha + \beta - \gamma) \neq 0\}, \tag{3.1}$$

$$H_1 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \mathrm{Re}\alpha \cdot \mathrm{Re}\beta \cdot \mathrm{Re}(\gamma - \alpha) \cdot \mathrm{Re}(\gamma - \beta) \neq 0\}, \tag{3.2}$$

$$H_2 = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \mathrm{Re}(\alpha - \beta) \cdot \mathrm{Re}(\alpha + \beta - \gamma) \cdot \mathrm{Re}\gamma \neq 0\}. \tag{3.3}$$

If (α, β, γ) is contained in H_0, the turning points and the singular points of (2.4) are mutually distinct. Moreover, if (α, β, γ) is not contained in $H_1 \cup H_2$, then the Stokes geometry is degenerate.

We assume that (α, β, γ) is contained in the sets $H_0 \cap H_1 \cap H_2$. Stokes graphs can be classified by its order sequence $\hat{n} = (n_0, n_1, n_2)$, where n_0, n_1 and n_2 are numbers of Stokes curves that flow into $0, 1$ and ∞, respectively. Next we define the sets ω_k $(k = 1, 2, 3, 4)$ of the parameters α, β and γ as follows:

$$\begin{aligned}
\omega_1 &= \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\alpha < \mathrm{Re}\gamma < \mathrm{Re}\beta\}, \\
\omega_2 &= \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\alpha < \mathrm{Re}\beta < \mathrm{Re}\gamma < \mathrm{Re}\alpha + \mathrm{Re}\beta\}, \\
\omega_3 &= \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid 0 < \mathrm{Re}\gamma < \mathrm{Re}\alpha < \mathrm{Re}\beta\}, \\
\omega_4 &= \{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \mid \mathrm{Re}\gamma - \mathrm{Re}\beta < \mathrm{Re}\alpha < 0\}
\end{aligned}$$

and involutions ι_j $(j = 0, 1, 2)$ in the space of parameters as follows:

$$\begin{aligned}
\iota_0 &: (\alpha, \beta, \gamma) \mapsto (\beta, \alpha, \gamma), \\
\iota_1 &: (\alpha, \beta, \gamma) \mapsto (\gamma - \beta, \gamma - \alpha, \gamma), \\
\iota_2 &: (\alpha, \beta, \gamma) \mapsto (-\alpha, -\beta, -\gamma).
\end{aligned}$$

The potential Q is invariant under those involutions. Moreover, we define Π_k as follows:

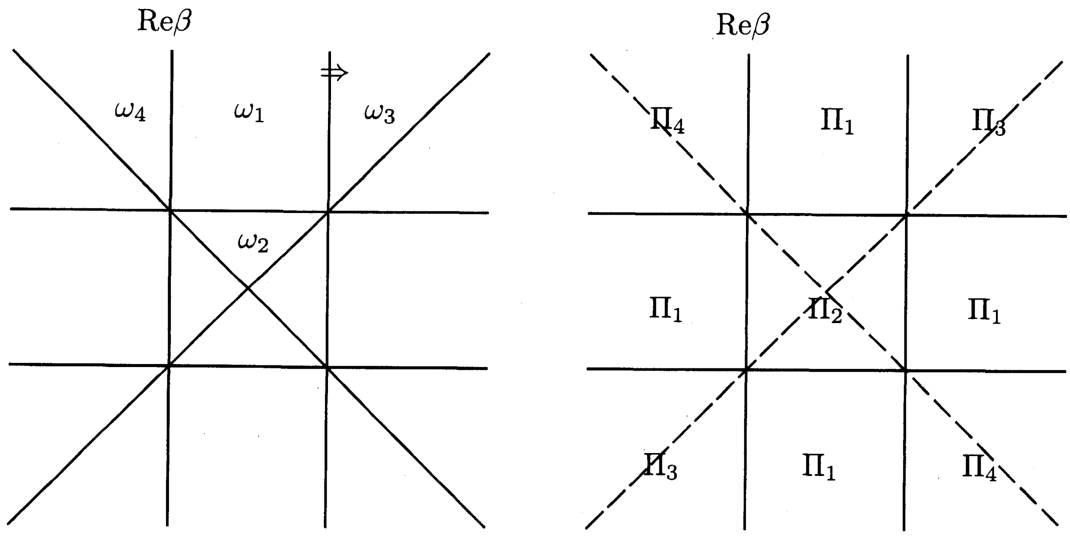
$$\Pi_k = \bigcup_{r \in G} r(\omega_k) \qquad (k = 1, 2, 3, 4). \tag{3.4}$$

Here G is the group generated by ι_j $(j = 0, 1, 2)$. We characterize the types of Stokes graphs in terms of the parameters. The following Theorem is proved in [2] (Theorem 3.2) (See all so [3], [10].)

Theorem 3.1. *Let $\hat{n}$ denote the order sequence of the Stokes graph with parameters (α, β, γ).*

(1) *If* $(\alpha, \beta, \gamma) \in \Pi_1$, *then* $\hat{n} = (2, 2, 2)$.
(2) *If* $(\alpha, \beta, \gamma) \in \Pi_2$, *then* $\hat{n} = (4, 1, 1)$.
(3) *If* $(\alpha, \beta, \gamma) \in \Pi_3$, *then* $\hat{n} = (1, 4, 1)$.
(4) *If* $(\alpha, \beta, \gamma) \in \Pi_4$, *then* $\hat{n} = (1, 1, 4)$.

Remark. For a fixed Re $\gamma > 0$, configurations of ω_k's and Π_k's in the real α-β plane are shown in Fig. 3.1.



Fih. 3.1

We will consider the Borel sums of Voros coefficients in ω_1 and in ω_3 in the next section. We show some example of Stokes curves in Fig. 3.2.

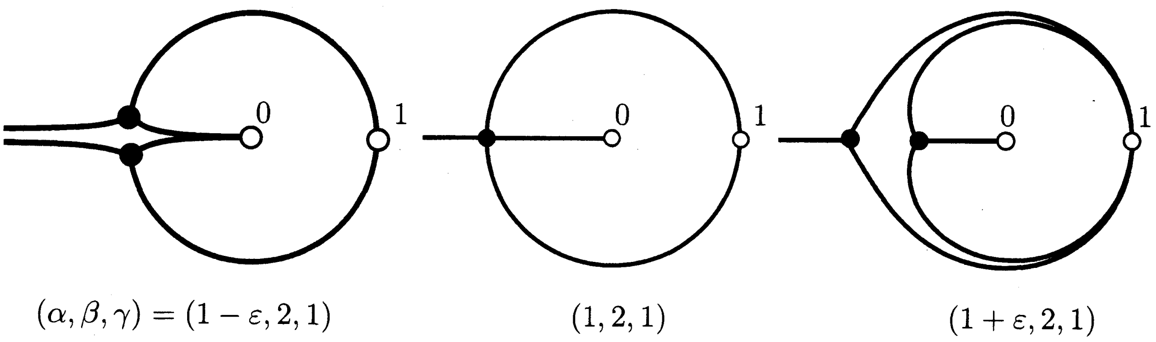


Fig. 3.2

Here bullets and white bullets designate turning points and singular points, respectively and $\varepsilon > 0$. If we take $(\alpha,\beta,\gamma) = (1,2,1)$, which is located on the boundary between ω_1 and ω_3, turning points coincide (cf. Fig. 3.2). If we take $(\alpha,\beta,\gamma) = (1-\varepsilon,2,1)$ (resp. $(1+\varepsilon,2,1)$), i.e, parameters are contained in ω_1 (resp. ω_3), we have $\hat{n} = (2,2,2)$ (resp. $\hat{n} = (1,4,1)$) in left-hand side (resp. right-hand side) of Fig. 3.2.

§ 4. Borel sums of Voros coefficients

In this section we consider the relation between Borel sums of Voros coefficients in ω_1 and ω_3. Let $V^1_{0,B}$ (resp. $V^3_{0,B}$) and V^1_j (resp. V^3_j) $(j = 0,1,2)$ denote the Borel transforms and the Borel sums of the Voros coefficients V_j in ω_1 (resp. ω_3), respectively. To clarify the relations between Borel sums V^1_j and V^3_j $(j = 0,1,2)$, we need the concrete forms of V^1_j and V^3_j. They are given as follows.

Theorem 4.1. *Borel sums V^1_j (resp. V^3_j) of Voros coefficients have following forms:*

$$V^1_0 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}}, \tag{4.1}$$

$$V^3_0 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+(\alpha-\gamma)\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}\eta}{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)(\alpha-\gamma)^{(\alpha-\gamma)\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}2\pi}, \tag{4.2}$$

$$V^1_1 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\Gamma^2((\alpha+\beta-\gamma)\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\eta}{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\alpha+\beta-\gamma)^{2(\alpha+\beta-\gamma)\eta-1}}, \tag{4.3}$$

$$V^3_1 = \frac{1}{2}\log\frac{2\pi\Gamma^2((\alpha+\beta-\gamma)\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\alpha-\gamma)^{(\alpha-\gamma)\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\eta}{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\alpha-\gamma)\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)(\alpha+\beta-\gamma)^{2(\alpha+\beta-\gamma)\eta-1}}, \tag{4.4}$$

$$V^1_2 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\alpha^{\alpha\eta}(\beta-\alpha)^{2(\beta-\alpha)\eta-1}}{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma^2((\beta-\alpha)\eta)\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\eta}, \tag{4.5}$$

$$V^3_2 = \frac{1}{2}\log\frac{2\pi\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\alpha^{\alpha\eta}(\alpha-\gamma)^{(\alpha-\gamma)\eta}(\beta-\alpha)^{2(\beta-\alpha)\eta-1}}{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+(\alpha-\gamma)\eta)\Gamma^2((\beta-\alpha)\eta)\beta^{\beta\eta}(\beta-\gamma)^{(\beta-\gamma)\eta}\eta}. \tag{4.6}$$

Outline of the proof. To compute the Borel sums V^1_0 and V^3_0, we first take the Borel transforms $V^1_{0,B}$ and $V^3_{0,B}$ of Voros coefficient V_0.

Proposition 4.2. *Borel transforms $V^1_{0,B}$ and $V^3_{0,B}$ of Voros coefficients V_0 have following forms:*

$$V^1_{0,B} = -g(\alpha) - g(\beta) - g(\gamma-\alpha) + g(\beta-\gamma) + \frac{1}{y}\Big(\frac{1}{\exp\frac{y}{\gamma}-1} - \frac{\gamma}{y} + \frac{1}{2}\Big)$$

and

$$V^3_{0,B} = -g(\alpha) - g(\beta) + g(\alpha-\gamma) + g(\beta-\gamma) + \frac{1}{y}\Big(\frac{1}{\exp\frac{y}{\gamma}-1} - \frac{\gamma}{y} + \frac{1}{2}\Big)$$

Here $g(s) = \frac{1}{2y}\exp(-\frac{y}{2s})(\frac{1}{e^{\frac{y}{s}}-1} + \frac{1}{2} - \frac{s}{y})$.

The Borel sums of V_0, are obtained by using the following integral representation of the logarithm of the Γ-function.

Lemma 4.3. *We heve the formula:*

$$\int_0^\infty \left(\frac{1}{e^t-1}+\frac{1}{2}-\frac{1}{t}\right)\frac{e^{-\theta t}}{t}\,dt = \log\frac{\Gamma(\theta)}{\sqrt{2\pi}}-(\theta-\frac{1}{2})\log\theta+\theta.$$

Next we consider the relation between V_0^1 and V_0^3. Borel sums of Voros coefficient V_0^1 is analytically continued over ω_3. We compare it with V_0^3. If Im $(\alpha-\gamma)>0$, then we rewrite V_0^1 as follows.

$$\begin{aligned} V_0^1 &= \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta}\eta}{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}} \\ &= \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\alpha-\gamma)^{(\gamma-\alpha)\eta}\eta}{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}}+\frac{(\gamma-\alpha)\eta\pi i}{2}. \end{aligned} \tag{4.7}$$

Subtracting (4.7) from (4.2), we have

$$\begin{aligned} V_0^1-V_0^3 &= \frac{1}{2}\log\frac{2\pi}{\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)\Gamma(\frac{1}{2}+(\alpha-\gamma)\eta)}+\frac{(\gamma-\alpha)\eta\pi i}{2} \\ &= \frac{1}{2}\log(e^{2(\gamma-\alpha)\eta\pi i}+1). \end{aligned}$$

On other hand, if $\mathrm{Im}(\alpha-\gamma)<0$, we rewrite V_0^1 as follows.

$$V_0^1 = \frac{1}{2}\log\frac{\Gamma(\frac{1}{2}+(\beta-\gamma)\eta)\Gamma^2(\gamma\eta)\alpha^{\alpha\eta}\beta^{\beta\eta}(\alpha-\gamma)^{(\gamma-\alpha)\eta}\eta}{\Gamma(\frac{1}{2}+\alpha\eta)\Gamma(\frac{1}{2}+\beta\eta)\Gamma(\frac{1}{2}+(\gamma-\alpha)\eta)(\beta-\gamma)^{(\beta-\gamma)\eta}\gamma^{2\gamma\eta-1}}-\frac{(\gamma-\alpha)\eta\pi i}{2}.$$

Hence we have the following relation:

$$V_0^1-V_0^3=\frac{1}{2}\log(e^{2(\alpha-\gamma)\eta\pi i}+1).$$

In the same way, we obtain formulas for the other cases. Summing up, we have the following.

Theorem 4.4. *The relations between Borel sums V_j^1 and V_j^3 $(j=0,1,2.)$ of Voros coefficients have following forms:*
(1) *If* Im $(\alpha-\gamma)>0$, *then we have*

$$\begin{aligned} V_0^1 &= V_0^3+\frac{1}{2}\log(e^{2(\gamma-\alpha)\eta\pi i}+1), \\ V_1^1 &= V_1^3-\frac{1}{2}\log(e^{2(\gamma-\alpha)\eta\pi i}+1) \end{aligned}$$

and

$$V_2^1 = V_2^3 - \frac{1}{2}\log(e^{2(\gamma-\alpha)\eta\pi i}+1).$$

(2) *If* Im $(\alpha-\gamma)<0$, *then we have*

$$V_0^1 = V_0^3 + \frac{1}{2}\log(e^{2(\alpha-\gamma)\eta\pi i}+1),$$
$$V_1^1 = V_1^3 - \frac{1}{2}\log(e^{2(\alpha-\gamma)\eta\pi i}+1)$$

and

$$V_2^1 = V_2^3 - \frac{1}{2}\log(e^{2(\alpha-\gamma)\eta\pi i}+1).$$

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