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On the boundary values of continuous functions, respectively hyperfunctions, various settings and some relations between them

By

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Abstract

The main argument in this note is on the boundary behavior of relatively smooth functions. In the first part we will assume that the functions under consideration are solutions to some hypoelliptic partial differential operator, whereas in the second part we shall deal with the question of how to define boundary values of continuous functions in a situation in which we look for such boundary values in hyperfunctions. In particular, we want to discuss some results concerning the relation of two rather distinct approaches to this problem: one is by the theory of mild hyperfunctions of K. Kataoka and T. Oaku and the other is by the theory of so called "strong boundary values" developed by J.P. Rosay and L.E. Stout.

§ 1. Temperate growth at the boundary

Let U be open in $\mathbb{R}^n$ and consider a point x^0 in the boundary ∂U of U. We say that a measurable function $u : U \to \mathbb{C}$ is of temperate growth at ∂U near x^0 if we can find $c > 0, k \geq 0$, and a neighborhood W of x^0 such that

$$|u(x)| \leq c\,\mathrm{dist}(x, \partial U)^{-k} \text{ for almost all } x \in U \cap W. \tag{1.1}$$

We also say that u is extendible across the boundary at x^0 (in distributions) if we can find a neighborhood W of x^0 and a distribution v on W such that v coincides with u on $U \cap W$.

Our aim in this section is to discuss the relation between temperate growth and extendibility across the boundary in the case when u satisfies satisfies a given linear hypoelliptic partial differential equation. At the end of the section, we will also make

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some comments concerning cases when either the equation is not hypoelliptic or when u does not satisfy any equation near x^0.

Consider then (with standard multiindex notation and conventions), at first a linear partial differential operator $p(x,D) = \sum_{|\alpha|\le m} a_\alpha(x) D_x^\alpha$ with $\mathcal{C}^\infty$ coefficients a_α defined in a neighborhood V of x^0. We recall that an operator p is called hypoelliptic if for every distribution u it follows from $p(x,D)u \in \mathcal{C}^\infty(W)$ for some open set $W \subset V$ that also u must be $\mathcal{C}^\infty$ on W. We will assume in this section that p admits a right parametrix given by a pseudodifferential operator associated with a symbol in a symbol class of type $S^\mu_{\rho,\rho}$ for some $\rho < 1$ and that this parametrix is defined in a full neighborhood of x^0. (We will recall the definition of the symbol classes $S^\mu_{\rho,\rho}$ in a moment.) Many conditions on the symbol $p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha = \exp[-i\langle x,\xi\rangle] p(x,D) \exp[i\langle x,\xi\rangle]$ of $p(x,D)$ are known to imply the existence of such parametrices, foremost the cases considered by L. Hörmander, [7], or L.Boutet de Monvel in [2]. Actually, somewhat less than a parametrix in a good symbol class is needed in the argument: the main result will remain valid whenever $p(x,D)$ admits a parametrix which satisfies the property (1.4) below. Indeed, we could have worked with symbols in the classes of R.Beals, [1], as well, and our preference for the symbol classes $S_{\rho,\rho}$ is purely opportunistic: the definition of symbol the classes $S^\mu_{\rho,\rho}$ is directly accessible also to non-specialists, whereas the symbol classes in [1] have rather complicated definitions and the meaning of the choices in the definitions needs probably to be explained by some comments. Anyway, we will make some comments on symbol classes later on.

The main result of this section is

Theorem 1.1. *Assume, under the above assumptions, that $u \in \mathcal{D}'(V)$ is a solution of $p(x,D)u = 0$ on $U \cap V$. Then u is of temperate growth at the boundary of U near x^0.*

We next recall some terminology related to "parametrices". A right parametrix for $p(x,D)$ on an open set V is a linear continuous operator $T\colon \mathcal{E}'(V) \to \mathcal{D}'(V)$ such that $T \circ p(x,D) = I + K$ on $\mathcal{E}'(V)$ where I is the identity operator and $K\colon \mathcal{E}'(V) \to \mathcal{C}^\infty(V)$ is an integral operator with a $\mathcal{C}^\infty$ kernel. If the operator $p(x,D)$ is hypoelliptic, then from $u \in \mathcal{E}'(V)$ and from the fact that u is $\mathcal{C}^\infty$ near some point $\tilde{x}$ it must follow that also Tu is $\mathcal{C}^\infty$ near $\tilde{x}$. It is then easy to see that the kernel $F \in \mathcal{D}'(V \times V)$ of T (given, if not already known explicitly, by the Schwartz kernel theorem) must be $\mathcal{C}^\infty$ outside the diagonal $\{(x,x); x \in V\}$ of $V \times V$. (We say that a distribution $F \in \mathcal{D}'(V \times V)$ is the kernel of an operator $T\colon \mathcal{E}'(V) \to \mathcal{D}'(V)$ if we have that $F(\varphi \otimes \psi) = T(\varphi)(\psi)$ for every $\varphi, \psi \in \mathcal{C}_0^\infty(V)$.) Actually, this can be seen by abstract functional analysis, and there is no need to know by what kind of argument the hypoellipticity of $p(x,D)$ was established, but if the parametrix is given by a pseudodifferential operator, then the fact that F must be $\mathcal{C}^\infty$ outside the diagonal $\{(x,x); x \in V\}$ of $V \times V$ is trivial.

We further recall the definition of the symbol classes $S^\mu_{\rho,\delta}$: let V be open in $\mathbb{R}^n$

and consider $0 < \delta \leq \rho \leq 1$ and $\mu \in \mathbb{R}$. We denote by $S^{\mu}_{\rho,\delta}(V \times V)$ (or by $S^{\mu}_{\rho,\delta}(V)$ when only one space variable is present), V open in $\mathbb{R}^n$, the class of $\mathcal{C}^\infty$ functions $q\colon V \times V \times \mathbb{R}^n \to \mathbb{C}$ such that for every compact $K \subset V$ and for every two multiindices α, β we can find constants $c_{\alpha,\beta}$ such that

$$(1.2) \qquad |\partial_\xi^\alpha \partial_{x,y}^\beta q(x,y,\xi)| \leq c_{\alpha,\beta}(1+|\xi|)^{\mu-\rho|\alpha|+\delta|\beta|}, \text{ if } (x,y) \in K \times K, \xi \in \mathbb{R}^n.$$

We also recall that when $\delta < \rho$, then the calculus of pseudodifferential operators associated with symbols in the class $S^{\mu}_{\rho,\delta}(V \times V)$ is rather simple and in particular a parametrix will exist if p can be inverted in the symbol algebra $S^{\mu}_{\rho,\delta}(V)$. By this we mean that there is $\mu \in \mathbb{R}$, and $q \in S^{\mu}_{\rho,\delta}(V)$ (note in particular that here q is assumed to be independent of the variable y) and a sequence μ_j which tends to $-\infty$, such that for every k we have

$$(1.3) \qquad 1 - \sum_{|\alpha|<k} \frac{i^{|\alpha|}}{\alpha!} \partial_\xi^\alpha q(x,\xi) \partial_x^\alpha p(x,\xi) \in S^{\mu_k}_{\rho,\delta}(V).$$

This is the situation in L.Hörmander [7]. While this case already covers many classes of pseudodifferential operators (including all hypoelliptic operators with constant coefficients), L. Boutet de Monvel in [2] was the first to consider operators with parametrices in the classes $S^{\mu}_{\delta,\rho}$, for the case $\rho = \delta = 1/2$. In a parallel development, with many intersections with the work of L.Boutet de Monvel, R.Beals in [1], considered symbols in very general classes $S^{\lambda}_{\varphi,\Phi}$, for suitable pairs of weight functions (φ, Φ). The symbol classes in [1] in particular contain parametrices for many examples of hypoelliptic operators (foremost perhaps the examples of V. Grushin [5]), which were not in the classical symbol classes of [7]. In Theorem 1.1 we will for simplicity work with symbols in the classes $S^{m}_{\rho,\delta}$, $\delta \leq \rho$. In the case $\delta = \rho$ the symbolic calculus will work efficiently only if some additional information on the symbols involved is available, as was in fact the case in [2], and in a large number of papers written about the same time or later. Since the number of papers in which parametrices associated with symbols close to the classes $S^{\mu}_{1/2,1/2}$ is huge, we only mention a number of papers written immediately after 1974: we cite L.Boutet de Monvel-F.Treves [4], L.Boutet de Monvel-A.Grigis-B.Helffer [3], J.Sjöstrand [18]. Also see L.Hörmander [8]. On the other hand, the number of papers where parametrices are constructed in $S^{\mu}_{\rho,\rho}$, $\rho < 1/2$, is apparently much smaller and we mention here M.Mughetti-F.Nicola [13], together with the references there. We also mention that the parametrices associated with symbols in the classes of [1] are of the type needed in this paper, provided that the pair (φ, Φ) is "localizable" in the sense of [1], page 5. In view of the complicated nature of the symbols in [1], we will not give details.

We now want to make some comments on the main assumption in Theorem 1.1, viz. the type of hypoellipticity of $p(x, D)$. We have assumed above for commodity that hypoellipticity comes from the existence of a parametrix in the symbol class $S^{\mu}_{\rho,\rho}$. Actually, all we need is that we can find a parametrix for p with a kernel F which satisfies the following condition:

let k be given and let m be the order of $p(x,D)$. Then there is k' such that $|x-y|^{k'}F(x,y)$ is $\mathcal{C}^{k+m}(V\times V)$. In particular, we have when $\overline{W}\subset V$, W a bounded neighborhood of x^0 and $\overline{W}$ its closure, the following result:

$$\sup_{x\in W, y\in W} |\partial_y^{\alpha}(|x-y|^{k'}F(x,y))| < \infty, \text{ for } |\alpha|\le k+m. \tag{1.4}$$

In some sense we could say that the conclusion in the theorem is more of a statement on the parametrices involved in the argument than on the operator $p(x,D)$ itself.

We continue this section with a number of remarks concerning temperate growth and extendibility at the boundary.

We observe at first that it does not follow from $E\in\mathcal{D}'(\mathbb{R}^n)$ and $\operatorname{singsupp} E=\{0\}$ that the restriction of E to $\mathbb{R}^n\setminus\{0\}$ has to be temperate at 0. Indeed, e.g., for $n=1$, $\sin(\exp[1/x])$, is bounded on $\mathbb{R}\setminus\{0\}$ and defines therefore a distribution on all of $\mathbb{R}$. Therefore also $E=(d/dx)\sin(\exp[1/x])$ defines a distribution on all of $\mathbb{R}$, but the restriction of E to $\mathbb{R}\setminus\{0\}$ is not temperate at 0. Now, to continue this example, denote for a given $\varphi\in\mathcal{C}_0^\infty(\mathbb{R})$ which is identically one in a neighborhood of $0\in\mathbb{R}$ by $S=\varphi E$ and by $T\colon\mathcal{E}'(\mathbb{R})\to\mathcal{D}'(\mathbb{R})$ the operator $Tu=S*u$. Then T has the kernel $F(x,y)=S(x-y)$. F has $(\mathcal{C}^\infty)$ singular support on the diagonal $\{(x,x);x\in\mathbb{R}\}$ of $\mathbb{R}\times\mathbb{R}$ and therefore shrinks singular supports (as the parametrices of hypoelliptic operators would do). T is thus an example of an operator which shrinks singular supports, but which has a kernel which is not temperate at the diagonal of $\mathbb{R}\times\mathbb{R}$.

In the other direction we have

Lemma 1.2. *Consider U open in $\mathbb{R}^n$ and let $x^0\in\partial U$. Assume that after a $\mathcal{C}^\infty$ change of variables in a neighborhood of x^0, $\partial U\cap V$ is of form $\{x\in V;q(x)=0\}$ where V is an open neighborhood of x^0 and q is a real analytic function defined on V. Also let $f\colon U\to\mathbb{C}$ be a measurable function which is of temperate growth at the boundary of U near x^0. Then f extends to a distribution defined on a neighborhood of x^0.*

(The lemma is an immediate consequence of results on the division of distributions by real analytic functions due to S.Lojasiewicz, respectively by polynomials, due to L.Hörmander. See [12], [6].)

Theorem 1.1 also has a bearing on the structure of the singularities of solutions of $p(D)u=0$, $p(D)$ a hypoelliptic constant coefficients operator. We assume that u is defined outside 0, say on $\{x\in\mathbb{R}^n;0<|x|<1\}$ and are interested in the structure of u near 0 when u has temperate growth at 0. We next fix in an arbitrary way a fundamental solution E of $p(D)$. We then have

Proposition 1.3. *Under the above assumptions, we can find a constant coefficient linear partial differential operator $R(D)$ and a C^∞ function G which satisfies $p(D)G = 0$ on $U = \{x; 0 < |x| < 1\}$ such that $u = G + R(D)E$ for $0 < |x| < 1$. Conversely, every function u on U of this form is a solution of $p(D)u = 0$ with temperate growth at 0.*

In the opposite direction, it is not difficult to see that if $p(D)$ is a given constant coefficient hypoelliptic partial differential operator, then we can find solutions of $p(D)u = 0$ defined for $x \neq 0$, which do not have temperate growth at 0.

A slightly more interesting example of the same type, but for a much bigger boundary, is the following. We consider the Laplace operator $\Delta = (\partial/\partial x)^2 + (\partial/\partial y)^2$ as an operator on $U = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$ and consider an analytic functional v on $x^2 + y^2 = 1$ which is not a distribution. Let u be a solution on U of the Dirichlet problem: $\Delta u = 0$ in $\{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$, $u_{|x^2+y^2=1} = v$. Note that a solution for this problem is given (e.g.) by Poisson's formula. It is then classical, and it also follows from results above, that u cannot have temperate growth at the boundary. (Otherwise, it were extendible across the boundary, and the boundary value would have to be a distribution, e.g., by Theorem B.2.9 in Hörmander [8], vol. III).

§ 2. Strong boundary values

In this section we recall some definitions introduced by J.P.Rosay and E.L.Stout in [15], [16], in which they have defined boundary values of continuous functions which do not necessarily satisfy growth conditions at the boundary. The relevant part of the boundaries from now on must be real-analytic and the boundary values in question will be hyperfunctions. The theory of Rosay-Stout is in some sense (for a much more restricted set of objects) alternative to the theory of mild hyperfunctions of K. Kataoka, T,Oaku, [9], [14]. It is our aim in this second part of the paper to review the relevant definitions and to say something about the relation between the two approaches. The definitions of Rosay-Stout come in a local and in a global variant. We start with the global variant, since it is more intuitive.

We consider a m-dimensional real-analytic compact manifold $\mathcal{M}$ without boundary. To avoid technical discussions about complexifications, we will assume that $\mathcal{M}$ has already been embedded into $\mathbb{R}^n$ for some large n and will regard $\mathcal{M}$ as a Riemannian manifold with the metric induced on it from the ambient $\mathbb{R}^n$.

Next consider a continuous function u on $\mathcal{M} \times (0, 1)$. A distribution v on $\mathcal{M} \times (0, 1)$ is associated with this u by

$$g \to v(g) = \int_{\mathcal{M}\times(0,1)} u(z,t)g(z,t)d\sigma(z)dt, \; g \in C_0^\infty(\mathcal{M} \times (0,1)), \tag{2.1}$$

where $d\sigma$ is the volume element of $\mathcal{M}$ (with respect to the metric on $\mathcal{M}$). We will be interested in boundary values of u at $t = 0$, which is regarded as part of the boundary of $\mathcal{M} \times (0,1)$. (Note that, after a rescaling, we could equally well work with functions defined on $\mathcal{M} \times (0, d)$ for some $d > 0$.)

Definition 2.1. We will say that u admits strong boundary values (in the sense of Rosay-Stout) at $t = 0+$ if for every real-analytic function g defined on $\mathcal{M} \times (-1,1)$ the function

$$\psi_g(t) = \int_{\mathcal{M}} u(z,t)g(z,t)d\sigma(z), \tag{2.2}$$

initially defined for $t \in (0,1)$, extends to a holomorphic function in a neighborhood of 0. (We can of course define $\psi_g(t)$ also for functions which depend only on z, and not on (z, t). Note that integration in (2.2) is anyway only in z.)

The local variant of the above is:

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ and let u be a continuous function defined on $\bar{\Omega} \times (0,1) \subset \mathbb{R}^{n+1}$. For $t \in (0,1)$ we let $I_{u,t}$ be the analytic functional defined by

$$I_{u,t}(f) = \int_{\Omega} f(x)u(x,t)\,dx, f \text{ real analytic on } \Omega. \tag{2.3}$$

The function u is said to admit a "strong boundary value" on Ω if for every neighborhood V of $\bar{\Omega} \setminus \Omega$ in $\mathbb{C}^{n+1}$, there is a family $\{E_{V,t}\}_{t\in(0,1)}$ of analytic functionals, each carried by V, such that for each $f \in \mathcal{A}(\mathbb{C}^{n+1})$ the function

$$t \mapsto I_{u,t}(f) - E_{V,t}(f) \tag{2.4}$$

extends to be real analytic on a neighborhood of $0 \in \mathbb{R}$.

The above definitions are discussed to great length in [15]. Having specified which functions on $\bar{\Omega} \times (0,1)$ admit strong boundary values, we need to explain what these boundary values are. (We only consider the local case.) In fact, the boundary value will be a hyperfunction on Ω. We specify it in one of the standard representations of hyperfunctions on bounded open domains, namely as equivalence classes of real-analytic functionals on $\bar{\Omega}$ modulo real-analytic functionals on $\partial\Omega$:

Definition 2.3 (Cf. Definition 5.5 in [15]). Let u be as in Definition 2.2 and consider a real-analytic functional v carried by $\bar{\Omega}$. We say that the equivalence class, modulo real-analytic functionals carried by $\partial\Omega$, of v is the boundary value of u on $\Omega \times \{t = 0\}$ if and only if for every neighborhood V of $\partial\Omega \times \{0\}$ in $\mathbb{C}^{n+1}$, there are analytic functionals $E_{V,t}$ as in definition 2.2 with the property that $\bar{V}$ is a carrier for the analytic functional

$$f \mapsto v(f(\cdot,0)) - \lim_{t\to 0+}(I_{u,t}(f) - E_{V,t}(f)), \text{ for every } f \in \mathcal{A}(\mathbb{C}^{n+1}). \tag{2.5}$$

§ 3. Mildness at a smooth boundary

"Mild" hyperfunctions have first been considered by K.Kataoka, see [9]. They are, together with their generalization by T.Oaku, [14], a very natural frame for defining boundary values of hyperfunctions. Before the advent of the theory of mild hyperfunctions, boundary values for hyperfunctional solutions of linear partial differential operators had been defined by H. Komatsu-T. Kawai, [10], [11] and P.Schapira, [17], using the operator in an essential way. In the theory of mild hyperfunctions, boundary values are defined in a way which does not require the hyperfunction under consideration to satisfy an equation.

Definition 3.1. Let u be a germ of a hyperfunction defined near $(x^0, 0) \in \mathbb{R}^{n+1}_{x,t}$ in the region $t > 0$.
a) (K.Kataoka) u is called "mild" from the positive side of $t = 0$ (or, mild at $t = 0+$, for short) near x^0, if there are $\varepsilon > 0$, $c > 0$, s, open convex cones G_j, $j = 1, \dots, s$, in $\mathbb{R}^n \setminus \{0\}$, and holomorphic functions h_j defined on the sets

$$D_j = \{(x,t) \in \mathbb{C}^{n+1}; |t| + |x - x^0| < \varepsilon, \operatorname{Im} x \in G_j, |\operatorname{Im} t| + \max(0, -\operatorname{Re} t) < (1/c)|\operatorname{Im} x|\}$$

such that

$$u = \sum_{j=1}^{s} b(h_j), \text{ for } t > 0 \text{ near } 0. \tag{3.1}$$

b) (T.Oaku) u is called F-mild at $t = 0+$ near x^0, if it can be represented as in (3.1), but where, this time, the h_j are holomorphic functions defined in a neighborhood in $\mathbb{C}^{n+1}$ of the set

$$D'_j = \{(x,t); |t| + |x - x^0| < \varepsilon, \operatorname{Re} t \geq 0, \operatorname{Im} t = 0, \operatorname{Im} x \in G_j\}. \tag{3.2}$$

c) u is called temperately F-mild at $t = 0+$ near x^0, if it can be represented as in (3.1) with $\varepsilon > 0$, cones G_j and h_j as in part b) of this definition, but with the additional property that for every $\varepsilon' < \varepsilon$ and every $\delta > 0$ there are constants c, k for which

$$|h_j(x,t)| < c|\operatorname{Im} x|^{-k}, \text{ if } |\operatorname{Re} x - x^0| < \varepsilon', \operatorname{Re} t > \delta. \tag{3.3}$$

The boundary values $b(h_j)$ are here taken of course in hyperfunctions.

If u is F-mild (or more regular) at $t = 0+$, then there is a very natural definition of the boundary value $u(\cdot, 0)$ at $t = 0$: we observe that the intersection of the domain D'_j with $t = 0$ is the wedge $W_j = \{x \in \mathbb{C}^n; |\operatorname{Re} x - x^0| < \varepsilon, \operatorname{Im} x \in G_j\}$. The restriction of h_j to W_j is then a holomorphic function defined on a wedge in $\mathbb{C}^n$ and as such defines a hyperfunction v_j on the set $\{x \in \mathbb{R}^n; |x - x^0| < \varepsilon\}$. The boundary value of u at $t = 0+$ is then defined to be $\sum_{j=1}^{s} v_j$.

Closely related to the preceding remark is the fact that if we fix $t^0 > 0$ small, then we can restrict u to $t = t^0$ for x near x^0 by standard analytic microlocalization, in that it is immediate to see (using Sato's definition of WF_a) that

$$((x^0, t^0), (0, \pm 1)) \notin WF_a u. \tag{3.4}$$

Part c) of the definition is not taken from [9] or [14]. It is one of the variants of mildness which this author found convenient to introduce while looking into the relation between mildness and the existence of strong boundary values as considered in the next section. The main property of the representation functions h_j which appear in the preceding definition is of course that their restrictions to the boundary $t = 0$ are holomorphic functions defined on wedges in the x-variables, and it may well be that situations different from the ones in part a) and b) in which this property is present, can turn out interesting. We have however not explored the relation between the variants of "mildness" which we will encounter. Anyway, it can be seen on examples that the "k", for a temperately F-mild hyperfunction will in general effectively depend on the choice of δ. It can moreover be shown that every F-mild continuous function on a set of form $I \times (0, d)$, $I \subset \mathbb{R}$ an interval, is temperately F-mild. Elementary examples seem to indicate that temperate F-mildness is a reasonable assumption also in higher dimensions when we study mildness at the boundary for continuous functions or distributions.

§ 4. Mildness versus strong boundary values

Our first result on the relation between mildness and strong boundary values is

Proposition 4.1. *If h is temperately mild at $t = 0$, then it admits strong boundary values at $t = 0+$.*

We have not succeeded in proving a result which is converse to the preceding proposition. The following results are however indications that the relation between mildness and the existence of strong boundary values is not trivial. The first of these results shows in fact already that global strong boundary values are related to analytic microlocalization. We assume that $\mathcal{M}$ is a compact real-analytic manifold embedded in $\mathbb{R}^n$ as in the definition of strong global boundary values. Also let u be a continuous function on $\mathcal{M} \times (0, 1)$. We have:

Proposition 4.2. *We assume that u admits strong global boundary values at $t = 0+$ and consider the distribution v on $\mathcal{M} \times (0, 1)$ associated with u in (2.1). Then for small $t^0 > 0$ it follows that*

$$((z^0, t^0), (0, \pm 1)) \notin WF_a v. \tag{4.1}$$

More complete results on the relation between strong boundary values and mildness seem difficult to obtain. A case study when results are more advanced is when $\mathcal{M}$ is the unit disk Z in the complex plane. To be able to use the structure of Z fully, we will have to work for global strong boundary values. We should nevertheless state beforehand that in these results "mildness" will be understood in a sense a little bit different from the initial definition. A convenient frame for the case $\mathcal{M} = Z$ is the following:
the variable in $\mathbb{C}$ will be denoted by z. Z is then the one-dimensional real-analytic variety $Z = \{z \in \mathbb{C}; |z| = 1\}$. Accordingly the variables in $Z \times (0,1)$ are (z,t). If $z^0 \in Z$ is fixed, we can parametrize Z near z^0 with analytic coordinate patches. The standard parametrization is of course by the mapping $\varphi \mapsto \exp[i\varphi]$. φ is here real and we can for example let φ vary in two open overlapping intervals, denoted J_1, J_2, which together cover $[-\pi, \pi)$, both intervals being of length strictly smaller than 2π. More generally, if ψ is real and $\varphi \in J_1 \cup J_2$, we map $x = \varphi + i\psi$ to $z = \exp[-\psi + i\varphi]$. Points $\varphi + i\psi$ with $\mathrm{Im}\, x = \psi > 0$ are mapped in this way to points z with $|z| < 1$ and points with $\mathrm{Im}\, x = \psi < 0$ are mapped to points z with $|z| > 1$. Moreover, $\psi \mapsto \varphi + i\psi$ moves for fixed φ and $\psi \in (-\infty, \infty)$, on the radius vector $r \mapsto r\exp[i\varphi]$. It is in the coordinates x that we should "work", but it is more instructive to write down statements directly in the variable $z = \exp[ix]$. For the inverse relation some care has to be taken in order to avoid the non-unicity of the complex logarithm, but the inverse relation is very simple for the part $\mathrm{Im}\, x$ of x, in that $\mathrm{Im}\, x = -\ln|z|$. In particular, $\mathrm{Im}\, x > 0$ corresponds to $|z| < 1$ and $\mathrm{Im}\, x < 0$ to $|z| > 1$.

Also note that a function h is real-analytic on Z, precisely if $\chi(\varphi) = h(\exp[i\varphi])$ is real-analytic in φ near every fixed φ^0, and $h(z)$, defined near z^0 with $|z^0| = 1$, corresponds in the parameter x to an analytic function defined in a neighborhood of $x^0 = \ln[z^0]$ in $\mathbb{C}_+ = \{x \in \mathbb{C}; \mathrm{Im}\, x > 0\}$, precisely if it is analytic on a set of form $\{z; |z - z^0| < c', 1 - d < |z| < 1\}$. Further note that for $0 < \mathrm{Im}\, x$ small, $\mathrm{Im}\, x$ has the order of magnitude of $1 - |z|$, $z = \exp[-\mathrm{Im}\, x]$, in the sense that there are constants c_i, $i = 1, 2$, with $1 - |z| \le c_1|\mathrm{Im}\, x| \le c_2(1 - |z|)$. It is by this type of arguments that we can relate in the following results variants of F-mildness to the existence of global strong boundary values.

At first we want to see that the existence of global strong boundary values implies that u is very close to being F-mild at $t = 0+$.

Theorem 4.3. *Assume that for each $g(z,t)$ which is real analytic in a neighborhood of the $Z \times \{0\}$, the function $t \to \int_Z u(z,t)g(z,t)\,dz = \psi_g(t)$ extends to be real-analytic in a neighborhood of $0 \in \mathbb{R}$. Then we can write u near $Z \times \{0\}$ in the form $u = b(h^+) + b(h^-)$ where h^+, h^-, are analytic functions on domains D^+, D^-, which have for a suitable continuous strictly positive function ρ the following form: $D^+ = \bigcup_{\delta > 0}[\{(z,t) \in$*

$\mathbb{C}^2$; $\operatorname{Im} x > 0, |x| < d, \exists\theta > 0, |\operatorname{Im} t| < \theta \operatorname{Im} x, \delta < \operatorname{Re} t < d\} \cup \{(x,t) \in \mathbb{C}^2; \operatorname{Im} x > 0, |x| < d, |t| < \rho(\operatorname{Im} x)\}]$ *and a similar expression for* D^-.

(Observe that the conclusion is very close to F-mildness: the only problem is, when speaking for example about the case of D^+, that the opening of the sector "$|\operatorname{Im} t| < \theta \operatorname{Im} x$" may shrink when $\operatorname{Re} t$ tends to $0+$).

In our final result we shall see that a condition a little bit stronger than mildness implies the existence of strong global boundary values.

Theorem 4.4. *Denote by* Z *the unit circle in* $\mathbb{C}$, *regarded as a real-analytic manifold, and assume that we are given a hyperfunction* u *on* $Z \times (0,1)$ *with the following property. For every* $z^0 \in Z$ *there are* h^+, h^-, *defined and analytic respectively on sets of form* D^+, D^-, *where, for some* $c > 0, \delta > 0$, *and for a suitable continuous strictly positive function* ρ,

$$D^+ = \{(z,t); 1-\delta < |z| < 1, |z - z^0| < c, |t| < c, -\operatorname{Re} t < \rho(1-|z|)\},$$
$$D^- = \{(z,t); 1 < |z| < 1+\delta, |z - z^0| < c, |t| < c, -\operatorname{Re} t < \rho(|z|-1)\}, \tag{4.2}$$

and with $u = b(h^+) + b(h^-)$ *near* $(z^0, 0)$ *in* $t > 0$.

Then the following conclusion holds:

- *for every fixed* $g \in \mathcal{A}(Z \times \{0\})$ *the map* $t \mapsto u(\cdot,t)(g) = \psi_g(t)$ *is well-defined for small* t *and extends to an analytic function in a neighborhood of* $0 \in \mathbb{R}$.

With respect to F-mildness, we have replaced the condition $|\operatorname{Im} t| < \rho(1-|z|)$, respectively $|\operatorname{Im} t| < \rho(|z|-1)$ (which together with $-\operatorname{Re} t < \rho(1-|z|)$, $-\operatorname{Re} t < \rho(|z|-1)$, are reformulations of the condition for F-mildness), in the domain of definition of the functions h^+, respectively h^-, by the "stronger" assumption that these functions exist on $|\operatorname{Im} t| < c$. We call the assumption "stronger" since the functions $h^\pm$ must exist on larger domains.

Proofs of the results in this note will appear elsewhere.

References

[1] Beals, R., A general calculus of pseudodifferential operators, *Duke Math. J.* **42** (1975), 1–42.

[2] Boutet de Monvel, L., Hypoelliptic operators with double characteristics and related pseudo-differential operators, *Comm. Pure Appl. Math.* **27** (1974), 585–639.

[3] Boutet de Monvel, L., Grigis, A. and Helffer, B., Parametrixes d'opérateurs pseudo-différentiels à caractéristiques multiples, *Journées Équations aux Dérivées Partielles de Rennes* (1975), *Astérisque* **34–35**, 1976, pp. 93–121.

[4] Boutet de Monvel, L. and Trèves, F., On a class of systems of pseudodifferential equations with double characteristics, *Comm. Pure Appl. Math.* **27** (1974), 59–89.

[5] Grushin, V. V., Hypoelliptic differential equations and pseudodifferential operators with operator-valued symbols, (Russian) *Mat. Sb. (N.S.)* **88** (130) (1972), 504–521.

[6] Hörmander, L., On the division of distributions by polynomials, *Ark. Mat.* **3** (1958), 555–568.

[7] ———, Pseudo-differential operators and hypoelliptic equations, *Singular Integrals, Proc. Sympos. Pure Math.* **10**, *Chicago, Ill.*, 1966, Amer. Math. Soc., Providence, R.I., 1967, pp. 138–183.

[8] ———, *The Analysis of Linear Partial Differential Operators*: I, III, *Pseudodifferential Operators*, Grundlehren Math. Wiss. **256** and **274**, Springer Verlag, 1983, 1985.

[9] Kataoka, K., Microlocal theory of boundary value problems I, II, *J. Fac. Sci. Univ. Tokyo, Sec. IA* **27** (1980), 355–399 and **28** (1981), 31–56.

[10] Komatsu, K., Boundary values for solutions of elliptic equations, *Proc. Int. Conf. Funct. An. Rel. Topics*, Univ. of Tokyo Press, Tokyo, 1970, pp. 107–121.

[11] Komatsu, H. and Kawai, T., Boundary values of hyperfunction solutions of linear partial differential equations, *Publ. Res. Inst. Math. Sci.* **7** (1971), 95–104.

[12] S. Lojasiewicz, S., Sur le problème de la division, *Studia Math.* **18** (1959), 87–136.

[13] Mughetti, M. and Nicola, F., Hypoellipticity for a class of operators with multiple characteristics, *J. Anal. Math.* **103** (2007), 377–396.

[14] Oaku, T., Microlocal boundary value problem for Fuchsian operators. I. F-mild microfunctions and uniqueness theorem, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **32** (1985), 287–317.

[15] Rosay, J. P. and Stout, L. E., *Strong Boundary Values, Analytic Functionals and Nonlinear Paley-Wiener Theory*, *Mem. Amer. Math. Soc.* **153**, 2001.

[16] ———, Strong boundary values: independence of the defining function and spaces of test functions, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **1** (2002), 13–31.

[17] Schapira, P., Hyperfonctions et problème aux limites elliptiques, *Bull. Soc. Math. France* **99** (1970), 113–141.

[18] Sjöstrand, J., Parametrices for pseudodifferential operators with multiple characteristics, *Ark. för Mat.* **12** (1974), 85–130.