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Phase space path integrals as analysis on path space

By

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Abstract

This survey is based on the talk at RIMS about of our papers [11], [12].

§ 1. Introduction

Let $T>0$ and $x\in \mathbf{R}^{d}$. Let $U(T,0)$ be the fundamental solution for the Schrödinger equation with the Planck parameter $0<\hbar<1$ such that

$$(1.1)\qquad \left(i\hbar\partial_{T}-H(T,x,\frac{\hbar}{i}\partial_{x})\right)U(T,0)=0\,,\quad U(0,0)=I\,.$$

By the Fourier transform with respect to $x_{0}\in \mathbf{R}^{d}$ and the inverse Fourier transform with respect to $\xi_{0}\in \mathbf{R}^{d}$, we can write

$$\begin{aligned}
Iv(x)\equiv v(x)&=\left(\frac{1}{2\pi\hbar}\right)^{d}\int_{\mathbf{R}^{2d}}e^{\frac{i}{\hbar}(x-x_{0})\cdot\xi_{0}}v(x_{0})dx_{0}d\xi_{0}\,,\\
\frac{\hbar}{i}\partial_{x}v(x)&=\left(\frac{1}{2\pi\hbar}\right)^{d}\int_{\mathbf{R}^{2d}}e^{\frac{i}{\hbar}(x-x_{0})\cdot\xi_{0}}\xi_{0}v(x_{0})dx_{0}d\xi_{0}\,,\\
H(T,x,\frac{\hbar}{i}\partial_{x})v(x)&=\left(\frac{1}{2\pi\hbar}\right)^{d}\int_{\mathbf{R}^{2d}}e^{\frac{i}{\hbar}(x-x_{0})\cdot\xi_{0}}H(T,x,\xi_{0})v(x_{0})dx_{0}d\xi_{0}\,.
\end{aligned}$$

When T is small, we consider the function $U(T,0,x,\xi_{0})$ satisfying

$$(1.2)\qquad U(T,0)v(x)\equiv\left(\frac{1}{2\pi\hbar}\right)^{d}\int_{\mathbf{R}^{2d}}e^{\frac{i}{\hbar}(x-x_{0})\cdot\xi_{0}}U(T,0,x,\xi_{0})v(x_{0})dx_{0}d\xi_{0}\,.$$

According to R. P. Feynman [5, Appendix B], we formally write

$$(1.3)\qquad e^{\frac{i}{\hbar}(x-x_{0})\cdot\xi_{0}}U(T,0,x,\xi_{0})=\int e^{\frac{i}{\hbar}\phi[q,p]}\mathcal{D}[q,p]\,.$$

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Here $q(T)$ is a position path with $q(T) = x$ and $q(0) = x_0$, and $p(t)$ is a momentum path with $p(0) = \xi_0$, $\phi[q,p]$ is the phase space action defined by

$$\phi[q,p] = \int_{[0,T)} p(t) \cdot dq(t) - \int_{[0,T)} H(t,q(t),p(t))dt\,. \tag{1.4}$$

and the phase space path integral $\int \sim \mathcal{D}[q,p]$ is a new sum over all the paths (q,p).

As mathematical treatments of the phase space path integrals, H. Kumano-go–H. Kitada [8], N. Kumano-go [10] and W. Ichinose [7] discussed (1.3) via Fourier integral operators. I. Daubechies-J. R. Klauder [4] formulated the phase space path integral via analytic continuation from measure. S. Albeverio-G. Guatteri-S. Mazzucchi [2], [1, §10.5.3], [13, §3.3] defined it via Fresnel integral transform. O. G. Smolyanov-A. G. Tokarev-A. Truman [15] treated it via Chernoff formula. However, in the sense of mathematics, the measure $\mathcal{D}[q,p]$ of the path integral (1.3) does not exist. Why can we say (1.3) is a kind of integral ? Even in the sense of physics, by the uncertain principle, we can not have the position $q(t)$ and the momentum $p(t)$ at the same time t. Why can we say these are phase space paths? Furthermore, as L. S. Schulman says in his book [14], 'in this method, formal tricks of great power can give just plain wrong answer.'

In [11], when T is small, using piecewise constant paths, we proved the existence of the phase space Feynman path integrals

$$\int e^{\frac{i}{\hbar}\phi[q,p]} F[q,p] \mathcal{D}[q,p]\,, \tag{1.5}$$

with general functional $F[q,p]$ as integrand. More precisely, we gave the two general classes $\mathcal{F}_{\mathcal{Q}}$, $\mathcal{F}_{\mathcal{P}}$ of functionals such that for any $F[q,p] \in \mathcal{F}_{\mathcal{Q}}$ or $\mathcal{F}_{\mathcal{P}}$, the time slicing approximation of (1.5) converges uniformly on compact subsets with respect to the endpoint x of position and the starting point ξ_0 of momentum. Furthermore, we proved some properties of the path integrals (1.5) similar to some properties of integrals.

Remark. (1) We treat (1.3) as one case with $F[q,p] \equiv 1$ of (1.5).

(2) Using polygonal paths of position and piecewise constant paths of momentum, W. Ichinose [7] discussed for the functionals $F[q,p] = \prod_{k=1}^{K} B_k(q(\tau_k), p(\tau_k))$, $0 < \tau_1 < \tau_2 < \cdots < \tau_K < T$ of cylinder type and showed that the time slicing approximation of (1.5) does not converge when $F[q,p] = q(t) \cdot p(t)$. We exclude the functionals of this type from our classes $\mathcal{F}_{\mathcal{Q}}$, $\mathcal{F}_{\mathcal{P}}$ to avoid the uncertain principle.

(3) Inspired by the forward and backward approach of J.-C. Zambrini [3, Part 2], we use left-continuous paths and right-continuous paths. Furthermore, inspired by L. S. Schulman [14, §31], we pay attention to the operations which are valid in the phase space path integrals.

§ 2. Phase space path integrals exist

Our assumption for the Hamiltonian function $H(t,x,\xi)$ of (1.1) is the following.

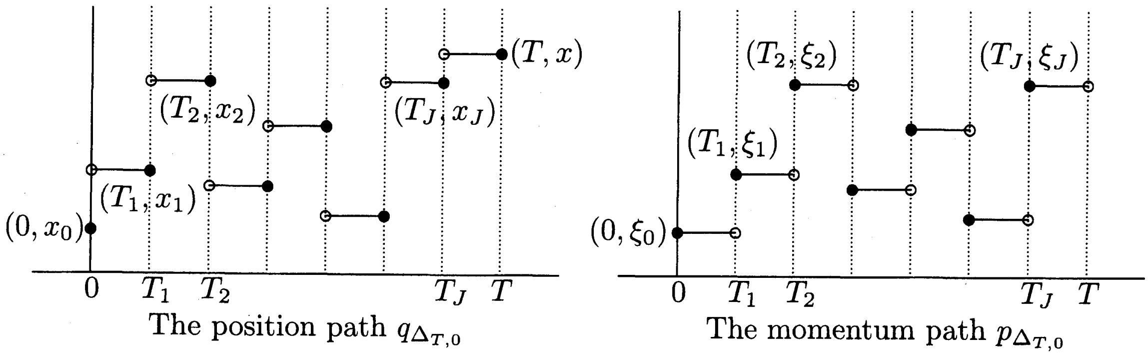


Figure 1.

Assumption 1 (Hamiltonian function). $H(t,x,\xi)$ is a real valued function of (t,x,ξ) in $\mathbf{R}\times\mathbf{R}^d\times\mathbf{R}^d$. For any multi-indices α, β, $\partial_x^\alpha\partial_\xi^\beta H(t,x,\xi)$ is continuous and there exists a positive constant $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha\partial_\xi^\beta H(t,x,\xi)| \le C_{\alpha,\beta}(1+|x|+|\xi|)^{\max(2-|\alpha+\beta|,0)}\,.$$

Example 1 (Hamiltonian operator).

$$\begin{aligned}H(t,x,\frac{\hbar}{i}\partial_x) = &\sum_{j,k=1}^{d}\left(a_{j,k}(t)\frac{\hbar}{i}\partial_{x_j}\frac{\hbar}{i}\partial_{x_k}+b_{j,k}(t)x_j\frac{\hbar}{i}\partial_{x_k}+c_{j,k}(t)x_jx_k\right)\\ &+\sum_{j=1}^{d}\left(a_j(t)\frac{\hbar}{i}\partial_{x_j}+b_j(t)x_j\right)+c(t,x)\,.\end{aligned}$$

Here $a_{j,k}(t)$, $b_{j,k}(t)$, $c_{j,k}(t)$, $a_j(t)$, $b_j(t)$ and $\partial_x^\alpha c(t,x)$ with any multi-index α are real-valued, continuous and bounded functions.

Let $\Delta_{T,0}=(T_{J+1},T_J,\ldots,T_1,T_0)$ be any division of the interval $[0,T]$ given by

$$\Delta_{T,0}: T=T_{J+1}>T_J>\cdots>T_1>T_0=0\,.$$

Set $x_{J+1}=x$. Let $x_j\in\mathbf{R}^d$ and $\xi_j\in\mathbf{R}^d$ for $j=1,2,\ldots,J$. We define the position path

$$q_{\Delta_{T,0}}=q_{\Delta_{T,0}}(t,x_{J+1},x_J,\ldots,x_1,x_0)$$

by $q_{\Delta_{T,0}}(0)=x_0$, $q_{\Delta_{T,0}}(t)=x_j$, $T_{j-1}<t\le T_j$ and the momentum path

$$p_{\Delta_{T,0}}=p_{\Delta_{T,0}}(t,\xi_J,\ldots,\xi_1,\xi_0)$$

by $p_{\Delta_{T,0}}(t)=\xi_{j-1}$, $T_{j-1}\le t<T_j$ for $j=1,2,\ldots,J,J+1$ (Figure 1).

Definition 1 (Two spaces $\mathcal{Q}$, $\mathcal{P}$ of piecewise constant paths).

(1) We write $q \in \mathcal{Q}$ if q is left-continuous and piecewise constant, i.e., $q = q_{\Delta_{T,0}}$.
(2) We write $p \in \mathcal{P}$ if p is right-continuous and piecewise constant, i.e., $p = p_{\Delta_{T,0}}$.

Definition 2.1 (Two classes $\mathcal{F}_{\mathcal{Q}}$, $\mathcal{F}_{\mathcal{P}}$ of functionals $F[q,p]$). Let $F[q,p]$ be a functional of $q \in \mathcal{Q}$ and $p \in \mathcal{P}$.
(1) We write $F[q,p] \in \mathcal{F}_{\mathcal{Q}}$ if $F[q,p]$ satisfies Assumption 3 (1).
(2) We write $F[q,p] \in \mathcal{F}_{\mathcal{P}}$ if $F[q,p]$ satisfies Assumption 3 (2).

Remark. For simplicity, we will state Assumption 3 (1)(2) in §13.

Then $\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$, $F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$ are the functions $\phi_{\Delta_{T,0}}$, $F_{\Delta_{T,0}}$ given by

$$\begin{aligned}
\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}] &= \sum_{j=1}^{J+1} \left(\int_{[T_{j-1},T_j)} p_{\Delta_{T,0}} \cdot dq_{\Delta_{T,0}}(t) - \int_{[T_{j-1},T_j)} H(t, q_{\Delta_{T,0}}, p_{\Delta_{T,0}}) dt \right) \\
&= \sum_{j=1}^{J+1} \left((x_j - x_{j-1}) \cdot \xi_{j-1} - \int_{[T_{j-1},T_j)} H(t, x_j, \xi_{j-1}) dt \right) \\
&\equiv \phi_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \dots, \xi_1, x_1, \xi_0, x_0), \\
F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}] &\equiv F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \dots, \xi_1, x_1, \xi_0, x_0).
\end{aligned}$$

Let $t_j = T_j - T_{j-1}$ and $|\Delta_{T,0}| = \max_{1 \le j \le J+1} t_j$.

Theorem 1 (Existence of phase space path integrals). *Let T be sufficiently small. Then, for any $F[q,p] \in \mathcal{F}_{\mathcal{Q}}$ or $\mathcal{F}_{\mathcal{P}}$,*

$$\begin{aligned}
(2.1) \quad & \int e^{\frac{i}{\hbar}\phi[q,p]} F[q,p] \mathcal{D}[q,p] \\
& \equiv \lim_{|\Delta_{T,0}| \to 0} \left(\frac{1}{2\pi\hbar} \right)^{dJ} \int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]} F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}] \prod_{j=1}^{J} dx_j d\xi_j,
\end{aligned}$$

converges uniformly on compact sets of $\mathbf{R}^{3d}$ with respect to (x, ξ_0, x_0), i.e., the phase space path integral (2.1) *is well-defined.*

Remark. Even when $F[q,p] \equiv 1$, each integral of the right hand side

$$(2.2) \quad \lim_{|\Delta_{T,0}| \to 0} \left(\frac{1}{2\pi\hbar} \right)^{dJ} \int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar} \sum_{j=1}^{J+1} ((x_j - x_{j-1}) \cdot \xi_{j-1} - \int_{T_{j-1}}^{T_j} H(t, x_j, \xi_{j-1}) dt)} \prod_{j=1}^{J} dx_j d\xi_j,$$

of (2.1) does not converge absolutely, i.e., $\int_{\mathbf{R}^{2d}} d\xi_j dx_j = \infty$. Furthermore, the number J of integrals (division points) tends to ∞, i.e., $\infty \times \infty \times \infty \times \infty \times \cdots\cdots$, $J \to \infty$. Therefore, we treat the multiple integral of (2.1) as an oscillatory integral (cf. H. Kumano-go [9, §1.6]) to use the forms $q_{\Delta_{T,0}}$, $p_{\Delta_{T,0}}$ of paths in the multiple integral.

Remark. If $d = 1$, $H(t,x,\xi) = x^2/2 + \xi^2/2$ and $F[q,p] \equiv 1$, we have

$$e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0) = \int e^{\frac{i}{\hbar}\phi[q,p]}\mathcal{D}[q,p]$$
$$= \frac{1}{(\cos T)^{1/2}} \exp\frac{i}{\hbar}\Big(- x_0\cdot\xi_0 + \frac{2x\cdot\xi_0 - (x^2+\xi_0^2)\sin T}{2\cos T}\Big).$$

As we will see in §12, if we use piecewise the bicharacteristic paths of [12] instead of the piecewise constant paths of [11], we calculate $U(T,0,x,\xi_0)$ directly.

§ 3. We can produce many functionals $F[q,p] \in \mathcal{F}_{\mathcal{Q}}$ or $\mathcal{F}_{\mathcal{P}}$

Typical examples of $F[q,p] \in \mathcal{F}_{\mathcal{Q}}$ or $\mathcal{F}_{\mathcal{P}}$ are the following.

Example 2 ($F[q,p] \in \mathcal{F}_{\mathcal{Q}}$ or $\mathcal{F}_{\mathcal{P}}$)**.** Let m be a non-negative integer.

(a) Assume that for any multi-index α, $\partial_x^\alpha B(t,x)$ is continuous in $\mathbf{R}\times\mathbf{R}^d$ and there exists a positive constant C_α such that $|\partial_x^\alpha B(t,x)| \le C_\alpha(1+|x|)^m$. Then the values at the fixed time t, $0 \le t \le T$,

$$F[q] = B(t,q(t)) \in \mathcal{F}_{\mathcal{Q}}\,, \quad F[p] = B(t,p(t)) \in \mathcal{F}_{\mathcal{P}}\,.$$

In particular, $F[q,p] \equiv 1 \in \mathcal{F}_{\mathcal{Q}} \cap \mathcal{F}_{\mathcal{P}}$.

(b) Let $0 \le T' \le T'' \le T$. Assume that for any multi-indices α, β, $\partial_x^\alpha\partial_\xi^\beta B(t,x,\xi)$ is continuous in $\mathbf{R}\times\mathbf{R}^d\times\mathbf{R}^d$ and there exists a positive constant $C_{\alpha,\beta}$ such that $|\partial_x^\alpha\partial_\xi^\beta B(t,x,\xi)| \le C_{\alpha,\beta}(1+|x|+|\xi|)^m$. Then the integral

$$F[q,p] = \int_{[T',T'')} B(t,q(t),p(t))dt \in \mathcal{F}_{\mathcal{Q}} \cap \mathcal{F}_{\mathcal{P}}\,.$$

(c) Assume that for any multi-indices α, β, $\partial_x^\alpha\partial_\xi^\beta B(t,x,\xi)$ is continuous on $\mathbf{R}\times\mathbf{R}^d\times\mathbf{R}^d$ and there exists a positive constant $C_{\alpha,\beta}$ such that $|\partial_x^\alpha\partial_\xi^\beta B(t,x,\xi)| \le C_{\alpha,\beta}$. Then

$$F[q,p] = e^{\int_{[T',T'')} B(t,q(t),p(t))dt} \in \mathcal{F}_{\mathcal{Q}} \cap \mathcal{F}_{\mathcal{P}}\,.$$

Remark. To avoid the uncertain principle, we do not treat the position $q(t)$ and the momentum $p(t)$ at the same time t, i.e., $q(t) \in \mathcal{F}_{\mathcal{Q}}$, $p(t) \notin \mathcal{F}_{\mathcal{Q}}$ and $q(t) \notin \mathcal{F}_{\mathcal{P}}$, $p(t) \in \mathcal{F}_{\mathcal{P}}$.

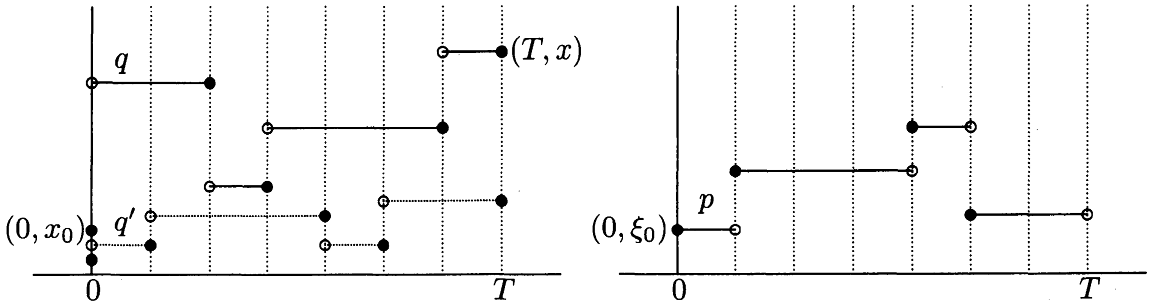
To state the algebra on the classes $\mathcal{F}_{\mathcal{Q}}$, $\mathcal{F}_{\mathcal{P}}$, we explain the functional derivatives.

Definition 2 (Fuctional derivatives)**.** For any division $\Delta_{T,0}$, we assume that

$$F[q_{\Delta_{T,0}},p_{\Delta_{T,0}}] = F_{\Delta_{T,0}}(x_{J+1},\xi_J,x_J,\ldots,\xi_0,x_0) \in C^\infty(\mathbf{R}^{d(2J+3)})\,.$$

For any q, $q' \in \mathcal{Q}$ and any p, $p' \in \mathcal{P}$, we define the functional derivatives $D_{q'}F[q,p]$ and $D_{p'}F[q,p]$ by

$$D_{q'}F[q,p] = \frac{\partial}{\partial\theta}F[q+\theta q',p]\Big|_{\theta=0}\,, \quad D_{p'}F[q,p] = \frac{\partial}{\partial\theta}F[q,p+\theta p']\Big|_{\theta=0}\,.$$



The position paths q and q' The momentum path p

Figure 2.

Remark. For any q, $q' \in \mathcal{Q}$ and $p \in \mathcal{P}$, choose $\Delta_{T,0}$ which contains all times when q, q' or p jumps (Figure 2). Set $q(T_j) = x_j$, $q'(T_j) = x'_j$ and $p(T_{j-1}) = \xi_{j-1}$. Since $(q+\theta q')(0) = x_0+\theta x'_0$, $(q+\theta q')(t) = x_j+\theta x'_j$ on $(T_{j-1}, T_j]$ and $p(t) = \xi_{j-1}$ on $[T_{j-1}, T_j)$, we have

$$F[q+\theta q', p] = F_{\Delta_{T,0}}(x_{J+1}+\theta x'_{J+1}, \xi_J, x_J+\theta x'_J, \dots, \xi_0, x_0+\theta x'_0)\,.$$

Hence we can treat $D_{q'}F[q,p]$ as a finite sum of functions, i.e.,

$$D_{q'}F[q,p] = \frac{\partial}{\partial\theta}F[q+\theta q', p]\bigg|_{\theta=0} = \sum_{j=0}^{J+1}(\partial_{x_j}F_{\Delta_{T,0}})(x_{J+1}, \xi_J, \dots, \xi_0, x_0)\cdot x'_j\,.$$

Theorem 2 (Smooth algebra on $\mathcal{F}_{\mathcal{Q}}$, $\mathcal{F}_{\mathcal{P}}$).

(1) *For any $F[q,p]$, $G[q,p] \in \mathcal{F}_{\mathcal{Q}}$, any $q' \in \mathcal{Q}$, any $p' \in \mathcal{P}$ and any real $d\times d$ matrices A, B, we have*

$$F[q,p]+G[q,p] \in \mathcal{F}_{\mathcal{Q}}\,,\ F[q,p]G[q,p] \in \mathcal{F}_{\mathcal{Q}}\,,\ F[q+q', p+p'] \in \mathcal{F}_{\mathcal{Q}}\,,$$
$$F[Aq, Bp] \in \mathcal{F}_{\mathcal{Q}}\,,\ D_{q'}F[q,p] \in \mathcal{F}_{\mathcal{Q}}\,,\ D_{p'}F[q,p] \in \mathcal{F}_{\mathcal{Q}}\,.$$

(2) *For any $F[q,p]$, $G[q,p] \in \mathcal{F}_{\mathcal{P}}$, any $q' \in \mathcal{Q}$, any $p' \in \mathcal{P}$ and any real $d\times d$ matrices A, B, we have*

$$F[q,p]+G[q,p] \in \mathcal{F}_{\mathcal{P}}\,,\ F[q,p]G[q,p] \in \mathcal{F}_{\mathcal{P}}\,,\ F[q+q', p+p'] \in \mathcal{F}_{\mathcal{P}}\,,$$
$$F[Aq, Bp] \in \mathcal{F}_{\mathcal{P}}\,,\ D_{q'}F[q,p] \in \mathcal{F}_{\mathcal{P}}\,,\ D_{p'}F[q,p] \in \mathcal{F}_{\mathcal{P}}\,.$$

Remark. The two classes $\mathcal{F}_{\mathcal{Q}}$, $\mathcal{F}_{\mathcal{P}}$ are closed under addition, multiplication, translation, real linear transformation and functional differentiation. Therefore, if we apply Theorem 2 to Example 2, we can produce many functionals $F[q,p] \in \mathcal{F}_{\mathcal{Q}}$ or $\mathcal{F}_{\mathcal{P}}$ which are phase space path integrable.

§ 4. However, we must note which operations are valid

As we will see in Theorems 3 and 5, because $q' \in \mathcal{Q}$, $p' \in \mathcal{P}$ are piecewise constant, the part $\int_{[0,T)} p(t) \cdot dq(t)$ of $\phi[q,p]$ does not always have good properties under the operations in Theorem 2. Therefore, we must pay attention to which operations are valid in the phase space path integrals $\int e^{\frac{i}{\hbar}\phi[q,p]} F[q,p] \mathcal{D}[q,p]$.

§ 5. Translation

Theorem 3 (Translation).

(1) *For any* $p' \in \mathcal{P}$, *we have* $e^{\frac{i}{\hbar}(\phi[q,p+p']-\phi[q,p])} \in \mathcal{F}_{\mathcal{Q}}$.
Furthermore, let T *be sufficiently small. Then for any* $F[q,p] \in \mathcal{F}_{\mathcal{Q}}$, *we have*

$$\int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p+p']} F[q,p+p'] \mathcal{D}[q,p] = \int_{q(T)=x,p(0)=\xi_0+p'(0),q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p]} F[q,p] \mathcal{D}[q,p] .$$

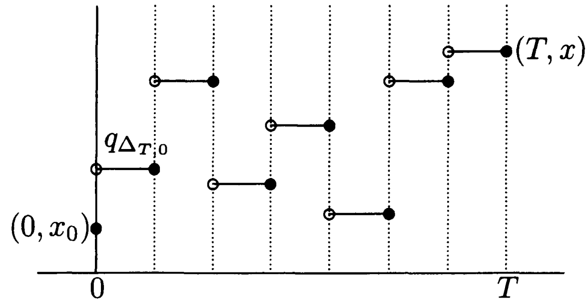
(2) *For any* $q' \in \mathcal{Q}$, *we have* $e^{\frac{i}{\hbar}(\phi[q+q',p]-\phi[q,p])} \in \mathcal{F}_{\mathcal{P}}$.
Furthermore, let T *be sufficiently small. Then for any* $F[q,p] \in \mathcal{F}_{\mathcal{P}}$, *we have*

$$\int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{\frac{i}{\hbar}\phi[q+q',p]} F[q+q',p] \mathcal{D}[q,p] = \int_{q(T)=x+q'(T),p(0)=\xi_0,q(0)=x_0+q'(0)} e^{\frac{i}{\hbar}\phi[q,p]} F[q,p] \mathcal{D}[q,p] .$$

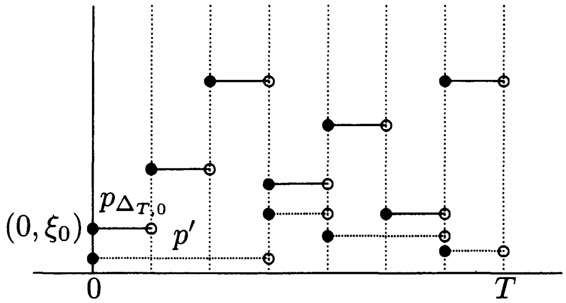
Proof of Theorem 3 (1). For simplicity, we omit the proof of $e^{\frac{i}{\hbar}(\phi[q,p+p']-\phi[q,p])} \in \mathcal{F}_{\mathcal{Q}}$. By Theorem 1 and 2 (1), we have

$$
\begin{aligned}
(5.1)\quad & \int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p+p']} F[q,p+p'] \mathcal{D}[q,p] \\
&= \int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p]} e^{\frac{i}{\hbar}(\phi[q,p+p']-\phi[q,p])} F[q,p+p'] \mathcal{D}[q,p] \\
&\equiv \lim_{|\Delta_{T,0}|\to 0} \left(\frac{1}{2\pi\hbar}\right)^{dJ} \int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}+p']} F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}+p'] \prod_{j=1}^{J} d\xi_j dx_j ,
\end{aligned}
$$

with $q_{\Delta_{T,0}}(T_j) = x_j$ and $p_{\Delta_{T,0}}(T_j) = \xi_j$. Choose $\Delta_{T,0}$ which contains all times when the path p' jumps (Figure 3). Set $p'(t) = \xi'_{j-1}$ on $[T_{j-1}, T_j)$. Since



The position path $q_{\Delta_{T,0}}$



The momentum paths $p_{\Delta_{T,0}}$ and p'

Figure 3.

$(p_{\Delta_{T,0}}+p')(t)=\xi_{j-1}+\xi'_{j-1}$ on $[T_{j-1},T_j)$, we can write

$$= \lim_{|\Delta_{T,0}|\to 0}\left(\frac{1}{2\pi\hbar}\right)^{dJ}\int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}(x_{J+1},\xi_J+\xi'_J,x_J,\ldots,\xi_1+\xi'_1,x_1,\xi_0+\xi'_0,x_0)}$$

$$\times F_{\Delta_{T,0}}(x_{J+1},\xi_J+\xi'_J,x_J,\ldots,\xi_1+\xi'_1,x_1,\xi_0+\xi'_0,x_0)\prod_{j=1}^{J} d\xi_j dx_j\,,$$

By the change of variables: $\xi_j+\xi'_j\to\xi_j$, $j=1,2,\ldots,J$, we have

$$= \lim_{|\Delta_{T,0}|\to 0}\left(\frac{1}{2\pi\hbar}\right)^{dJ}\int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}(x_{J+1},\xi_J,x_J,\ldots,\xi_1,x_1,\xi_0+\xi'_0,x_0)}$$

$$\times F_{\Delta_{T,0}}(x_{J+1},\xi_J,x_J,\ldots,\xi_1,x_1,\xi_0+\xi'_0,x_0)\prod_{j=1}^{J} d\xi_j dx_j$$

$$= \int_{q(T)=x,p(0)=\xi_0+p'(0),q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p]}F[q,p]\mathcal{D}[q,p]\,.\ \square$$

Remark. By $e^{\frac{i}{\hbar}(\phi[q,p+p']-\phi[q,p])}\in\mathcal{F}_Q$, Theorem 1 guarantees the existence of the phase space path integral of (5.1), i.e., the definition "$\equiv$" of (5.1) for any $\Delta_{T,0}$ with $|\Delta_{T,0}|\to 0$. Note that we do not treat the case with $e^{\frac{i}{\hbar}(\phi[q+q',p+p']-\phi[q,p])}$.

§ 6. Orthogonal transformation

Theorem 4 (Orthogonal transformation). *Let T be sufficiently small. Then for any $F[q,p]\in\mathcal{F}_Q$ or $\mathcal{F}_P$ and any $d\times d$ orthogonal matrix Q,*

$$\int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{\frac{i}{\hbar}\phi[Qq,Qp]}F[Qq,Qp]\mathcal{D}[q,p]$$

$$= \int_{q(T)=Qx,p(0)=Q\xi_0,q(0)=Qx_0} e^{\frac{i}{\hbar}\phi[q,p]}F[q,p]\mathcal{D}[q,p]\,.$$

§ 7. Integration by parts with respect to functional differentiation

Theorem 5 (Integration by parts).
(1) *For any $p' \in \mathcal{P}$, we have $D_{p'}\phi[q,p] \in \mathcal{F}_{\mathcal{Q}}$. Furthermore, let T be sufficiently small. Then for any $F[q,p] \in \mathcal{F}_{\mathcal{Q}}$ and any $p' \in \mathcal{P}$ with $p'(0) = 0$,*

$$\int e^{\frac{i}{\hbar}\phi[q,p]}(D_{p'}F)[q,p]\mathcal{D}[q,p] = -\frac{i}{\hbar}\int e^{\frac{i}{\hbar}\phi[q,p]}(D_{p'}\phi)[q,p]F[q,p]\mathcal{D}[q,p]\,.$$

(2) *For any $q' \in \mathcal{Q}$, we have $D_{q'}\phi[q,p] \in \mathcal{F}_{\mathcal{P}}$. Furthermore, let T be sufficiently small. Then for any $F[q,p] \in \mathcal{F}_{\mathcal{P}}$ and any $q' \in \mathcal{Q}$ with $q'(T) = q'(0) = 0$,*

$$\int e^{\frac{i}{\hbar}\phi[q,p]}(D_{q'}F)[q,p]\mathcal{D}[q,p] = -\frac{i}{\hbar}\int e^{\frac{i}{\hbar}\phi[q,p]}(D_{q'}\phi)[q,p]F[q,p]\mathcal{D}[q,p]\,.$$

Remark (Analogues of canonical equations). Set $F[q,p] \equiv 1$. Note that

$$\phi[q,p] = \int_{[0,T)} p(t)\cdot dq(t) - \int_{[0,T)} H(t,q(t),p(t))dt\,.$$

Then we can rewrite Theorem 5 as follows:
(1) For any $p' \in \mathcal{P}$ with $p'(0) = 0$, we have

$$0 = \int e^{\frac{i}{\hbar}\phi[q,p]}\left(\int_{[0,T)} p'dq - (\partial_\xi H)(t,q,p)p'dt\right)\mathcal{D}[q,p]\,.$$

(2) For any $q' \in \mathcal{Q}$ with $q'(T) = q'(0) = 0$, we have

$$0 = \int e^{\frac{i}{\hbar}\phi[q,p]}\left(\int_{[0,T)} pdq' - (\partial_x H)(t,q,p)q'dt\right)\mathcal{D}[q,p]\,.$$

Note that the inner parts of the phase space path integrals are similar to the canonical equations: $\partial_t q(t) = (\partial_\xi H)(t,q,p)$, $\partial_t p(t) = -(\partial_x H)(t,q,p)$.

§ 8. Theorem of Fubini's type

Because the measure of (2.1) does not exist, we state a theorem of Fubini-type.

Theorem 6 (Fubini-type). *Let m be a non-negative integer. Assume that for any multi-index α, $\partial_x^\alpha B(t,x)$ is continuous in $\mathbf{R}\times\mathbf{R}^d$ and there exists a positive constant C_α such that $|\partial_x^\alpha B(t,x)| \le C_\alpha(1+|x|)^m$. Furthermore let T be sufficiently small. Let $0 \le T' \le T'' \le T$. Then we have the following:*

(1) *For any* $F[q,p]\in\mathcal{F}_Q$ *including* $F[q,p]\equiv 1$, *we have*

$$\int e^{\frac{i}{\hbar}\phi[q,p]}\int_{[T',T'')}B(t,q(t))dtF[q,p]\mathcal{D}[q,p] = \int_{[T',T'')}\int e^{\frac{i}{\hbar}\phi[q,p]}B(t,q(t))F[q,p]\mathcal{D}[q,p]dt\,.$$

(2) *For any* $F[q,p]\in\mathcal{F}_{\mathcal{P}}$ *including* $F[q,p]\equiv 1$, *we have*

$$\int e^{\frac{i}{\hbar}\phi[q,p]}\int_{[T',T'')}B(t,p(t))dtF[q,p]\mathcal{D}[q,p] = \int_{[T',T'')}\int e^{\frac{i}{\hbar}\phi[q,p]}B(t,p(t))F[q,p]\mathcal{D}[q,p]dt\,.$$

Remark. To avoid the uncertain principle, we do not treat the position $q(t)$ and the momentum $p(t)$ at the same time t.

Remark. If $|\partial_x^\alpha B(t,x)|\le C_\alpha$, we have the perturbation expansion:

$$\begin{aligned}\int & e^{\frac{i}{\hbar}\phi[q,p]+\frac{i}{\hbar}\int_{[0,T)}B(\tau,q(\tau))d\tau}\mathcal{D}[q,p]\\ &=\sum_{n=0}^{\infty}\left(\frac{i}{\hbar}\right)^n\int_{[0,T)}d\tau_n\int_{[0,\tau_n)}d\tau_{n-1}\cdots\int_{[0,\tau_2)}d\tau_1\\ &\quad\times\int e^{\frac{i}{\hbar}\phi[q,p]}B(\tau_n,q(\tau_n))B(\tau_{n-1},q(\tau_{n-1}))\cdots B(\tau_1,q(\tau_1))\mathcal{D}[q,p]\,.\end{aligned}$$

§ 9. Semiclassical approximation of Hamiltonian type as $\hbar\downarrow 0$

Let T be sufficiently small. Let $\bar q(t)=\bar q(t,x,\xi_0)$ and $\bar p(t)=\bar p(t,x,\xi_0)$ be the solution of the canonical equations

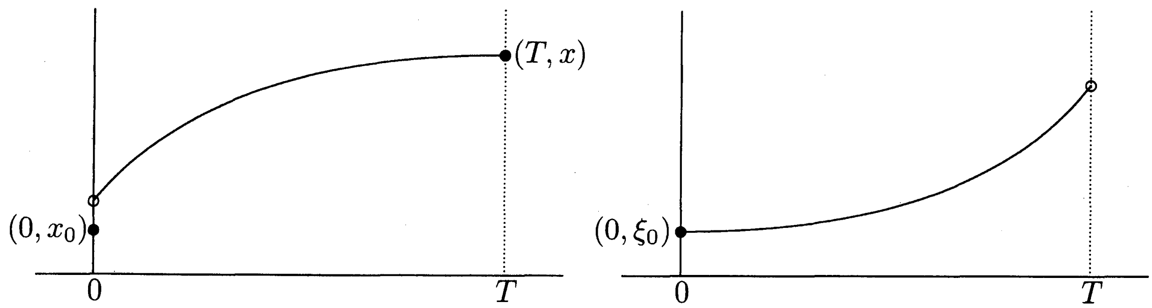
$$\partial_t\bar q(t)=(\partial_\xi H)(t,\bar q(t),\bar p(t))\,,\quad \partial_t\bar p(t)=-(\partial_x H)(t,\bar q(t),\bar p(t))\,,\quad 0\le t\le T\,,$$

with the boundary conditions $\bar q(T)=x$ and $\bar p(0)=\xi_0$. We define the bicharacteristic paths $q^\flat=q^\flat(t,x,\xi_0,x_0)$ and $p^\flat=p^\flat(t,x,\xi_0)$ by

$$\begin{aligned} q^\flat(0)=x_0,\quad q^\flat(t)&=\bar q(t,x,\xi_0)\,,\quad 0<t\le T\,,\\ p^\flat(t)&=\bar p(t,x,\xi_0)\,,\quad 0\le t<T\end{aligned}$$

(Figure 4). Let $(x_J^*,\xi_J^*,\dots,x_1^*,\xi_1^*)$ be the stationary point of $\phi_{\Delta_{T,0}}$ given by

$$(\partial_{(\xi_J,x_J,\dots,\xi_1,x_1)}\phi_{\Delta_{T,0}})(x_{J+1},\xi_J^*,x_J^*,\dots,\xi_1^*,x_1^*,\xi_0)=0\,.$$



The bicharacteristic path $q^\flat$ The bicharacteristic path $p^\flat$

Figure 4.

Set $x = x_{J+1}$. We define $D(T, x, \xi_0)$ by

$$D(T,x,\xi_0) = \lim_{|\Delta_{T,0}|\to 0} (-1)^{dJ} \det(\partial^2_{(\xi_J, x_J, \dots, \xi_1, x_1)} \phi_{\Delta_{T,0}})(x_{J+1}, x_J^*, \xi_J^*, \dots, x_1^*, \xi_1^*, \xi_0)\,.$$

Theorem 7 (Semiclassical approximation of Hamiltonian type as $\hbar \downarrow 0$). *Let T be sufficiently small. Then, for any $F[q,p] \in \mathcal{F}_{\mathcal{Q}}$ or $\mathcal{F}_{\mathcal{P}}$, we have*

$$\int e^{\frac{i}{\hbar}\phi[q,p]} F[q,p] \mathcal{D}[q,p] = e^{\frac{i}{\hbar}\phi[q^\flat, p^\flat]} \Big(D(T,x,\xi_0)^{-1/2} F[q^\flat, p^\flat] + \hbar \Upsilon(\hbar, T, x, \xi_0, x_0) \Big)\,.$$

Here for any multi-indices α, β, there exists a positive constant $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha \partial_{\xi_0}^\beta \Upsilon(\hbar, T, x, \xi_0, x_0)| \le C_{\alpha,\beta}(1 + |x| + |\xi_0| + |x_0|)^m\,.$$

§ 10. Proof for Theorems 1 and 2

In order to prove the convergence of the multiple integral

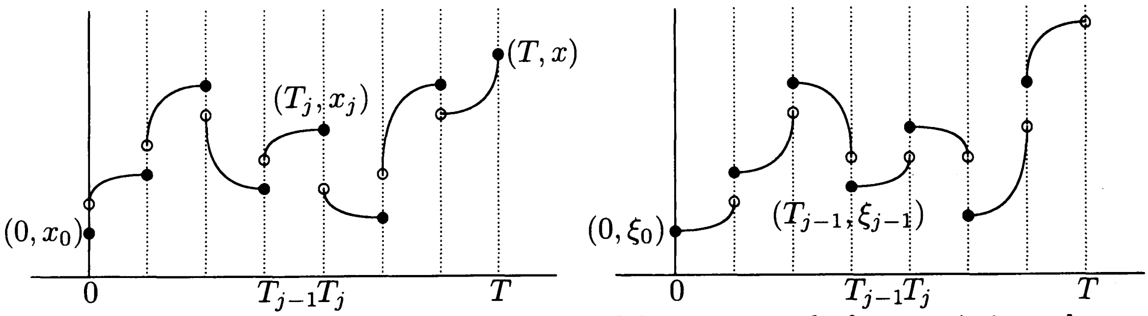
$$\left(\frac{1}{2\pi\hbar}\right)^{dJ} \int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]} F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}] \prod_{j=1}^{J} d\xi_j dx_j\,, \tag{10.1}$$

as $|\Delta_{T,0}| \to 0$, we have only to add many assumptions to the function

$$F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \dots, x_1, \xi_0, x_0) = F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]\,.$$

and define $\mathcal{F}_{\mathcal{Q}}$, $\mathcal{F}_{\mathcal{P}}$ by them. Do not consider other things. Then $\mathcal{F}_{\mathcal{Q}}$, $\mathcal{F}_{\mathcal{P}}$ will be larger as a set. If lucky, $\mathcal{F}_{\mathcal{Q}}$, $\mathcal{F}_{\mathcal{P}}$ will contain at least one example $F[q,p] \equiv 1$.

Our proof consists of 3 steps: As the first step, by an estimate of H. Kumano-go-Taniguchi's type [9, p.360, (6.94)], we control the multiple integral (10.1) by C^J with a positive constant C as $J \to \infty$. As the second step, by a stationary phase method of Fujiwara's type [6], we control the multiple integral (10.1) by C with a positive constant



The piecewise bicharacteristic path $q_{\Delta_{T,0}}$ The piecewise bicharacteristic path $p_{\Delta_{T,0}}$

Figure 5.

C independent of $J \to \infty$. As the last step, we add assumptions so that the multiple integral (10.1) converges as $|\Delta_{T,0}| \to 0$.

For the properties of the phase space path integrals, we have only to prove the properties which we can prove.

§ 11. Assumption via piecewise bicharacteristic paths

The piecewise constant paths are rougher as an approximation. In order to make the calculation for the convergence more easily, we use the piecewise bicharacteristic paths instead of the piecewise constant paths.

Let $|\Delta_{T,0}|$ be small. We define the bicharacteristic paths $\bar{q}_{T_j,T_{j-1}} = \bar{q}_{T_j,T_{j-1}}(t,x_j,\xi_{j-1})$ and $\bar{p}_{T_j,T_{j-1}} = \bar{p}_{T_j,T_{j-1}}(t,x_j,\xi_{j-1})$, $T_{j-1} \le t \le T_j$ by the canonical equation

$$
\begin{aligned}
&\partial_t \bar{q}_{T_j,T_{j-1}}(t) = (\partial_\xi H)(t,\bar{q}_{T_j,T_{j-1}},\bar{p}_{T_j,T_{j-1}}),\\
&\partial_t \bar{p}_{T_j,T_{j-1}}(t) = -(\partial_x H)(t,\bar{q}_{T_j,T_{j-1}},\bar{p}_{T_j,T_{j-1}}),\quad T_{j-1} \le t \le T_j\,,
\end{aligned}
\tag{11.1}
$$

with $\bar{q}_{T_j,T_{j-1}}(T_j) = x_j$ and $\bar{p}_{T_j,T_{j-1}}(T_{j-1}) = \xi_{j-1}$. Using $\bar{q}_{T_j,T_{j-1}}$ and $\bar{p}_{T_j,T_{j-1}}$, we define the piecewise bicharacteristic paths $q_{\Delta_{T,0}} = q_{\Delta_{T,0}}(t,x_{J+1},\xi_J,x_J,\dots,\xi_1,x_1,\xi_0,x_0)$ and $p_{\Delta_{T,0}} = p_{\Delta_{T,0}}(t,x_{J+1},\xi_J,x_J,\dots,\xi_1,x_1,\xi_0)$ by

$$
\begin{aligned}
&q_{\Delta_{T,0}}(t) = \bar{q}_{T_j,T_{j-1}}(t,x_j,\xi_{j-1}),\quad T_{j-1} < t \le T_j,\quad q_{\Delta_{T,0}}(0) = x_0,\\
&p_{\Delta_{T,0}}(t) = \bar{p}_{T_j,T_{j-1}}(t,x_j,\xi_{j-1}),\quad T_{j-1} \le t < T_j
\end{aligned}
\tag{11.2}
$$

for $j = 1,2,\dots,J,J+1$ (Figure 5). Then the assumption via piecewise bicharacteristic paths corresponding to Assumption 3 (1) is the following:

Assumption 2 (via piecewise bicharateristic paths). Let $m \ge 0$. Let $u_j \ge 0$, $j = 1,2,\dots,J,J+1$ are non-negative parameters depending on the division $\Delta_{T,0}$ such that $\sum_{j=1}^{J+1} u_j = U < \infty$. For any integer $M \ge 0$, there exist positive constants A_M, X_M

such that

$$
\begin{aligned}
(11.3) \quad & |(\prod_{j=1}^{J+1} \partial_{x_j}^{\alpha_j} \partial_{\xi_{j-1}}^{\beta_{j-1}}) F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \dots, \xi_1, x_1, \xi_0, x_0)| \\
& \le A_M (X_M)^{J+1} (\prod_{j=1}^{J+1} (t_j)^{\min(|\beta_{j-1}|,1)}) (1 + \sum_{j=1}^{J+1} (|x_j| + |\xi_{j-1}|) + |x_0|)^m , \\
(11.4) \quad & |(\prod_{j=1}^{J+1} \partial_{x_j}^{\alpha_j} \partial_{\xi_{j-1}}^{\beta_{j-1}}) \partial_{x_k} F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \dots, \xi_1, x_1, \xi_0, x_0)| \\
& \le A_M (X_M)^{J+1} u_k (\prod_{j \ne k} (t_j)^{\min(|\beta_{j-1}|,1)}) (1 + \sum_{j=1}^{J+1} (|x_j| + |\xi_{j-1}|) + |x_0|)^m ,
\end{aligned}
$$

for any $\Delta_{T,0}$, any multi-indices α_j, β_{j-1} with $|\alpha_j|$, $|\beta_{j-1}| \le M$, $j = 1, 2, \dots, J, J+1$ and any $1 \le k \le J$.

Remark. We explain the mechanism of the convergence roughly. As the first step, we assume

$$
\begin{aligned}
(11.5) \quad & |(\prod_{j=1}^{J+1} \partial_{x_j}^{\alpha_j} \partial_{\xi_{j-1}}^{\beta_{j-1}}) F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \dots, \xi_1, x_1, \xi_0, x_0)| \\
& \le A_M (X_M)^{J+1} (1 + \sum_{j=1}^{J+1} (|x_j| + |\xi_{j-1}|) + |x_0|)^m ,
\end{aligned}
$$

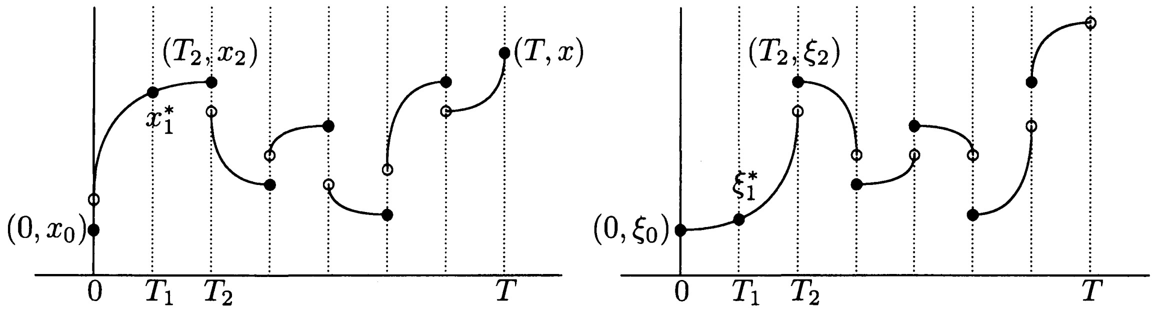
to control (10.1) by C^J with a positive constant C as $J \to \infty$. As the second step, we assume (11.3) to control (10.1) by C with a positive constant C independent of $J \to \infty$. As the last step, we add (11.4) so that (10.1) converges as $|\Delta_{T,0}| \to 0$. Roughly speaking, (11.4) implies that if the difference of two paths is small, then the difference of two heights is small.

§ 12. Calculation examples via piecewise bicharacteristic paths

If we use the piecewise bicharacteristic paths, then we can calculate the functions $U(T, 0, x, \xi)$ of the fundamental solutions $U(T, 0)$ for some equations directly.

Example 12.1. We calculate $U(T, 0, x, \xi)$ when $d = 1$, $H(t, x, \xi) = x^2/2 + \xi^2/2$ and $F[q, p] \equiv 1$. Note $(\partial_\xi H) = \xi$ and $(\partial_x H) = x$. By the canonical equation

$$
\partial_t \bar{q}_{T_j, T_{j-1}}(t) = \bar{p}_{T_j, T_{j-1}}(t) , \quad \partial_t \bar{p}_{T_j, T_{j-1}}(t) = -\bar{q}_{T_j, T_{j-1}}(t) , \quad T_{j-1} \le t \le T_j
$$



The path $q_{(\Delta_{T,T_2},0)}$ and x_1^* The path $p_{(\Delta_{T,T_2},0)}$ and ξ_1^*

Figure 6.

with $\bar{q}_{T_j,T_{j-1}}(T_j) = x_j$ and $\bar{p}_{T_j,T_{j-1}}(T_{j-1}) = \xi_{j-1}$, we have the bicharacteristic paths

$$\bar{q}_{T_j,T_{j-1}}(t) = \frac{x_j \cos(t - T_{j-1}) - \xi_{j-1} \sin(T_j - t)}{\cos(T_j - T_{j-1})},$$
$$\bar{p}_{T_j,T_{j-1}}(t) = \frac{-x_j \sin(t - T_{j-1}) + \xi_{j-1} \cos(T_j - t)}{\cos(T_j - T_{j-1})}.$$

Let $q_{\Delta_{T,0}}$, $p_{\Delta_{T,0}}$ be the piecewise bicharacteristic paths of (11.2) (Figure 5). Then the functional $\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$ becomes the function

$$\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}] = \phi_{\Delta_{T,0}} = \sum_{j=1}^{J+1} \phi_{T_j,T_{j-1}}(x_j, \xi_{j-1}, x_{j-1}),$$

where

$$\phi_{T_j,T_{j-1}}(x_j, \xi_{j-1}, x_{j-1}) = -x_{j-1} \cdot \xi_{j-1} + \frac{2x_j \cdot \xi_{j-1} - (x_j^2 + \xi_{j-1}^2) \sin(T_j - T_{j-1})}{2\cos(T_j - T_{j-1})}.$$

Let (ξ_1^*, x_1^*) be the solution of $\partial_{(\xi_1,x_1)}(\phi_{T_2,T_1} + \phi_{T_1,0})(x_2, \xi_1^*, x_1^*, \xi_0) = 0$ (Figure 6).

Then we have

$$\begin{aligned}
&\phi_{T_2,T_1}(x_2, \xi_1, x_1) + \phi_{T_1,0}(x_1, \xi_0, x_0) \\
&= \phi_{T_2,0}(x_2, \xi_0, x_0) + \frac{1}{2}\partial^2_{(\xi_1,x_1)}(\phi_{T_2,T_1} + \phi_{T_1,0}) \begin{bmatrix} \xi_1 - \xi_1^* \\ x_1 - x_1^* \end{bmatrix} \cdot \begin{bmatrix} \xi_1 - \xi_1^* \\ x_1 - x_1^* \end{bmatrix},
\end{aligned}$$

Note that

$$(-1)\det \partial^2_{(\xi_1,x_1)}(\phi_{T_2,T_1} + \phi_{T_1,0}) = (-1)\begin{vmatrix} -\frac{\sin(T_2-T_1)}{\cos(T_2-T_1)} & -1 \\ -1 & -\frac{\sin(T_1-0)}{\cos(T_1-0)} \end{vmatrix} = \frac{\cos T_2}{\cos t_2 \cos t_1}.$$

Using the formula

$$\int_{\mathbf{R}^2} e^{\frac{i}{\hbar}\frac{1}{2}Ax\cdot x}dx = \sqrt{\frac{(2\pi\hbar i)^2}{\det A}} = \frac{2\pi\hbar}{\sqrt{(-1)\det A}},$$

for any 2×2 real symmetric matrix A, we have

$$\left(\frac{1}{2\pi\hbar}\right)\int_{\mathbf{R}^2} e^{\frac{i}{\hbar}\phi_{T_2,T_1}(x_2,\xi_1,x_1)+\frac{i}{\hbar}\phi_{T_1,0}(x_1,\xi_0,x_0)}dx_1 d\xi_1 = e^{\frac{i}{\hbar}\phi_{T_2,0}(x_2,\xi_0,x_0)}\left(\frac{\cos t_2\cos t_1}{\cos T_2}\right)^{1/2}.$$

Using this relation inductively and taking $|\Delta_{T,0}| = \max_{1\le j\le J+1} t_j \to 0$, we have

$$\begin{aligned}
e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0) &= \int e^{\frac{i}{\hbar}\phi[q,p]}\mathcal{D}[q,p]\\
&= \lim_{|\Delta_{T,0}|\to 0}\left(\frac{1}{2\pi\hbar}\right)^J\int_{\mathbf{R}^{2J}} e^{\frac{i}{\hbar}\sum_{j=1}^{J+1}\phi_{T_j,T_{j-1}}(x_j,\xi_{j-1},x_{j-1})}\prod_{j=1}^{J}dx_j d\xi_j\\
&= \lim_{|\Delta_{T,0}|\to 0} e^{\frac{i}{\hbar}\phi_{T,0}(x,\xi_0,x_0)}\left(\frac{\prod_{j=1}^{J+1}\cos t_j}{\cos T}\right)^{1/2}\\
&= \frac{1}{(\cos T)^{1/2}}\exp\frac{i}{\hbar}\Big(-x_0\cdot\xi_0+\frac{2x\cdot\xi_0-(x^2+\xi_0^2)\sin T}{2\cos T}\Big).
\end{aligned}$$

Example 12.2. If $d=1$, $H(t,x,\xi)=\xi^2/2+x\cdot\xi+x^2/2$ and $F[q,p]\equiv 1$, we have

$$e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0) = \left(\frac{e^T}{1+T}\right)^{1/2}\exp\frac{i}{\hbar}\left(-x_0\cdot\xi_0+\frac{2x\cdot\xi_0-T(x^2+\xi_0^2)}{2(1+T)}\right).$$

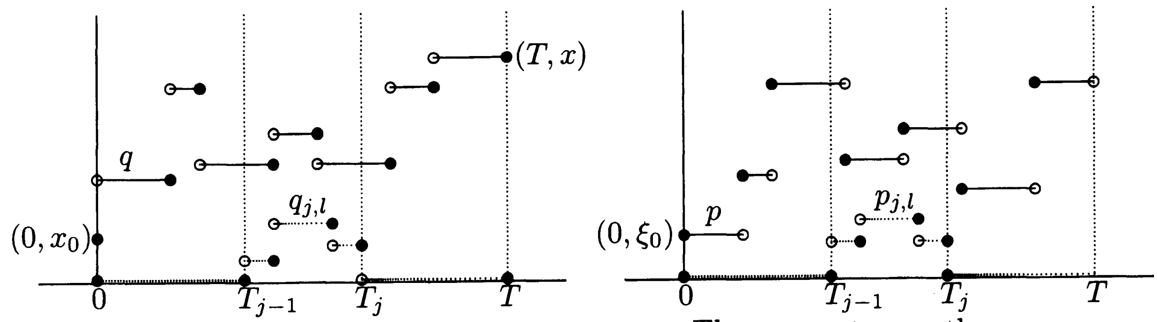
Example 12.3. Even when $d=1$, $H(t,x,\xi)=-ix^2/2-i\xi^2/2$ (complex-valued, i.e., a heat equation) and $F[q,p]\equiv 1$, in a similar way, we can calculate

$$e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0) = \frac{1}{(\cosh T)^{1/2}}\exp\frac{i}{\hbar}\left(-x_0\cdot\xi_0+\frac{2x\cdot\xi_0+i(x^2+\xi_0^2)\sinh T}{2\cosh T}\right).$$

§ 13. Assumption for two classes $\mathcal{F}_{\mathcal{Q}}$, $\mathcal{F}_{\mathcal{P}}$ of functionals $F[q,p]$

Using the functional derivatives of higher order, we rewrite Assumption 2 via the piecewise bicharacteristic paths to Assumption 3 (1) via piecewise constant paths.

Assumption 3. Let m be a non-negative integer. Let u_j, $j=1,2,\dots,J,J+1$ and U be non-negative parameters depending on $\Delta_{T,0}$ such that $\sum_{j=1}^{J+1}u_j = U<\infty$. Set $\|q\| = \sup_{0\le t\le T}|q(t)|$ and $\|p\| = \sup_{0\le t<T}|p(t)|$.



The position paths q, $q_{j,l}$ The momentum paths p, $p_{j,l}$

Figure 7.

(1) For any non-negative integer M, there exist positive constants A_M, X_M such that

$$\left|\Big(\prod_{j=0}^{J+1}\prod_{l=1}^{L_{\mathcal{Q},j}} D_{q_{j,l}}\Big)\Big(\prod_{j=1}^{J+1}\prod_{l=1}^{L_{\mathcal{P},j}} D_{p_{j,l}}\Big)F[q,p]\right| \le A_M (X_M)^{J+1}(1+\|q\|+\|p\|)^m$$

$$\times\Big(\prod_{j=1}^{J+1}(t_j)^{\min(L_{\mathcal{P},j},1)}\Big)\prod_{j=0}^{J+1}\prod_{l=1}^{L_{\mathcal{Q},j}}\|q_{j,l}\|\prod_{j=1}^{J+1}\prod_{l=1}^{L_{\mathcal{P},j}}\|p_{j,l}\|,$$

$$\left|\Big(\prod_{j=0}^{J+1}\prod_{l=1}^{L_{\mathcal{Q},j}} D_{q_{j,l}}\Big)\Big(\prod_{j=1}^{J+1}\prod_{l=1}^{L_{\mathcal{P},j}} D_{p_{j,l}}\Big)D_{q_k}F[q,p]\right| \le A_M (X_M)^{J+1}(1+\|q\|+\|p\|)^m$$

$$\times u_k\|q_k\|\Big(\prod_{j=1,j\neq k}^{J+1}(t_j)^{\min(L_{\mathcal{P},j},1)}\Big)\prod_{j=0}^{J+1}\prod_{l=1}^{L_{\mathcal{Q},j}}\|q_{j,l}\|\prod_{j=1}^{J+1}\prod_{l=1}^{L_{\mathcal{P},j}}\|p_{j,l}\|,$$

for any division $\Delta_{T,0}$, any $L_{\mathcal{Q},j}=0,1,\dots,M$, any $L_{\mathcal{P},j}=0,1,\dots,M$, any $q_{j,l}\in\mathcal{Q}$ with $q_{j,l}(t)=0$ outside $(T_{j-1},T_j]$, any $q_k\in\mathcal{Q}$ with $q_k(t)=0$ outside $(T_{k-1},T_k]$, and any $p_{j,l}\in\mathcal{P}$ with $p_{j,l}(t)=0$ outside $[T_{j-1},T_j)$ (Figure 7).

(2) is omitted (see [11]).

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