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Borel summability of WKB theoretic transformation to the Weber equation

By

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Abstract

We take a Schrödinger equation which has a pair of simple turning points connected by a Stokes line, and consider WKB theoretic transformation series to the Weber equation. Under suitable conditions, the transformation series is Borel summable, and analysis of so-called fixed singularities can be reduce to the Weber equation.

§ 1. Introduction

We consider a second order linear differential equation with a large parameter

$$\left(\frac{d^2}{dx^2} - \eta^2 Q(x)\right)\psi = 0. \tag{1.1}$$

The coefficient $Q(x)$ is a holomorphic function (typically rational function or polynomial). The equation has formal solutions (WKB solutions) of the form

$$\psi(x,\eta) = \exp\left(\int^x S(x,\eta)\,dx\right), \tag{1.2}$$

where $S(x,\eta) = \eta S_{-1} + S_0 + \eta^{-1}S_1 + \cdots$ is a formal power series satisfying the Riccati equation

$$S^2 + \frac{dS}{dx} = \eta^2 Q(x). \tag{1.3}$$

The WKB solutions $\psi(x,\eta)$ (or $S(x,\eta)$) are divergent in general, and we apply Borel resummation method. (See e.g., [12], [9].) Under generic assumptions, the WKB solutions (of suitable normalization) is Borel summble (see [4], [8]), but in some cases not.

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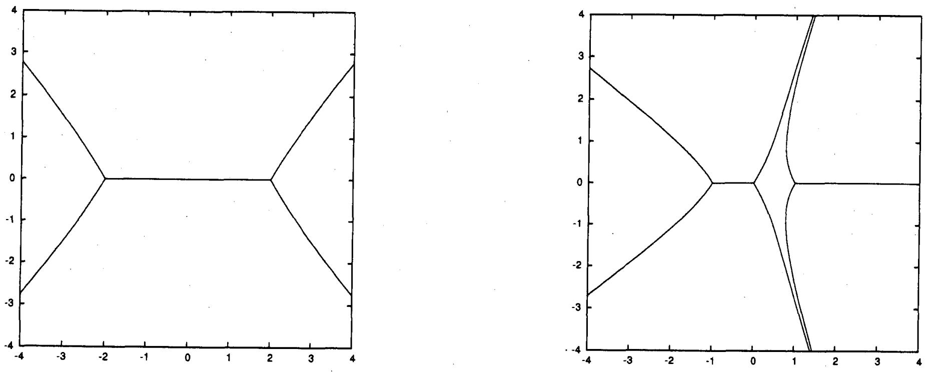


Figure 1. two examples in which Stokes lines connect simple turning points

For example if $Q(x) = E - x^2/4$ (namely the Weber equation) with a positive constant $E > 0$, WKB solutions are not Borel summable (depending on the normalization). See e.g., [11]. This is a general phenomenon if the equation has a pair of turing points (zeros of $Q(x)$) connected by a Stokes line $\Im \int^x \sqrt{Q(x)}\, dx = 0$. See e.g., [2], [3]. In Figure 1, we give examples of Stokes lines connecting turning points.

In such cases, the Borel transform of a WKB solution has singularities on the real axis (in the Borel plane), which we call "fixed singularites". They are fixed in the sence that the location is independ of x. To analyze such singularities, in [1], WKB theoretic transformation to the Weber equation is constructed, and Borel transformability is given. Here transformation series is a formal power series $x(q, \eta) = x_0(q) + \eta^{-1}x_1(q) + \eta^{-2}x_2(q) + \cdots$ which transforms the equation

$$\left(\frac{d^2}{dq^2} - \eta^2 Q(q)\right)\psi = 0 \tag{1.4}$$

to the Weber equation (with an infinite power series $E = E(\eta) = E_0 + \eta^{-1}E_1 + \eta^{-2}E_2 + \cdots$)

$$\left(\frac{d^2}{dx^2} - \eta^2\left(E - \frac{x^2}{4}\right)\right)\phi = 0, \tag{1.5}$$

with a gauge transform $\psi = x^{-1/2}\phi$. This is equivalent to that $x(q, \eta)$ satisfies the following:

$$Q(q) = \left(\frac{dx}{dq}\right)^2\left(E - \frac{x^2}{4}\right) - \frac{1}{2}\eta^{-2}\{x; q\}. \tag{1.6}$$

Here $\{x; q\}$ is the Schwarzian derivative. Though this is a transformation between equations, this also connect WKB solutions of certain normalization. (See [1] and the following section.)

The purpose of this paper is to present Borel summability of transformation series in a simple case, and add a brief explanation of the consequence (the following section).

In ending of this introduction, we refer to the work of Kamimoto and Koike ([5]), which shows Borel summability of transformation series to the Airy equation ($Q(x) = x$). The Airy equation has only one simple turning point, and is the simplest equation whose (Borel transformed) WKB solution has so-called "movable singularities". The basic idea of the proof of the Weber case follows the Airy case [5], while one additional problem arises which we should overcome. In this paper, we do not give a proof of Borel summability of transformation series to the Weber equation. A detailed proof will be given elsewhere.

§ 2. Borel summability of transformation series

In this section, for simplicity we assume that the coefficient $Q(q)$ in (1.4) is polynomial. Let $q_\pm$ be simple turning points of the equation (1.4). Assume $q_\pm$ are connected by a Stokes line and the other Stokes lines emanating from the two points tend to infinity. For example, if $Q(q) = q(q^2-1)$ and we take $q_+ = 0$ and $q_- = -1$, these conditions are satisfied (See Figure 1). Take a neighborhood

$$D = \left\{ \left| \int_{q_+}^{q} \sqrt{Q}\, dq \right| < d \right\} \cup \left\{ \left| \int_{q_-}^{q} \sqrt{Q}\, dq \right| < d \right\} \tag{2.1}$$

of $\{q_\pm\}$ and set

$$\hat{D} = \bigcup_{q' \in D} \left\{ \Im \int_{q'}^{q} \sqrt{Q}\, dq = 0 \right\}. \tag{2.2}$$

(cf. Figure 2.) We take d small enough so that $\hat{D}$ does not contain any turning points except for $q_\pm$. Then there exist formal power series $x(q,\eta) = x_0(q) + \eta^{-1}x_1(q) + \eta^{-2}x_2(q) + \cdots$ and $E(\eta) = E_0 + \eta^{-1}E_1 + \eta^{-2}E_2 + \cdots$ with $x_j(q)$ being holomorphic on $\hat{D}$ ($j = 0,1,2,\ldots$) which satisfy the equation (1.6) and $dx_0/dq \neq 0$. $x(q,\eta)$ and $E(\eta)$ are uniquely determined up to the choice of $x_0(q)$. See [1], [9].

Remark. $x_0(q)$ is a map which maps a turning point to a turning point, a level curve (Stokes line) $\Im \int^q \sqrt{Q}\, dq = 0$ to a level curve (Stokes line)$\Im \int^x \sqrt{(E_0 - x^2/4)}\, dx = 0$. There are two turning points $q_\pm$, and we have two choices of $x_0(q)$.

The Borel summability of $E(\eta)$ is known. See [8]. In addition we have the following theorem.

Theorem 2.1. *Under the assumptions above, the transformation series $x(q,\eta)$ is Borel summable uniformly on $\hat{D}$.*

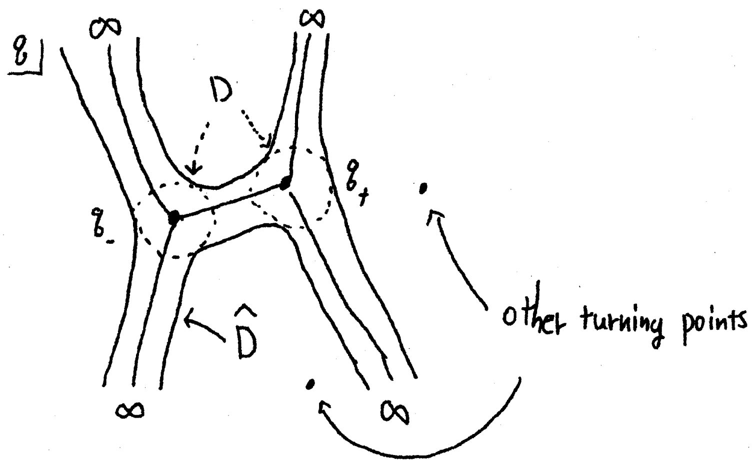


Figure 2. Domains D and $\hat{D}$.

Thus the equation (1.4) on $\hat{D}$ is transformed to the canonical equation (1.5) by two Borel summable series $x(q,\eta)$ and $E(\eta)$. Then as is explained in [1] and [9] (though mainly Airy case, not Weber case), a WKB solution of (1.4) is also transformed into a WKB solution of (1.5); Let $\psi(q,\eta)$ be a WKB solution of (1.4) normalized at q_+ $\phi(x,E,\eta)$ be a WKB solution of (1.5) normalized at $2\sqrt{E}$. Here we assume $x_0(q_+) = 2\sqrt{E_0}$. (For normalization, see e.g., [9].) Then the following relation holds:

$$\psi(q,\eta) = \left(\frac{dx}{dq}(q,\eta)\right)^{-1/2} \phi(x(q,\eta),E(\eta),\eta). \tag{2.3}$$

Though this is a formal relation, if Borel transformed, this becomes an analytic relation. Set $x(q,\eta) = x_0(q) + X(q,\eta)$ and $E(\eta) = E_0 + F(\eta)$. By Taylor expansion, we have

$$\begin{aligned} \psi(q,\eta) &= \left(\frac{dx}{dq}(q,\eta)\right)^{-1/2} \sum_{n=0}^{\infty} \frac{X^n(q,\eta)}{n!} \frac{\partial^n \phi}{\partial x^n}(x_0(q),E(\eta),\eta) \\ &= \left(\frac{dx}{dq}(q,\eta)\right)^{-1/2} \sum_{n=0}^{\infty} \frac{X^n(q,\eta)}{n!} \left(\sum_{m=0}^{\infty} \frac{F^m(\eta)}{m!} \frac{\partial^{n+m}\phi}{\partial E^m \partial x^n}(x_0(q),E_0,\eta)\right). \end{aligned} \tag{2.4}$$

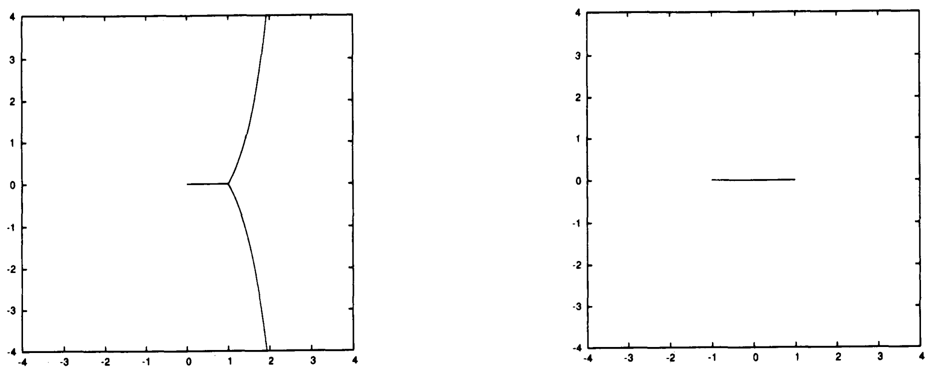
Then by Borel transform, we have

(2.5)

$$\psi_{\mathrm{B}}(q,y) = \left(\left(\frac{dx}{dq}\right)^{-1/2}\right)_{\mathrm{B}}(q,y) * \sum_{n=0}^{\infty} \frac{X_{\mathrm{B}}^{*n}(q,y)}{n!} * \left(\sum_{m=0}^{\infty} \frac{F_{\mathrm{B}}^{*m}(y)}{m!} * \frac{\partial^{n+m}\phi_{\mathrm{B}}}{\partial E^m \partial x^n}(x_0(q),E_0,y)\right),$$

where the subscript B means Borel transform and $*$ is convolution. Now let us take one term

$$\left(\left(\frac{dx}{dq}\right)^{-1/2}\right)_{\mathrm{B}}(q,y) * \frac{X_{\mathrm{B}}^{*n}(q,y)}{n!} * \frac{F_{\mathrm{B}}^{*m}(y)}{m!} * \frac{\partial^{n+m}\phi_{\mathrm{B}}}{\partial E^m \partial x^n}(x_0(q),E_0,y).$$



$Q(x) = (x-1)/x$ $\qquad$ $Q(x) = 1/(x^2-1)$

Figure 3. Stokes lines connecting a pair of a simple turning point and a simple pole(left), a pair of simple poles (right).

Since X and F are Borel summable, the front part

$$\left(\left(\frac{dx}{dq}\right)^{-1/2}\right)_{\mathrm{B}}(q,y) * \frac{X_{\mathrm{B}}^{*n}(q,y)}{n!} * \frac{F_{\mathrm{B}}^{*m}(y)}{m!}$$

is holomorphic in a strip region containing the positive real axis. Since we know well about ϕ_{B} (see e.g., [11], [10]), for this single term, we can see continuability avoiding singularities, disconitinuity at a singularity, etc. Then by summing up with respect to m and n (with care on convergence), we see continuablity etc. also for $\psi_{\mathrm{B}}(q,y)$.

Remark. $\phi_{\mathrm{B}}(x_0(q), E_0, y)$ has infinitely many singularites in the y-plane with real period $2\pi E_0$, and with Borel summability we can analyze all singularities through transformation. Thus Borel summability of transformation is important in the analysis of fixed singularities.

Remark. In this paper, we considered only two simple turning points problem. On the other hand, simple poles (of Q) are known to play a role similar to simple turning points ([6], [7]), and a pair of a simple turning point and a simple pole, or a pair of simple poles causes fixed singularities as well (cf. Figure 3). The former one can be treated in the same manner as a pair of simple turning points. However the latter one is difficult to treat with. Also, a sole simple turning point makes a pair in some sense, generating a loop of Stokes line (cf. Figure 4), and this has the same difficulty.

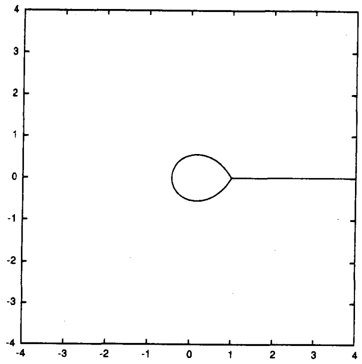


Figure 4. A loop of Stokes line ending a sole simple turning point.

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