数理解析研究所講究録
第 1866 巻 2013 年 68-81

Edge-state integrals on shaped triangulations

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Abstract

The partition functions of two state integral TQFT models of the Turaev–Viro type suggested in [11, 2] are briefly reviewed. It is also shown that their behavior under adding or removing a vertex is consistent with the conjectural relationship between two models. This text is partially based on author's talk given at the workshop "Intelligence of Low-dimensional Topology" which took place at RIMS, Kyoto University, on May 22-24, 2013.

1 Introduction

Recently, two state integral 3-dimensional TQFT models of the Turaev–Viro type were suggested in [11, 2]. In this paper, we review these models and analyze their behavior with respect to the Pachner moves which add or remove vertices. To distinguish between the models, we will call R-model the one of [11], and C-model the one of [2]. There are two main ingredients common to both models: the shaped triangulations and Faddeev's quantum dilogarithm. Let us start by briefly recalling these ingredients.

1.1 Shaped triangulations

A triangulation for us is identical to a cellular complex where all cells are simplices. A refined version of triangulation given by Δ-triangulations in the sense of Hatcher [9] is used in the C-model. If X is a triangulation, we denote by $\Delta_i(X)$ the set of i-dimensional cells of X, and for any $i > j$, we denote by $\Delta_{i,j}(X)$ the set of pairs (a, b) with $a \in \Delta_i(X)$ and $b \in \Delta_j(a)$ (here a is itself considered as a triangulation). If $\mathring{X}$ is the interior of X, we set $\Delta_i(\mathring{X}) := \Delta_i(X) \setminus \Delta_i(\partial X)$. For any two sets A and B, B^A will denote the set of all maps from A to B.

A *shaped triangulation* is a triangulation X of an oriented 3-dimensional pseudo manifold where each tetrahedron is provided with angles of an ideal hyperbolic tetrahedron. More formally, a *shape structure* (or simply a *shape*) is a map

$$\alpha\colon \Delta_{3,1}(X) \to\ ]0, \pi[\tag{1}$$

with the condition that the sum of the values at each vertex of each tetrahedron is π, or, equivalently, that it takes equal values on opposite edges of each tetrahedron

$$\alpha(T, e) = \alpha(T, e^{\mathrm{op}}), \quad \forall T \in \Delta_3(X), \tag{2}$$

Supported in part by Swiss National Science Foundation.

and the sum over all edges of each tetrahedron is 2π,

$$\sum_{e\in\Delta_1(T)} \alpha(T,e) = 2\pi, \quad \forall T \in \Delta_3(X). \tag{3}$$

The set of all shape structures on X is denoted $S(X)$.

By using the orientation of the triangulation, let us choose a cyclic order between three edges at each vertex in each tetrahedron by saying that f *follows* e if by departing from e in the clockwise direction around the common vertex one arrives to f. By using such cyclic order, we define skew-symmetric functions

$$\varepsilon_T \colon \Delta_1(X) \times \Delta_1(X) \to \{-1,0,1\}, \quad \forall T \in \Delta_3(X), \tag{4}$$

with $\varepsilon_T(e,f) = 1$ if and only if both e and f are edges of T which do not form a pair of opposite edges, and f follows e. The *Neumann–Zagier simplectic structure* in $S(X)$ is given by the following non-degenerate closed two-form [14]:

$$\omega_{NZ} = \sum_{T\in\Delta_3(X)} \sum_{e,f\in\Delta_1(T)} \varepsilon_T(e,f) d(T,e) \wedge d(T,f), \tag{5}$$

where any element (T,e) of the set $\Delta_{3,1}(X)$ is considered as a function on $S(X)$ defined by $(T,e)(\alpha) := \alpha(T,e)$.

Two shape structures on one and the same triangulation X are called *gauge equivalent* if they are related by the action of the additive group $\mathbb{R}^{\Delta_1(\mathring{X})}$ induced by the Poisson action of the total dihedral angles around the interior edges of the triangulation[1]. A formal characterization is as follows. Two shape structures α and β on a triangulation X are gauge equivalent if and only if there exists a function

$$g \colon \Delta_1(X) \to \mathbb{R} \tag{6}$$

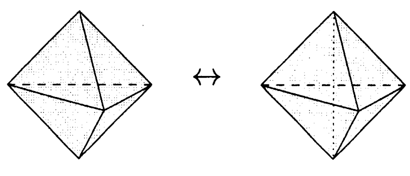
such that

$$g|_{\Delta_1(\partial X)} = 0, \tag{7}$$

and

$$\beta(T,e) = \alpha(T,e) + \sum_{f\in\Delta_1(T)} \varepsilon_T(e,f) g(f), \quad \forall (T,e) \in \Delta_{3,1}(X). \tag{8}$$

We say that two shaped triangulations are related by a *shaped* $2-3$ *Pachner move* if they differ only in the interior of a three dimensional ball as is shown in this picture



where it is assumed that the total dihedral angles on the corresponding boundary edges are pairwise equal. This implies that the only interior edge in the right hand side carries the

[1]In [1] this equivalence is called *based gauge equivalence*, while the (non-based) gauge equivalence is defined differently.

total dihedral angle 2π. In that case the edge is called *balanced.* A shaped triangulation where all edges are interior and balanced is known as a (strict) *angle structure* introduced by Casson, Rivin and Lackenby as a linearized version of the hyperbolic structure [15, 13].

A (quantum) *state* of a triangulation X is an assignment of real numbers to all edges, i.e. a map

$$x\colon \Delta_1(X) \to \mathbb{R}. \tag{9}$$

1.2 Faddeev's quantum dilogarithm

Following [7], *Faddeev's quantum dilogarithm* function [6] is defined by the formula

$$\Phi_\hbar(z) := \exp\left(\int_{\mathbb{R}+i\epsilon} \frac{e^{-i2xz}}{4\sinh(xb)\sinh(xb^{-1})}\frac{dx}{x}\right), \tag{10}$$

where $\hbar \in \mathbb{C} \setminus \mathbb{R}_{\le 0}$, b is any root of the equation

$$(b+b^{-1})^{-2} = \hbar, \tag{11}$$

and ϵ is an arbitrary sufficiently small positive real. From now on, we choose the unique b such that $0 \le \arg b < \pi/2$, and $\sqrt{\hbar}$ will be defined to be $(b+b^{-1})^{-1}$. This implies that $\Re\sqrt{\hbar} > 0$.

The integral in (10) is absolutely convergent only in the strip $|\Im z| < |\Re(\frac{1}{2\sqrt{\hbar}})|$ where it satisfies the functional equations

$$\Phi_\hbar(z - ib^{\pm1}/2) = (1 + e^{2\pi b^{\pm1} z})\Phi_\hbar(z + ib^{\pm1}/2) \tag{12}$$

which can be used for extending the domain of definition of $\Phi_\hbar(z)$ to the whole complex plane. One has the inversion relation

$$\Phi_\hbar(z)\Phi_\hbar(-z) = e^{i\pi z^2} e^{-\pi i(2+\hbar^{-1})/12}, \tag{13}$$

and in the case when $\arg b > 0$ (i.e. $\hbar \notin]0, \frac{1}{4}]$) one can show that

$$\Phi_\hbar(z) = \frac{(-qe^{2\pi bz}; q)_\infty}{(-\bar{q}e^{2\pi b^{-1}z}; \bar{q})_\infty}, \quad q := e^{2\pi i b^2}, \ \bar{q} := e^{-2\pi i b^{-2}}, \tag{14}$$

where we use the common notation in the theory of basic hypergeometric series

$$(x; y)_\infty := \prod_{n=0}^{\infty}(1 - xy^n). \tag{15}$$

Formula (14) and the continuity argument permit to extract the complete information about the analytic properties of Faddeev's quantum dilogarithm. Namely, it is a meromorphic function on $\mathbb{C}$ with the set of poles

$$P_\hbar = ib\left(\frac{1}{2} + \mathbb{Z}_{\ge 0}\right) + ib^{-1}\left(\frac{1}{2} + \mathbb{Z}_{\ge 0}\right), \tag{16}$$

the set of zeros $-P_\hbar$, i.e.

$$(\Phi_\hbar(z))^{\pm 1} = 0 \ \Leftrightarrow \ z \in \mp P_\hbar, \tag{17}$$

and the asymptotic behavior at infinity:

$$\Phi_\hbar(z)\big|_{|z|\to\infty} = \Phi_\hbar^{(\infty)}(z) + o(1), \tag{18}$$

where

$$\Phi_\hbar^{(\infty)}(z) := \begin{cases} 1 & \text{if } |\arg z| > \frac{\pi}{2} + \arg b \\ e^{i\pi z^2} e^{-\pi i(2+\hbar^{-1})/12} & \text{if } |\arg z| < \frac{\pi}{2} - \arg b \\ \frac{(\bar q;\bar q)_\infty}{\Theta(ib^{-1}z;-b^{-2})} & \text{if } |\arg z - \frac{\pi}{2}| < \arg b \\ \frac{\Theta(ibz;b^2)}{(q;q)_\infty} & \text{if } |\arg z + \frac{\pi}{2}| < \arg b \end{cases}$$

with the Jacobi theta-function

$$\Theta(z;\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i\tau n^2 + 2\pi i z n}, \quad \Im\tau > 0. \tag{19}$$

In the *unitary* case, corresponding to $\hbar \in \mathbb{R}_{>0}$, we have the complex conjugation property

$$\overline{\Phi_\hbar(z)}\Phi_\hbar(\bar z) = 1 \tag{20}$$

which implies that for any selfadjoint operator A in a Hilbert space, the operator $\Phi_\hbar(A)$ is unitary. Moreover, if $\mathbf{p}$ and $\mathbf{q}$ are selfadjoint operators satisfying Heisenberg's commutation relation

$$\mathbf{pq} - \mathbf{qp} = (2\pi i)^{-1}, \tag{21}$$

for example, the operators in $L^2(\mathbb{R})$ defined by

$$\mathbf{p}f(x) := \frac{1}{2\pi i}\frac{\partial f(x)}{\partial x}, \quad \mathbf{q}f(x) := xf(x), \tag{22}$$

then one has Faddeev's quantum five term identity [5]

$$\Phi_\hbar(\mathbf{p})\Phi_\hbar(\mathbf{q}) = \Phi_\hbar(\mathbf{q})\Phi_\hbar(\mathbf{p}+\mathbf{q})\Phi_\hbar(\mathbf{p}) \tag{23}$$

which can be shown to be equivalent to the integral version of Ramanujan's ${}_1\psi_1$ summation formula:

$$\int_{\mathbb{R}+i\epsilon} \frac{\Phi_\hbar(x+u)}{\Phi_\hbar\left(x - \frac{i}{2\sqrt{\hbar}}\right)} e^{-2\pi i w x}\, dx = \frac{\Phi_\hbar(u)\,\Phi_\hbar\left(\frac{i}{2\sqrt{\hbar}} - w\right)}{\Phi_\hbar(u-w)} e^{\pi i\left(1+\hbar^{-1}\right)/12} \tag{24}$$

where the integral is absolutely convergent under the conditions

$$0 < \Im w < \Im u < \Re\frac{1}{2\sqrt{\hbar}}. \tag{25}$$

As a particular limiting case of (24), we have the following Fourier transformation formula

$$\int_{\mathbb{R}+i\epsilon} \Phi_\hbar(x) e^{-2\pi i w x}\, dx = \Phi_\hbar\left(\frac{i}{2\sqrt{\hbar}} - w\right) e^{-\pi i w^2} e^{\pi i\left(1+\hbar^{-1}\right)/12}. \tag{26}$$

In what follows, for simplicity of presentation, we will always assume the unitary case, i.e. that $\hbar \in \mathbb{R}_{>0}$.

Remark 1. Faddeev's quantum dilogarithm is closely related to Shintani's double sine function [16, 12, 3], but the operator identity (23) seems not to be known before [5].

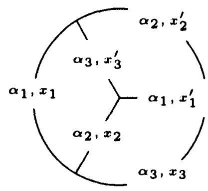
2 The R-model

2.1 Weight functions

For a tetrahedron T with shape α and in state x, one assigns a weight function

$$W_\hbar(T,\alpha,x) = \prod_{j=1}^{3} \Psi_\hbar\left(x_{j+1} + x'_{j+1} - x_{j-1} - x'_{j-1} + \frac{i}{\sqrt{\hbar}}\left(\frac{1}{2} - \frac{\alpha_j}{\pi}\right)\right) \tag{27}$$

where all indices are considered modulo 3, the notation for the shape and state variables corresponds to this picture



while the function $\Psi_\hbar(x)$ is a renormalized version of Faddeev's quantum dilogarithm:

$$\Psi_\hbar(x) = \frac{\Phi_\hbar(x)}{\Phi_\hbar(0)} e^{-i\pi x^2/2} \tag{28}$$

which corresponds to the simplest version of the inversion relation (13):

$$\Psi_\hbar(x)\Psi_\hbar(-x) = 1. \tag{29}$$

Remark 2. For calculational purposes, it is useful to write the integral Ramanujan formula (24) in the form

$$\begin{aligned}\varphi(x,y) &:= \int_{\mathbb{R}} \frac{\Phi_\hbar\left(t+\frac{x}{2}\right)}{\Phi_\hbar\left(t-\frac{x}{2}\right)} e^{2\pi i t y} dt \\ &= \Psi_\hbar\left(x - \frac{i}{2\sqrt{\hbar}}\right)\Psi_\hbar\left(y + \frac{i}{2\sqrt{\hbar}}\right)\Psi_\hbar\left(-x-y+\frac{i}{2\sqrt{\hbar}}\right)\end{aligned} \tag{30}$$

so that, denoting

$$x *_j \alpha := x_{j+1} + x'_{j+1} - x_{j-1} - x'_{j-1} - \frac{i\alpha_j}{\pi\sqrt{\hbar}}, \quad \forall j \in \{1,2,3\}, \tag{31}$$

the tetrahedral weight function (27) can be written

$$W_\hbar(T,\alpha,x) = \varphi\left(\frac{i}{\sqrt{\hbar}} + x *_j \alpha, x *_k \alpha\right), \tag{32}$$

where $\{j,k\}$ is any 2-element subset of $\{1,2,3\}$.

To a triangulation X in state x and with shape α, we associate the weight function given by the product of tetrahedral weight functions for all tetrahedra of X with induced states and shapes:

$$W_\hbar(X,\alpha,x) = \prod_{T\in\Delta_3(X)} W_\hbar\left(T, x|_T, \alpha|_T\right). \tag{33}$$

2.2 Partition functions

A *state gauge transformation* on a triangulation X is an element $g \in \mathbb{R}^{\Delta_0(X)}$ which acts in the space of states according to the map

$$\mathbb{R}^{\Delta_1(X)} \times \mathbb{R}^{\Delta_0(X)} \to \mathbb{R}^{\Delta_1(X)}, \quad (x,g) \mapsto x^g, \tag{34}$$

defined by the formula

$$x^g(e) = x(e) + g(v_1) + g(v_2), \quad \partial e = \{v_1, v_2\}. \tag{35}$$

For any subset $S \subset \Delta_0(X)$, an element $g \in \mathbb{R}^S$ will be thought of as a state gauge transformation vanishing on the complement of S. One can easily verify the state gauge invariance of the weight function (33):

$$W_\hbar(X,\alpha,x) = W_\hbar(X,\alpha,x^g), \quad \forall g \in \mathbb{R}^{\Delta_0(X)}. \tag{36}$$

A *state gauge fixing* λ on a triangulation X is an assignment to each interior vertex of X of a real valued linear form λ_v on the vector space of states $\mathbb{R}^{\Delta_1(X)}$ such that

$$\langle \lambda_v, x^g \rangle = \langle \lambda_v, x \rangle + g(v), \quad \forall g \in \mathbb{R}^{\Delta_0(\mathring{X})}. \tag{37}$$

To a triangulation X of shape α and boundary state $s \in \mathbb{R}^{\Delta_1(\partial X)}$, we associate the *partition function*

$$R_\hbar(X,\alpha,s) := \int_{x\in\mathbb{R}^{\Delta_1(X)}} W_\hbar(X,\alpha,x)\delta\left(s - x\big|_{\partial X}\right)\delta\left(\langle\lambda,x\rangle\right)dx. \tag{38}$$

where λ is a state gauge fixing on X,

$$\delta\left(s - x\big|_{\partial X}\right) := \prod_{e\in\Delta_1(\partial X)} \delta(s(e) - x(e)) \tag{39}$$

$$\delta(\langle\lambda,x\rangle) := \prod_{v\in\Delta_0(\mathring{X})} \delta\left(\langle\lambda_v,x\rangle\right), \tag{40}$$

$$dx := \prod_{e\in\Delta_1(X)} dx(e). \tag{41}$$

The main result of [11] is the following theorem.

Theorem 1 ([11]). *The partition function $R_\hbar(X,\alpha,s)$ is an absolutely convergent integral independent of the choice of the state gauge fixing λ and invariant under the shaped $2-3$ Pachner moves and the shape gauge transformations.*

3 The C-model

3.1 Leveled shaped triangulations

Similarly to known TQFT models of the Reshetikhin–Turaev type, there is a phase ambiguity in the Teichmüller TQFT of [1, 2], which is taken into account by assigning a real number called *level* to a shaped triangulation and extending the shaped gauge transformations and the shaped $2-3$ Pachner moves in a certain specific way. More precisely, a *leveled shape* on a triangulation X is a pair $(\alpha, u) \in S(X) \times \mathbb{R}$, and two leveled shapes (α, u) and (β, v) on a triangulation X are *gauge equivalent* if and only if α and β are gauge equivalent through a function g in equations (6)–(8) and

$$v = u + \frac{1}{4\pi} \sum_{(T,e)\in\Delta_{3,1}(X)} g(e)\left(\alpha(T,e) - \frac{\pi}{3}\right). \tag{42}$$

We also say that leveled shaped triangulations (X, α, u) and (Y, β, v) are related by a *leveled shaped* $2-3$ *Pachner move* if the underlying shaped triangulations are related by a shaped $2-3$ Pachner move, where, for example, Y is obtained from X by removing an edge $e \in \Delta_1(X)$ and

$$v = u + \frac{1}{48} \sum_{(T,f)\in\Delta_{3,1}(X)} \varepsilon_T(e,f)\alpha(T,f). \tag{43}$$

3.2 Weight functions

Let $T \in \mathbb{R}^3$ be an Euclidean tetrahedron with vertex ordering mapping

$$v\colon \{0,1,2,3\} \ni i \mapsto v_i \in \Delta_0(T). \tag{44}$$

We say that T is *positive* (respectively *negative*) if the vectors v_0v_1, v_0v_2, v_0v_3 form a right-handed (respectively left-handed) frame in $\mathbb{R}^3$. For any shape $\alpha \in S(T)$ and state $x \in \mathbb{R}^{\Delta_1(T)}$, we associate a weight function

$$B_\hbar(T,\alpha,x) = g_{\alpha_1,\alpha_3}(x_{02}+x_{13}-x_{03}-x_{12},\, x_{02}+x_{13}-x_{01}-x_{23})$$

if T is positive, and the same but complex conjugate expression if T is negative. Here

$$x_{ij} := x(v_iv_j), \quad \alpha_i := \alpha_T(v_0v_i)/2\pi, \tag{45}$$

$$g_{a,c}(s,t) := \sum_{m\in\mathbb{Z}} \tilde{\psi}'_{a,c}(s+m)e^{\pi i t(s+2m)}, \tag{46}$$

$$\tilde{\psi}'_{a,c}(s) := e^{-\pi i s^2}\tilde{\psi}_{a,c}(s), \tag{47}$$

$$\tilde{\psi}_{a,c}(s) := \int_{\mathbb{R}} \psi_{a,c}(t)e^{-2\pi i s t}dt, \tag{48}$$

and

$$\psi_{a,c}(t) := \frac{e^{2\pi a t/\sqrt{\hbar}}e^{\pi i(-a(a+c)+(4(a-c)+1)/24)/\hbar}}{\Phi_\hbar\left(t - i(a+c)/\sqrt{\hbar}\right)}. \tag{49}$$

Remark 3. In formula (46), we use a transformation which associates to functions on $\mathbb{R}$ quasi-periodic functions on $\mathbb{R}^2$. This transform by itself has a number of remarkable properties, and in the literature, it can be traced back to the works of Weil [17], Gel′fand [8], and Zak [18].

Let X be a Δ-triangulation of an oriented pseudo 3-manifold, where we distinguish between positive and negative tetrahedra. For any shape $\alpha \in S(X)$ and state $x \in \mathbb{R}^{\Delta_1(X)}$, we define the weight function

$$B_\hbar(X,\alpha,x) := \prod_{T\in\Delta_3(X)} B_\hbar\left(T, \alpha|_{\Delta_{3,1}(T)}, x|_{\Delta_1(T)}\right). \tag{50}$$

3.3 Partition functions

Let (X,α) be a shaped Δ-triangulation in a fixed boundary state $s \in \mathbb{R}^{\Delta_1(\partial X)}$. We define a partition function by the formula

$$C_\hbar(X,\alpha,s) := \int_{\mathring{x}\in[0,1]^{\Delta_1(\mathring{X})}} \int_{\dot{x}\in\mathbb{R}^{\Delta_1(\partial X)}} B_\hbar(X,\alpha,x)\delta(\dot{x}-s)dx \tag{51}$$

where

$$\mathring{x} := x|_{\Delta_1(\mathring{X})}, \quad \dot{x} := x|_{\Delta_1(\partial X)}, \tag{52}$$

$$\delta(\dot{x}-s) := \prod_{e\in\Delta_1(\partial X)} \delta(x(e)-s(e)), \quad dx := \prod_{e\in\Delta_1(X)} dx(e). \tag{53}$$

Theorem 2 ([2]). *The partition function $C_\hbar(X,\alpha,s)$ is absolutely convergent integral which admits complex analytic continuation to a meromorphic function of its arguments (complex shapes and boundary states) and, for any leveled shape $(\alpha,u) \in S(X)\times\mathbb{R}$, the quantity $e^{-iu/\hbar}C_\hbar(X,\alpha,s)$ is invariant under the leveled shaped $2-3$ Pachner moves and the leveled shape gauge transformations.*

4 Calculations with two tetrahedra

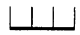
It is expected that R- and C-models are related according to the following conjecture.

Conjecture 1 ([11, 2]). For any shaped Δ-triangulation (X,α) of a closed 3-manifold, one has the equality

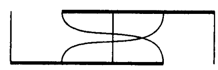
$$R_\hbar(X,\alpha) = 2\left|C_\hbar(X,\alpha)\right|^2. \tag{54}$$

Here we give an additional justification for this conjecture by calculating and comparing the two partition functions for triangular pillowcases triangulated into two tetrahedra with one interior vertex.

It is convenient to encode Δ-triangulations graphically by associating to each tetrahedron T an element as in this picture

$T =$ 

where the vertical segments, ordered from left to right, correspond to the faces $\partial_i T$ for $i \in \{0,1,2,3\}$, and by joining the corresponding segments for identified faces. Let us consider the diagram

$Y =$  (55)

which represents a Δ-triangulation composed of two tetrahedra $T^{\pm}$ glued along three common faces:

$$\partial_i T^- \sim \partial_i T^+, \quad i \in \{1,2,3\}, \tag{56}$$

so that the 0-skeleton consists of four vertices v_i, $i \in \{0,1,2,3\}$, where the vertex v_0 is in the interior, and all others are on the boundary. Thus, Y is a triangulation of a 3-ball with one interior vertex. Topologically, it can be conveniently visualized as a cellular decomposition of the standard unit ball in $\mathbb{R}^3 \simeq \mathbb{C} \times \mathbb{R}$ with all four vertices in the plane $\mathbb{C} \times \{0\}$ at the positions

$$v_0 = (0,0), \quad v_1 = (1,0), \quad v_2 = (e^{2\pi i/3}, 0), \quad v_3 = (e^{-2\pi i/3}, 0), \tag{57}$$

with the six edges being given by three arcs of the equator and three straight radii of the equatorial unit disc.

Define a meromorphic function on $\mathbb{C}^3$ by the formula

$$f_\hbar(z) := \prod_{j=1}^{3} \Psi_\hbar\left(\frac{i}{2\sqrt{\hbar}}\left(1 - \frac{z_j}{\pi}\right)\right), \quad z = (z_1, z_2, z_3), \tag{58}$$

and a map $w\colon S(Y) \to \mathbb{R}^3_{>0} \subset \mathbb{C}^3$ which associates total dihedral angles to interior edges of Y:

$$w_j(\alpha) := \alpha(T^+, v_0 v_j) + \alpha(T^-, v_0 v_j), \quad \forall j \in \{1,2,3\}. \tag{59}$$

Theorem 3. *The following equalities hold true:*

$$R_\hbar(Y, \alpha, s) = R_\hbar(Y, \alpha) := \left|f_\hbar(w(\alpha))\right|^2, \tag{60}$$

$$C_\hbar(Y, \alpha, s) = C_\hbar(Y, \alpha) := f_\hbar(w(\alpha)) e^{\frac{i}{4\pi\hbar}\chi(\alpha)}, \tag{61}$$

where

$$\chi(\alpha) := c_1^- c_3^+ - c_3^- c_1^+, \quad c_j^{\pm} := \alpha(T^{\pm}, v_0 v_j) - \frac{\pi}{3}, \quad \forall j \in \{1,3\}. \tag{62}$$

Proof. **Calculation of $R_\hbar(Y, \alpha, s)$.** After performing the (trivial) integration over boundary edge states in (38), the partition function becomes the following triple integral:

$$\begin{aligned} &R_\hbar(Y, \alpha, s) \\ &= \int_{\mathbb{R}^3} \varphi(2c_\hbar + x_2 + s_2 - x_3 - s_3 - 2c_\hbar\alpha_1^+, x_3 + s_3 - x_1 - s_1 - 2c_\hbar\alpha_2^+) \\ &\quad \times \varphi(2c_\hbar + x_3 + s_3 - x_2 - s_2 - 2c_\hbar\alpha_1^-, x_1 + s_1 - x_3 - s_3 - 2c_\hbar\alpha_2^-)\delta(x_3) dx_1 dx_2 dx_3 \end{aligned} \tag{63}$$

where

$$x_j := x(v_0 v_j), \quad s_j := s(v_{j+1} v_{j-1}), \quad j \in \{1,2,3\}, \tag{64}$$

$$c_\hbar := \frac{i}{2\sqrt{\hbar}}, \quad \alpha_j^{\pm} := \alpha(T^{\pm}, v_0 v_j)/\pi, \quad j \in \{1,2,3\}, \tag{65}$$

and the state gauge fixing at the only interior vertex v_0 is chosen to be $\langle \lambda_{v_0}, x \rangle = x_3$. Integrating over x_3 and shifting two other variables as follows:

$$x_1 \mapsto x_1 + s_3 - s_1, \quad x_2 \mapsto x_2 + s_3 - s_2, \tag{66}$$

we observe that the integral is manifestly independent of the boundary state:

$$\text{r.h.s of (63)} \\ = \int_{\mathbb{R}^2} \varphi(2c_\hbar + x_2 - 2c_\hbar\alpha_1^+, -x_1 - 2c_\hbar\alpha_2^+)\varphi(2c_\hbar - x_2 - 2c_\hbar\alpha_1^-, x_1 - 2c_\hbar\alpha_2^-)dx_1dx_2. \quad (67)$$

Futhermore, by using the analyticity of the integrand, we can make complex shifts

$$x_1 \mapsto x_1 + c_\hbar(\alpha_2^- - \alpha_2^+), \quad x_2 \mapsto x_2 + c_\hbar(\alpha_1^+ - \alpha_1^-) \quad (68)$$

without changing the value of the integral so that, introducing the normalized total dihedral angles

$$\omega_i := c_\hbar(\alpha_i^+ + \alpha_i^-), \quad i \in \{1,2,3\}, \quad \sum_{i=1}^{3} \omega_i = 2c_\hbar, \quad (69)$$

we arrive at a manifestly shape gauge invariant form

$$\text{r.h.s of (67)} = \int_{\mathbb{R}^2} \varphi(2c_\hbar + x_2 - \omega_1, -x_1 - \omega_2)\varphi(2c_\hbar - x_2 - \omega_1, x_1 - \omega_2)dx_1dx_2. \quad (70)$$

Now, if we use the integral representation (30), we obtain

$$\text{r.h.s of (70)} = \int_{\mathbb{R}^4} \frac{\Phi_\hbar\left(t_1 + c_\hbar + \frac{1}{2}(x_2 - \omega_1)\right)\Phi_\hbar\left(t_2 + c_\hbar - \frac{1}{2}(x_2 + \omega_1)\right)}{\Phi_\hbar\left(t_1 - c_\hbar - \frac{1}{2}(x_2 - \omega_1)\right)\Phi_\hbar\left(t_2 - c_\hbar + \frac{1}{2}(x_2 + \omega_1)\right)} \\ \times e^{2\pi i(t_1(-x_1-\omega_2)+t_2(x_1-\omega_2))}dt_1dt_2dx_1dx_2. \quad (71)$$

Integration over x_1 produces a delta-function which removes one of the t-variables bringing us back to a 2-dimensional integral:

$$\text{r.h.s of (71)} = \int_{\mathbb{R}^2} \frac{\Phi_\hbar\left(t + c_\hbar + \frac{1}{2}(x - \omega_1)\right)\Phi_\hbar\left(t + c_\hbar - \frac{1}{2}(x + \omega_1)\right)}{\Phi_\hbar\left(t - c_\hbar - \frac{1}{2}(x - \omega_1)\right)\Phi_\hbar\left(t - c_\hbar + \frac{1}{2}(x + \omega_1)\right)} e^{-4\pi i t\omega_2}dtdx, \quad (72)$$

where we have removed the indices of the integration variables. Now, we make two consecutive shifts of integrations variables, $t \mapsto t + \frac{1}{2}x$ followed by $x \mapsto x - t$:

$$\begin{aligned} \text{r.h.s of (72)} &= \int_{\mathbb{R}^2} \frac{\Phi_\hbar\left(t + c_\hbar + x - \frac{1}{2}\omega_1\right)\Phi_\hbar\left(t + c_\hbar - \frac{1}{2}\omega_1\right)}{\Phi_\hbar\left(t - c_\hbar + \frac{1}{2}\omega_1\right)\Phi_\hbar\left(t - c_\hbar + x + \frac{1}{2}\omega_1\right)} e^{-4\pi i(t+\frac{1}{2}x)\omega_2}dtdx \\ &= \int_{\mathbb{R}^2} \frac{\Phi_\hbar\left(c_\hbar + x - \frac{1}{2}\omega_1\right)\Phi_\hbar\left(t + c_\hbar - \frac{1}{2}\omega_1\right)}{\Phi_\hbar\left(t - c_\hbar + \frac{1}{2}\omega_1\right)\Phi_\hbar\left(-c_\hbar + x + \frac{1}{2}\omega_1\right)} e^{-2\pi i(t+x)\omega_2}dtdx \\ &= \left(\int_{\mathbb{R}} \frac{\Phi_\hbar\left(c_\hbar + x - \frac{1}{2}\omega_1\right)}{\Phi_\hbar\left(-c_\hbar + x + \frac{1}{2}\omega_1\right)} e^{-2\pi i x\omega_2}dx\right)^2 = \left(\varphi(2c_\hbar - \omega_1, -\omega_2)\right)^2 \\ &= \left(\prod_{i=1}^{3} \Psi_\hbar(c_\hbar - \omega_i)\right)^2 = \left(f_\hbar(w(\alpha))\right)^2 = \left|f_\hbar(w(\alpha))\right|^2, \end{aligned} \quad (73)$$

where the last equality is valid because the quantities $\Psi_\hbar(c_\hbar - \omega_i)$ are real numbers.

Calculation of $C_\hbar(Y,\alpha,s)$. After performing the (trivial) integration over boundary edge states in (51), the partition function is the following triple integral:

$$C_\hbar(Y,\alpha,s) = \int_{[0,1]^3} g_{a^+,c^+}(x_2+s_2-x_3-s_3, x_2+s_2-x_1-s_1) \\ \times \bar{g}_{a^-,c^-}(x_2+s_2-x_3-s_3, x_2+s_2-x_1-s_1)dx_1x_2dx_3, \quad (74)$$

where the notation in (64) is used and

$$a^\pm := \frac{\alpha(T^\pm, v_0v_1)}{2\pi}, \quad c^\pm := \frac{\alpha(T^\pm, v_0v_3)}{2\pi}. \quad (75)$$

After the change of the integration variables

$$(x_1, x_3) = (-y-s_1+x_2+s_2, -x-s_3+x_2+s_2) \quad (76)$$

the integrand becomes independent of the boundary states s_i and the variable x_2 so that integration over x_2 gives the multiplicative factor 1:

$$\text{r.h.s of (74)} = \int_{[0,1]^2} g_{a^+,c^+}(x,y) \times \bar{g}_{a^-,c^-}(x,y)dxdy. \quad (77)$$

Substituting the definition (46), we have

$$\begin{aligned}\text{r.h.s of (77)} &= \int_{[0,1]^2} \sum_{m,n\in\mathbb{Z}} \tilde{\psi}'_{a^+,c^+}(x+m)e^{\pi i y(2m+x)}\overline{\tilde{\psi}'}_{a^-,c^-}(x+n)e^{-\pi i y(2n+x)}dxdy \\ &= \int_{[0,1]} \sum_{m,n\in\mathbb{Z}} \tilde{\psi}'_{a^+,c^+}(x+m)\overline{\tilde{\psi}'}_{a^-,c^-}(x+n)\delta_{m,n}dx \\ &= \int_{[0,1]} \sum_{m\in\mathbb{Z}} \tilde{\psi}'_{a^+,c^+}(x+m)\overline{\tilde{\psi}'}_{a^-,c^-}(x+m)dx \\ &= \int_{\mathbb{R}} \tilde{\psi}'_{a^+,c^+}(x)\overline{\tilde{\psi}'}_{a^-,c^-}(x)dx = \langle\tilde{\psi}'_{a^+,c^+}|\tilde{\psi}'_{a^-,c^-}\rangle = \langle\psi_{a^+,c^+}|\psi_{a^-,c^-}\rangle, \quad (78)\end{aligned}$$

where in the last two equalities we use Dirac's bra-ket notation for the scalar product in the Hilbert space $L^2(\mathbb{R})$ and unitarity of the transformations in (47) and (48) with respect to this scalar product. Substituting now (49) and, by using analyticity of Faddeev's quantum dilogarithm, shifting the integration variable in imaginary direction, we finally obtain

$$\begin{aligned}&\langle\psi_{a^+,c^+}|\psi_{a^-,c^-}\rangle e^{\pi i(a^-+c^+-a^+-c^-)/6\hbar} \\ &= e^{\pi i(a^-(a^-+c^-)-a^+(a^++c^+))/\hbar} \int_{\mathbb{R}} \frac{\Phi_\hbar\left(x+\frac{i}{\sqrt{\hbar}}(a^-+c^-)\right)}{\Phi_\hbar\left(x-\frac{i}{\sqrt{\hbar}}(a^++c^+)\right)} e^{2\pi(a^++a^-)x/\sqrt{\hbar}}dx \\ &= e^{\pi i(a^-c^+-a^+c^-)/\hbar} \int_{\mathbb{R}} \frac{\Phi_\hbar\left(x+\frac{i}{2\sqrt{\hbar}}(a^++a^-+c^++c^-)\right)}{\Phi_\hbar\left(x-\frac{i}{2\sqrt{\hbar}}(a^++a^-+c^++c^-)\right)} e^{2\pi(a^++a^-)x/\sqrt{\hbar}}dx\end{aligned}$$

$$= e^{\pi i(a^-c^+-a^+c^-)/\hbar}\varphi\left(\frac{i}{\sqrt{\hbar}}(a^++a^-+c^++c^-), -\frac{i}{\sqrt{\hbar}}(a^++a^-)\right)$$
$$= e^{\pi i(a^-c^+-a^+c^-)}f_\hbar(w(\alpha)). \tag{79}$$

□

Remark 4. Let (X,α) be a shaped Δ-triangulation containing Y as a sub-triangulation, and let $F(X,\alpha,s)$ be any one of the two partition functions. Then, from Theorem 3 it easily follows that one has the following factorization

$$F(X,\alpha,s) = F(Y,\alpha)F(X',\alpha',s), \tag{80}$$

where (X',α') is obtained from (X,α) by applying a *shaped* $2-0$ *Pachner move*: X' is obtained from X by removing the interior part of Y and identifying the remaining two triangular faces with each other, and

$$\alpha' = \alpha\big|_{\Delta_{3,1}(X\setminus\mathring{Y})}. \tag{81}$$

As Theorem 3 also implies that $R_\hbar(Y,\alpha) = |C_\hbar(Y,\alpha)|^2$, factorization (80) is consistent with Conjecture 1.

4.1 Application to knot invariants

An H-triangulation is a pair (X,H) where X is a triangulation of a closed compact oriented 3-manifold M and H is a Hamiltonian 1-dimensional sub-complex of X representing a link $L\subset M$. Pachner moves of types $2-3$, $0-2$ and their inverses can be extended to *relative Pachner moves* of H-triangulations as follows. Let (X,H) be an H-triangulation. All moves which do not involve the edges of H are applicable and stay the same as in the case of usual triangulations. The moves which do not add or remove vertices and which involve edges from H are forbidden. A $2-0$ move which removes a vertex necessarily involves at least two edges of H which form two sides of a unique triangle t, and the move is applicable if and only if it does not involve any other edge from H, and upon application of the move one replaces those two edges of H by the third side of triangle t. Topologically, this move corresponds to an elementary isotopy of a polygonal link. Originally, relative Pachner moves of H-triangulations were introduced in [10] and it was conjectured that their finite sequences can relate any two topologically equivalent H-triangulations. The conjecture subsequently was proven in [4].

The shaped version of relative Pachner moves is straightforward, but under the moves which add or remove edges, initially balanced edges can become unbalanced so that topological significance of such moves is yet not clear. However, in the case of generalized shapes where the angles can be any real numbers, positive or negative, shapes become a very convenient means to encode Hamiltonian paths: one imposes the condition that the total dihedral angle around an edge is zero if the edge is from the Hamiltonian path and 2π otherwise. In that case, all shaped relative Pacher moves preserve this class of shapes. This scheme was used in [1] for defining renormalized partition functions of 1-vertex H-triangulations. Theorem 3 allows us to extend this definition to the case of H-triangulations with arbitrary number of vertices as follows.

Let (X, H, α) be a shaped H-Δ-triangulation. Let

$$\omega_\alpha \colon \Delta_1(X) \to \mathbb{R}_{>0}, \quad e \mapsto \sum_{T \in \Delta_3(X)} \alpha(T, e), \tag{82}$$

be the map which associates to each edge the total dihedral angle around it. Let F be either the partition function $R_\hbar$ or $|C_\hbar|$. Define

$$\nu_F(x) := \left| \Psi_\hbar \left(\frac{i}{2\sqrt{\hbar}} \left(1 - \frac{x}{\pi}\right)\right)\right|^{n_F} \tag{83}$$

where

$$n_F := \begin{cases} 1 & \text{if } F = |C_\hbar| \\ 2 & \text{if } F = R_\hbar \end{cases} \tag{84}$$

Let α_H be a generalized (i.e. possibly non-positive real valued) shape structure on X such that

$$\omega_{\alpha_H}(e) = \begin{cases} 0 & \text{if } e \in \Delta_1(H) \\ 2\pi & \text{otherwise} \end{cases} \tag{85}$$

We define the *renormalized partition function* by the following formula:

$$\tilde{F}(X, H) := \lim_{\alpha \to \alpha_H} \frac{F(X, \alpha)}{\prod_{e \in \Delta_1(H)} \nu_F(\omega_\alpha(e))} \tag{86}$$

Theorem 4. *Under the condition that the limit in* (86) *exists, the renormalized partition function* $\tilde{F}(X, H)$ *does not depend on the choice of* α_H *and is invariant under shaped relative Pachner moves of the pair* (X, H).

Proof. The proof follows from a straightforward verification of the shaped relative Pachner moves and shape gauge invariance. □

Acknowledgements

This work is based on joint works with Jørgen Ellegaard Andersen, Feng Luo and Grigory Vartanov, and I would like thank them for collaboration and fruitful discussions. This work was completed during my visit to MSC of Tsignhua University, and it is a great pleasure to thank people of MSC for their hospitality. I would also like to thank Tomotada Ohtsuki for inviting me to write this article and his hospitality at RIMS, Kyoto University. This work is supported in part by Swiss National Science Foundation.

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