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Computing equivariant characteristic classes of singular varieties

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Abstract

Starting from the classical theory we describe Hirzebruch class and the related Todd genus of a complex singular algebraic varieties. When the variety is equipped with an action of an algebraic torus we localize the Hirzebruch class at the fixed points of the action. We give some examples of computations.

The paper is based on the talk given in Kyoto, November 2012.

1 Riemann-Roch theorem

More than 150 years ego Bernhard Riemann [14] proved certain inequality, which in the contemporary language can be stated as follows: Let C be a smooth complete complex curve (i.e. a Riemann surface) of genus g and let L be a line bundle over C. Then

$$h^0(C;L) \geq \deg(L) + 1 - g\,,$$

where $h^0(C;L)$ denotes the dimension of the space of the holomorphic sections $H^0(C;L)$ and $\deg(L)$ is the degree of the line bundle which is equal to the integral of its first Chern class

$$\deg(L) = \int_C c_1(L)\,.$$

Few years later a student of Riemann, Gustav Roch computed the error term in the inequality. Now due to Serre duality we can write

Theorem 1

$$h^0(C;L) - h^1(C;L) = \deg(L) + 1 - g\,.$$

This number is equal to the Euler characteristic $\chi(C;L)$ of the line bundle L. The Riemann-Roch theorem became an indispensable tool for algebraic geometers. It was generalized to higher dimensions by Hirzebruch: Let X be a projective smooth algebraic variety, E an algebraic vector bundle over X, then

Theorem 2

$$\chi(X;E) = \int_X ch(E) \cup td(TX).$$

In this formula $ch(-)$ and $td(-)$ are expressions in Chern classes of a vector bundle. The first ingredient of the formula is the Chern character of a vector bundle. It satisfies two conditions:

- for a line bundle

$$ch(L) = exp(c_1(L)) = 1 + c_1(L) + \frac{c_1(L)^2}{2!} + \frac{c_1(L)^3}{3!} + \frac{c_1(L)^4}{4!} + \dots,$$

- Chern character is additive

$$ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2).$$

The second ingredient is the Todd class of the tangent bundle. This characteristic class satisfies:

- for a line bundle the Todd class is given by the formula

$$td(L) = \frac{c_1(L)}{1 - exp(-c_1(L))} = 1 + \frac{c_1(L)}{2} + \frac{c_1(L)^2}{12} - \frac{c_1(L)^4}{720} + \frac{c_1(L)^6}{30240} - \dots,$$

- Todd class is multiplicative

$$td(E_1 \oplus E_2) = td(E_1) \cup td(E_2).$$

The final elegant form of Riemann–Roch theorem was established by Alexander Grothendieck, who added the functorial flavour. The Grothendieck-Riemann-Roch deals with $K(X)$, the K-theory of coherent sheaves on X. The elements of $K(X)$ are formal differences of isomorphism classes of coherent sheaves $[E] - [F]$. If X is smooth, then the *coherent sheaves* in the definition of $K(X)$ can be replaced by the *locally free sheaves*, i.e. holomorphic vector bundles. For a proper map of smooth projective varieties $f : X \to Y$ let

$$f_!(E) := \sum_{i=0}^{\dim(X)} (-1)^i R^i f_*(E) \in K(Y).$$

Here $R^i f_*(E)$ is a sheaf with the stalk over y equal to $H^i(f^{-1}(y);E)$. In general $ch(f_!(E)) \neq f_* ch(E)$, but the following diagram commutes

Theorem 3

$$\begin{array}{ccccc} & & f_! & & \\ & K(X) & \to & K(Y) & \\ ch(-)\cup td(TX) & \downarrow & & \downarrow & ch(-)\cup td(TY) \\ & H^*(X) & \to & H^*(Y) & \\ & & f_* & & \end{array}$$

The functorial point of view allowed to extend the theory to singular varieties. Note that in the singular case there is no tangent bundle TX. If the functorial Todd class $td(X)$ existed for singular X then for an embedding $f : X \hookrightarrow M$ into a smooth variety the diagram of Grothendieck-Riemann-Roch would be commutative:

$$f_*\left(ch(E)\cup td(X)\right) = ch(f_!(E))\cup td(TM)\,.$$

For the trivial bundle $E = \mathcal{O}_X$ whose Chern character is equal to 1, we would obtain.

$$f_*\left(td(X)\right) = ch(f_!(\mathcal{O}_X))\cup td(TM) \in H^*(M)\,.$$

In fact this expression may be taken as the definition of the Todd class, or at least the definition of its image in $H^*(M)$. But it takes a lot of work (done by Baum–Fulton–MacPherson, [3]) to refine the construction of the Chern character to obtain the Todd class localized on X, i.e. to have $td^{BFM}(X) \in H^*(M, M\setminus X) \simeq H_*(X)$. From the practical point of view the difficulty is hidden in the computation of $ch(f_!(\mathcal{O}_X))$. Not only one has to resolve the sheaf $f_!(\mathcal{O}_X)$ in the sense of the homological algebra, i.e. find an exact sequence of vector bundles on M:

$$0 \to E_{\dim(M)} \to \cdots \to E_1 \to E_0 \to f_!(\mathcal{O}_X) \to 0\,.$$

Next one has to apply further nontrivial operations like "graph construction" to that sequence.

Another, nonequivariant definition of a Todd class for singular varieties was proposed by Brasselet-Schuermann-Yokura [6]. Now one imposes a "motivic" condition. One considers not a variety itself, but a map of varieties. To any map $f : X \to Y$ one associates a homology class $td(f : X \to Y) \in H_*(Y)$. One demands that

- if X is smooth, then

$$td(f : X \to M) = f_*(td(X) \cap [X] \in H_*(M)\,,$$

- if $Z \subset X$ is a closed subset, $U = X \setminus Z$, then

$$td(f : X \to M) = td(f_{|Z} : Z \to M) + td(f_{|U} : U \to M)\,.$$

The existence of a Todd class is nontrivial, but the computations are quite effective, provided, that one can resolve the singularities of X in a geometric sense. We will reserve the notation $td(-)$ for the motivic Todd class, while the Baum-Fulton-MacPherson construction will appear only occasionally.

2 Equivariant version

We will study the singular subvarieties admitting an action of a torus. We assume that the torus $\mathbb{T} = (\mathbb{C}^*)^r$ acts on smooth complex variety M, and X is preserved. For $T = \mathbb{C}^*$ we can say that we consider quasihomogeneous singularities, but actions of a bigger tori appear naturally and then more information about the variety is involved. We compute the discussed invariants in the equivariant cohomology $H^*_{\mathbb{T}}(M)$ instead of the usual cohomology. This cohomology group has reacher structure and it is a refinement of the usual cohomology. Let us consider cohomology with rational coefficients. For a point we have

$$H^*_{\mathbb{T}}(pt) = Sym(\mathbb{T}^\vee \otimes \mathbb{Q}) \simeq \mathbb{Q}[t_1, t_2 \dots t_r],$$

where $\mathbb{T}^\vee = \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$ is the character group. The group $H^*_{\mathbb{T}}(M)$ is and algebra over $H^*_{\mathbb{T}}(pt)$.

Main advantages of the equivariant cohomology are

- If M is a complex smooth compact algebraic variety, then
  $$H^*_{\mathbb{T}}(M) \simeq H^*_{\mathbb{T}}(pt) \otimes H^*(M)$$
  as modules over $H^*_{\mathbb{T}}(pt)$, see e.g. [8].
- Localization theorem: the kernel and cokernel of the restriction map $H^*_{\mathbb{T}}(M) \to H^*_{\mathbb{T}}(M^{\mathbb{T}})$ is a torsion $H^*_{\mathbb{T}}(pt)$-module, [13, 8]. If M is a complex smooth compact algebraic variety, then the restiction map is in addition injective.

The second property is called the Localization Theorem, in fact it is due to Borel (as pointed by Quillen [13]), but it was not formulated by him in that form. The conclusion is that almost everything about equivariant cohomology can be read from some data concentrated at the fixed points. The tip of this iceberg is the well known fact: the Euler characteristic $\chi(M)$ is equal to the Euler characteristic of the fixed point set $\chi(M^{\mathbb{T}})$. The situation when the fixed point set is finite is of particular interest.

The topological version of the localization theorem was transformed into a formula which is handy for computations. Suppose that M is a compact manifold of dimension n and the torus $(S^1)^r$ acts smoothly with a finite fixed point set. For a fixed point $p \in M^{\mathbb{T}}$ define the Euler class $e(p) \in H^{2n}_{\mathbb{T}}(\{p\})$ as the product of characters of $\mathbb{T}$ appearing in the tangent representation at p. The localization theorem implies that integral of a cohomology class over M can be expressed by the local data. Precisely

Theorem 4 (Atiyah-Bott[2] or Berline-Vergne [4]) *For a class $a \in H^*_{\mathbb{T}}(M)$ the integral is the sum of fractions*

$$\int_M a = \sum_{p \in M^{\mathbb{T}}} \frac{a_{|p}}{e(p)}.$$

Example 5 Suppose $M = \mathbb{P}^2$, $T = (\mathbb{C}^*)^3$. Then

$$M^{\mathbb{T}} = \{\, p_0 = [1:0:0],\ p_1 = [0:1:0],\ p_2 = [0:0:1] \,\}$$

is the fixed point set. Let $h := c_1(\mathcal{O}(-1))$ be Chern class of the tautological bundle. We apply Berline-Vergne formula to compute the integral for $a = h^2$

$$\int_{\mathbb{P}^2} h^2 = \frac{t_0^2}{(t_1-t_0)(t_2-t_0)} + \frac{t_1^2}{(t_0-t_1)(t_2-t_1)} + \frac{t_2^2}{(t_0-t_2)(t_1-t_2)} =$$

$$= -Res_{z=\infty}\frac{z^2}{(z-t_0)(z-t_1)(z-t_2)}$$

If we set $z = w^{-1}$, then the integral is equal to the coefficient of w in

$$\frac{w^{-2}}{(w^{-1}-t_0)(w^{-1}-t_1)(w^{-1}-t_2)} = 1$$

as it should be by obvious reasons. See [17] for further application of the residue method applied to computations based on Localization Theorem.

We plan to use Localization Theorem to compute some invariants of $\mathbb{T}$-invariant singular varieties $X \subset M$. The Todd class is of main interest here. Equally well we compute Hirzebruch class

$$td_y(X) \in H_*(X) \otimes \mathbb{Q}[y]\,,$$

which is equal to the formal combination of classes corresponding to sheaves of differential forms

$$\sum_{k=0}^{\dim X} (ch(\Omega_X^k) \cup td(TX)) \cap [X]\, y^k$$

for the smooth X case. The motivic generalizations of the Hirzebruch class to the singular case was proven to exist in [6]. Also there a relation with other characteristic classes was discussed. As a particular case $y = -1$ we obtain (after a suitable normalization) Chern-Schwartz-MacPherson class [10]. It is straightforward to obtain an equivariant version of the Hirzebruch class as it was done for the Todd class in [5] and for Chern classes in [12]. These classes live in the *equivariant homology*. Poincaré duality isomorphism identifies equivariant homology and cohomology of M. If $M^{\mathbb{T}}$ has isolated fixed points, then the class $td_y(X)$ without loss of information can be replaced by the image in $H^*_{\mathbb{T}}(M)$. In the notation we indicate the embedding. We write

$$td_y(X \to M) \in H^*_{\mathbb{T}}(M)\,.$$

Now the equivariant Todd class can be studied locally. One can consider the local Todd class

$$td(X \to M)_{|p} \in \prod_{k=0}^{\infty} H^k_{\mathbb{T}}(pt) = \mathbb{Q}[[t_1, t_2, \dots t_r]]$$

for a fixed point $p \in X$. More generally

$$td_y(X \to M)_{|p} \in \prod_{k=0}^{\infty} H^k_{\mathbb{T}}(pt)[y] = \mathbb{Q}[[t_1, t_2, \dots t_r]][y]$$

is an invariant of a $\mathbb{T}$-equivariant singularity germ. Let us concentrate on the case of td i.e. $y = 0$. Due to the integration formula of Theorem 4 every fixed point of the $\mathbb{T}$-action contributes to the integral

$$\int_X td(X) = \int_M td(X \to M) = \chi(X; \mathcal{O}_X),$$

that is to the Todd genus of X. The contribution of each point $p \in X^{\mathbb{T}}$ is equal to $\frac{td(X \to M)_{|p}}{e(p)}$. Regardless of the complicated definition the outcome is just a Laurent series in t_i. Moreover if we introduce new variables $T_i = e^{-t_i}$ the answer is of the form

$$\frac{W(T_1, T_2, \dots T_r)}{\prod_{i=1}^n (1 - e^{-w_i})} = \frac{W(T_1, T_2, \dots T_r)}{\prod_{i=1}^n (1 - \prod_{j=1}^r T_i^{a_i^j})}.$$

Here n is the dimension of the ambient space, and $w_i = \sum_{j=1}^r a_i^j t_j$ for $i = 1, 2, \dots, n$ are the weight vectors of the tangent representation. The numerator $W(T) = W(T_1, T_2, \dots T_r)$ is a *polynomial* in T_i. If the point $p \in X$ is smooth, then

$$W(T_1, T_2, \dots T_r) = 1,$$

In general the local contribution to the χ_y-genus of X is the local Hirzebruch class

$$\frac{td_y(X \to M)_{|p}}{e(p)} = \prod \frac{1 + y\, e^{-w_i}}{1 - e^{-w_i}},$$

A question arises for singular points: How to compute the invariant $W(T)$ effectively? Using motivic nature of Todd genus one can decompose X into smooth strata and compute the Todd class separately for each stratum. Then one has to sum up the strata contribution. We will also propose another method, based on localization theorem, which is effective e.g. for determinant varieties.

We note that for toric varieties the answer is due to Brion-Vergne [4] and Brylinski-Zhang [5]. The Hirzebruch class for toric varieties is studied in [11] (although not from the equivariant point of view) .

3 Computation by means of a resolution

Below we will present some computations just for Todd class, i.e. for $y=0$.

Example 6 *Whitney umbrella.* We will show step by step how to compute the equivariant Todd class localized at the origin.

Consider the torus $\mathbb{T}=(\mathbb{C}^*)^2$ acting on $\mathbb{C}^3$ by the formula

$$(T_1\,,T_2\,)\cdot(x_1,x_2,x_3)=(\,T_1T_2x_1\,,T_1x_2\,,T_2^2x_3\,).$$

Let us denote the characters $T\to\mathbb{C}^*$

$$t_1:(T_1,T_2)\to T_1,\quad t_2:(T_1,T_2)\to T_2\,.$$

The considered representation have the characters $t_1+t_2\,,t_1\,,2t_2$. The action preserves the Whitney umbrella

$$X=\{(x_1,x_2,x_3)\in\mathbb{C}^3\,|\,x_1^2-x_2^2x_3=0\}\,.$$

We will show that the local equivariant Todd class divided by the Euler class is equal to

$$\frac{td(X\to\mathbb{C}^3)_{|0}}{e(0)}=\frac{1+e^{-(t_1+t_2)}}{(1-e^{-t_1})\,(1-e^{-2t_2})}$$

We will compute the Todd class by additivity property. Let

$$Z=\{(x_1,x_2,x_3)\in\mathbb{C}^3\,|\,x_1=0,x_2=0\}\quad\text{and}\quad X^o=X\setminus Z\,.$$

We have

$$td(X\to\mathbb{C}^3)=td(X^o\to\mathbb{C}^3)+td(Z\to\mathbb{C}^3)\,.$$

Let

$$f:\widetilde{X}=\mathbb{C}^2\to\mathbb{C}^3\,,\quad f(u,v)=(uv,u,v^2)\,,$$

be the resolution of the Whitney Umbrela. The map f is proper and it is an isomorphism of the open subsets

$$\widetilde{X}^o=(\mathbb{C}\setminus\{0\})\times\mathbb{C}\to X^o.$$

The map f is equivariant provided that the characters of $\mathbb{C}^2$ are $t_1\,,t_2$. Therefore

$$\begin{aligned}td(X\to\mathbb{C}^3)_{|0}&=td(X^o\to\mathbb{C}^3)_{|0}+td(Z\to\mathbb{C}^3)_{|0}\\&=f_*td(\widetilde{X}^o\to\mathbb{C}^2)_{|0}+td(Z\to\mathbb{C}^3)_{|0}\,.\end{aligned}$$

Let $\mathbb{C}\hookrightarrow\mathbb{C}^2$ be the inclusion as the v-coordinate line. First let us compute

$$td(\widetilde{X}^o\to\mathbb{C}^2)_{|0}=td(\mathbb{C}^2)_{|0}-td(\mathbb{C}\hookrightarrow\mathbb{C}^2)_{|0}=\frac{t_1t_2}{(1-e^{-t_1})(1-e^{-t_2})}-t_1\frac{t_2}{1-e^{-t_2}}\,.$$

The map $f_*: H^k_{\mathbb{T}}(\mathbb{C}^2) \to H^{k+2}_{\mathbb{T}}(\mathbb{C}^3)$ is a map of $H^*_T(pt)$-modules sending 1 to $\deg(X) = 2(t_1 + t_2)$, therefore $td(X \to \mathbb{C}^3)$ is equal to

$$2(t_1+t_2)\left(\frac{t_1t_2}{(1-e^{-t_1})(1-e^{-t_2})} - t_1\frac{t_2}{1-e^{-t_2}}\right) + (t_1+t_2)t_1\frac{2t_2}{1-e^{-2t_2}} =$$

$$= \frac{2t_1t_2(t_1+t_2)\left(1-e^{-2(t_1+t_2)}\right)}{(1-e^{-t_1})\left(1-e^{-2t_2}\right)\left(1-e^{-(t_1+t_2)}\right)} = td(\mathbb{C}^3)_{|0}\left(1-e^{-2(t_1t_2)}\right).$$

The image of the Baum-Fulton-MacPherson class is also equal to

$$td(\mathbb{C}^3)_{|0} \cup ch(\mathcal{O}_X) = td(\mathbb{C}^3)_{|0}(1-e^{-2(t_1+t_2)}).$$

In general, if X is a complete intersection $X = \bigcap_{i=1}^k \{f_i = 0\}$ then we have

$$ch(\mathcal{O}_X) = \prod_{i=1}^k (1 - e^{-\deg(f_i)}),$$

where $\deg(f) \in \mathbb{T}^\vee = H^2_{\mathbb{T}}(pt)$ is understood as the multidegree of a $\mathbb{T}$-invariant function (if $\mathbb{T} = \mathbb{C}^*$ this is the quasihomogeneous degree of f). It happens in may cases that $td(X \to M) = td(M)ch(\mathcal{O}_X)$ but the equality does not hold in general. An easy counterexample is the cusp $x^3 = y^2$ in $\mathbb{C}^2$. In fact the equality is quite rare. For affine cones over a smooth hyperplane of degree d in $\mathbb{P}^{n-1}$ the equality holds if and only if $d \le n$. One can say that the difference between Baum-Fulton-MacPherson class and the Brasselet-Schuermann-Yokura class measures how difficult the singularity is.

There might be a similar relation for full Hirzebruch class. For the Chern-Schwartz-MacPherson class, i.e. for $y = -1$, the discrepancy between the expected value and the actual value of the discussed expression was widely studied. In the case of isolated singularities the difference is equal up to a sign to the Milnor number, [7, §14.1].

4 Procedure to compute $td(X)$ via localization

Now, we describe another a method of computing the local Hirzebruch class. As before we present the computations for Todd class. Suppose that $X \subset M$ is an invariant subvariety in a $\mathbb{T}$-manifold with isolated fixed points:

$$\int_M td(X \to M) = \sum_{p \in X^{\mathbb{T}}} \frac{td(X \to M)_{|p}}{e(p)}$$

If X has a decomposition into algebraic cells the integral is equal to the number of 0–cells. Suppose the point $p \in X^{\mathbb{T}}$ is smooth. Then the contribution to $\int_M td(X \to M)$ is equal to

$$td(X \to M)_{|p} = \prod \frac{w_i}{1 - exp(-w_i)} \cdot \prod n_j = e(p) \prod \frac{1}{1 - exp(-w_i)},$$

Here

- w_i are the characters of tangent representation T_pX,
- n_j are the characters of normal representation.

Suppose that all but one points of X are smooth. Then one can compute

$$td(X \to M)_{|p_{sing}} = e(p_{sing}) \left(\int_M td(X \to M) - \sum_{\text{smooth } p \in X^{\mathbb{T}}} \frac{td(X \to M)_{|p}}{e(p)} \right)$$

$$\frac{td(X \to M)_{|p_{sing}}}{e(p_{sing})} = \int_M td(X \to M) - \sum_{\text{smooth } p \in X^{\mathbb{T}}} \prod \frac{1}{1 - exp(-w_i)}$$

Amazingly, computing the local Todd class of a singular variety this way one does not use its definition but only the knowledge how it looks like at the smooth points.

We will show how to compute the local Todd class when X is the set of singular square matrices: X is a hypersurface in $M(n \times n; \mathbb{C}) = \mathbb{C}^{n^2}$ given by the equation $\det(A) = 0$. This set can be compactified: the matrix defines a linear map $\mathbb{C}^n \to \mathbb{C}^n$, and its graph is an element of the Grassmannian $Gr_n(\mathbb{C}^{2n})$. The closure of the set of singular matrices is the Schubert variety of codimension one.

Example 7 *Let X be the codimension one Schubert variety in $Gr_2(\mathbb{C}^4)$.*

The torus $T = (\mathbb{C}^*)^4$ acts on $Gr_2(\mathbb{C}^4)$. The action is induced from the action on $\mathbb{C}^4 = \mathbb{C}^2_s \oplus \mathbb{C}^2_t$. The space of linear maps $\mathrm{Hom}(\mathbb{C}^2_s, \mathbb{C}^2_t)$ is identified with a subset of $Gr_2(\mathbb{C}^4)$. Let $\overline{X}$ be the set of planes satisfying the Schubert condition

$$\overline{X} = \{V \in Gr_2(\mathbb{C}^4 : V \cap (\mathbb{C}^2_s \oplus 0) \neq 0\}.$$

Assume that the action on $\mathbb{C}^2_s$ component is through negative characters $-s_1$ and $-s_2$ and on the second component $\mathbb{C}^2_t$ is through t_1 and t_2. The action of $\mathbb{T}$ on $Gr_2(\mathbb{C}^4)$ has exactly six fixed points which are the coordinate planes. The variety $\overline{X}$ contains all fixed points except $0 \oplus \mathbb{C}^2_t$. The only singular point of $\overline{X}$ is $p_{sing} = \mathbb{C}^2_s \oplus 0$, which corresponds to $0 \in X \subset \mathrm{Hom}(\mathbb{C}^2_s, \mathbb{C}^2_t)$ in the affine neighbourhood. The action of $\mathbb{T}$ on the tangent space is through characters $s_i + t_j$. The equation of $X = \overline{X} \cap \mathrm{Hom}(\mathbb{C}^2_s, \mathbb{C}^2_t)$ is

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0.$$

The characters of the coordinates are the following:

$$\begin{array}{ll} \chi_a = s_1 + t_1, & \chi_b = s_1 + t_2, \\ \chi_c = s_2 + t_1, & \chi_d = s_2 + t_2. \end{array}$$

Let us introduce new variables $S_i = e^{-s_i}$ and $T_i = e^{-t_i}$, since in the formulas for $\frac{td(X \to \mathbb{C}^4)_{|p}}{e(p)} = \prod \frac{1}{1-e^{-x_i}}$ only e^{-x_i} appears, not the character x_i itself. We compute the contribution to Berline-Vergne formula at the smooth points. For example the fixed point $p = lin\{(1,0,0,0),(0,0,1,0)\}$ contributes with

$$\frac{1}{\left(1-\frac{S_1}{S_2}\right)\left(1-\frac{1}{S_2T_1}\right)\left(1-\frac{T_2}{T_1}\right)} = \frac{S_2^2T_1^2}{(S_1-S_2)(1-S_2T_1)(T_2-T_1)}.$$

Now let us compute the Todd class at the singular point:

$$\frac{td(X \to \mathbb{C}^4)_{|p_{12}}}{e(p_{12})} = \left(\chi(\mathcal{O}_X) - \sum_{\text{smooth } p \in X^T} td(X \to \mathbb{C}^4)_{|p}\right) =$$

$$1 - \frac{S_2^2T_1^2}{(S_1-S_2)(1-S_2T_1)(T_2-T_1)} - \frac{S_1^2T_1^2}{(S_2-S_1)(1-S_1T_1)(T_2-T_1)} -$$
$$- \frac{S_2^2T_2^2}{(S_1-S_2)(1-S_2T_2)(T_1-T_2)} - \frac{S_1^2T_2^2}{(S_2-S_1)(1-S_1T_2)(T_1-T_2)} =$$
$$= \frac{1-S_1S_2T_1T_2}{(1-S_1T_1)(1-S_1T_2)(1-S_2T_1)(1-S_2T_2)}$$

Again we see that

$$td(X \to \mathbb{C}^4) = td(\mathbb{C}^4)(1 - S_1S_2T_1T_2) =$$
$$= td(\mathbb{C}^4)(1 - e^{-(s_1+s_2+t_1+t_2)}) = td(\mathbb{C}^4)(1 - e^{-\deg(f)}),$$

where f is the function defining the variety X. The formula for $td_y(X)_{|p_{12}}$ is much more complicated. On the other hand the class $\frac{td_y}{e}$ of the open cell $Gr_2(\mathbb{C}^4) \setminus X$ is quite friendly

$$td(\mathbb{C}^4)(1+y)^2\, S_1S_2T_1T_2\Big((1-y)(1-y\, S_1S_2T_1T_2) + y(S_1+S_2)(T_1+T_2)\Big)$$

Applying this method inductively as described in [15] one can compute the local Hizebruch class of determinant varieties of higher dimensions. The result for all y are messy, but for the special values $y = 1$, i.e. the local L-class we always have

$$\frac{td_1((\mathbb{C}^{n^2} \setminus X) \to \mathbb{C}^{n^2})}{e(0)} = 2^n\, td(\mathbb{C}^{n^2}) \prod_{i<j}(S_i+S_j) \prod_{i<j}(T_i+T_j) \prod_i (S_iT_i)$$

with the convention $T_i = e^{-t_i}$ and $S_i = e^{-s_i}$. For the Todd class in higher dimensions we have

$$\frac{td(X \to \mathbb{C}^{n^2})}{e(0)} = td(\mathbb{C}^{n^2})(1 - \prod_{i=1}^{n} S_i T_i)).$$

To finish the list of computations let us display the td_y-class for the open cell for $n = 4$ but with the substitution $s_1 = s_2 = s_3 = s_4 = 0$, $t_1 = t_2 = t_3 = t_4 = t$, that is for the radial action of $\mathbb{C}^*$ on $\mathbb{C}^4$. The common factor is $td(\mathbb{C}^{16})(1+y)^4 T^4$. It should be multiplied by the sum

$$\begin{aligned}
&(1-y)^2\,(1+y^2)\,(1-y+y^2)+\\
&+T\,16\,(1-y)\,y\,(1-y+y^2)^2+\\
&+T^2\,12\,(1-y)^2\,y^2\,(10-13y+10y^2)+\\
&+T^3\,16\,(1-y)\,y^2\,(1+44y-79y^2+44y^3+y^4)+\\
&-T^4\,(1-y)^2\,y\,(1+29y+62y^2-2902y^3+62y^4+29y^5+y^6)+\\
&-T^5\,16\,(1-y)\,y^3\,(11+62y-492y^2+62y^3+11y^4)+\\
&+T^6\,4y^3\,(9-86y-1139y^2+3456y^3-1139y^4-86y^5+9y^6)+\\
&+T^7\,16\,(1-y)\,y^4\,(11+62y-492y^2+62y^3+11y^4)+\\
&-T^8\,(1-y)^2\,y^3\,(1+29y+62y^2-2902y^3+62y^4+29y^5+y^6)+\\
&-T^9\,16\,(1-y)\,y^5\,(1+44y-79y^2+44y^3+y^4)+\\
&+T^{10}\,12\,(1-y)^2\,y^6\,(10-13y+10y^2)+\\
&-T^{11}\,16\,(1-y)\,y^6\,(1-y+y^2)^2+\\
&+T^{12}\,(1-y)^2\,y^6\,(1+y^2)\,(1-y+y^2)
\end{aligned}$$

It remains to say that the meaning of this coefficients is not clear. When written in that form one can see certain symmetry, which comes from Poincaré duality. On the other hand after substitution $y = -1-\delta$, $T = 1+S$ one finds that all coefficients are nonnegative. This feature is related to some positivity of the logarithmic vector fields sheaves on the resolution of the determinant variety. In detail it will be explained in [16].

Also, by the substitution $T = e^{-(y+1)t}$ and taking the limit $y \to -1$ we obtain the Chern-Schwartz-MacPherson class studied in [1]. The corresponding positivity property in the nonequivariant case was proven in [9] Hopefully computing the entire Hirzebruch class one may gain better insight into Chern-Schwartz-MacPherson classes and find nontrivial relation between them.

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