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ON THE DIMENSION DATUM OF A SUBGROUP

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1. THE PROBLEMS

Langlands [6] has suggested to use the dimension datum of a subgroup as a key ingredient in his programme "Beyond endoscopy". In this expository article, we will survey recent advances in the theory of dimension data by the authors ([1], [12]). No proof is given here.

Definition 1.1. Let G be a compact Lie group and let H be a closed subgroup of G. We define the *dimension datum* of H (as a subgroup of G), to be the following function on the unitary dual $\widehat{G}$ of G:

$$\mathscr{D}_H : \widehat{G} \to \mathbb{Z}, \qquad V \mapsto \dim V^H.$$

1.2. **Variants.** We may consider the same notion when G and H are complex reductive groups, and $\widehat{G}$ is taken to be the set of equivalence classes of irreducible rational representations. By the relation between compact groups and complex reductive groups ([1, Section 8]), studying the dimension data in this context is exactly the same as studying them in the compact Lie group context. By the Lefschetz principle, we can further replace "complex reductive groups" by "reductive groups over F" for any algebraically closed field F of characteristic 0, such as $F = \bar{\mathbb{Q}}_\ell$ without changing the essence of this notion. These variants are what actually occur in Langlands' programme.

1.3. **Alternative formulations.** The dimension datum $\mathscr{D}_H$ is also equivalently encoded in the following objects:

- the equivalence class of the G-module $L^2(G/H)$;
- the push forward $\mu_H^\natural$ of μ_H by the composition $H \hookrightarrow G \to G^\natural$, where μ_H is the normalized Haar measure of H, and $G^\natural$ is the space of conjugacy classes of G.

The equivalences follow from the Frobenius reciprocity and the Peter-Weyl theorem. The convenience of using these alternative formulations is the reason why we choose to work in the compact Lie group context.

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1.4. **The role in the Langlands program I.** Let F be a number field and let L_F be the conjectural Langlands group. Let $\mathbf{G}$ be a connected reductive group over F and let ${}^L\mathbf{G}$ be the L-group of $\mathbf{G}$. Consider an L-homomorphism $\phi : L_F \to {}^L\mathbf{G}$. Then ϕ maps $\ker(L_F \to W_F)$ into $\mathbf{G}^\vee$, the (complex) dual group of $\mathbf{G}$. Let H_ϕ be the Zariski closure in $\mathbf{G}^\vee$ of $\phi(\ker(L_F \to W_F))$. Then H_ϕ is a complex reductive group normalized by $\phi(L_F)$. The group $\mathscr{H}_\phi := H_\phi\phi(L_F)$ is of great interests in the Langlands program [2].

Conjecturally [6], when ϕ is the Langlands parameter of an automorphic representation π of $\mathbf{G}(\mathbb{A}_F)$, $\mathscr{D}_{\mathscr{H}_\phi}(V)$ is the order of the pole of $L(s,\pi,V)$ at $s=1$. Therefore, Langlands suggested to pinpoint ${}^\lambda H_\pi := \mathscr{H}_\phi$ through its dimension datum. Thus it is important to investigate: *To what extent is H (up to G-conjugacy) determined by its dimension datum $\mathscr{D}_H$?*

In particular, Langlands wrote that it will be important to establish the following result, which we did in [1].

Theorem. *If the function $\mathscr{D}_H$ is given then there are only finitely many possibilities for the conjugacy class of H.*

It turns out that this number of possibilities is usually small when G and H are connected, and one may even consider those cases when this number is > 1 exceptional. Therefore it is natural to consider:

Problem. Assume that G is connected. Identify all (H,H') with H, H' connected such that $\mathscr{D}_H = \mathscr{D}_{H'}$ and H is not G-conjugate to H'.

See Theorems 2.1, 2.2, 3.1 for results about this problem. We will describe the complete solution to this problem by Jun Yu in Section 4.

1.5. **Linear relations.** Observe that $\mathscr{D}_H$ lives in the vector space $\mathbb{R}^{\hat{G}}$ of real-valued functions on $\hat{G}$, while $\mu_H^\natural$ is in the vector space $\mathscr{M}$ of real-valued measures on $G^\natural$. The Peter-Weyl theorem says that

$$D : \mathscr{M} \to \mathbb{R}^{\widehat{G}}, \qquad D(\mu)(V) = \int_{G^\natural} \mathrm{Tr}(g|V)d\mu(g)$$

is a linear injection sending $\mu_H^\natural$ to $\mathscr{D}_H$. Therefore, it makes sense to consider linear relations among $\mathscr{D}_H$'s for varying H's, or among $\mu_H^\natural$'s, and they are the same relations.

1.6. **The role in the Langlands program II.** There is another question raised by Langlands in [6, 1.1 and 1.6], which is related to linear relations. Denote by $\mathbb{R}[\widehat{G}]$ the free $\mathbb{R}$-module with basis $\widehat{G}$, so that its dual space $\mathrm{Hom}(\mathbb{R}[\widehat{G}],\mathbb{R})$ is $\mathbb{R}^{\widehat{G}}$.

Problem. Let $\mathscr{L}$ be a set of subgroups of G. Can we find a collection $\{a_H\}_{H\in\mathscr{L}}$ of elements in $\mathbb{R}[\widehat{G}]$ with the following property?

$$\text{For all } H, H' \in \mathscr{L}, \qquad (a_H, \mathscr{D}_{H'}) = \begin{cases} 1 & \text{if } H' \sim_{\mathrm{LP}} H, \\ 0 & \text{if } H' \not\sim_{\mathrm{LP}} H, \end{cases}$$

where $(-,-)$ is the natural pairing between $\mathbb{R}[\widehat{G}]$ and $\mathbb{R}^{\widehat{G}}$. We refer to [6] or [1] for the definition of $\prec_{\mathrm{LP}}$.

Langlands proposed that the existence of $\{a_H\}_{H\in\mathscr{L}}$ may facilitate a way to deal with the dimension data of ${}^{\lambda}H_{\pi}$ using the trace formula. It is very easy to observe:

Lemma. *If $\{a_H\}_{H\in\mathscr{L}}$ exists, then $\{\mathscr{D}_{H_1},\ldots,\mathscr{D}_{H_n}\}$ is linearly independent for any $H_1,\ldots,H_n\in\mathscr{L}$ such that $\mathscr{D}_{H_i}\neq\mathscr{D}_{H_j}$ for $i\neq j$.*

Therefore, non-trivial linear relations are obstructions to what Langlands proposed. In [6, 1.2], Langlands started with the class $\mathscr{L}_1 = \{H\subset G : H\to G/G^\circ \text{ is surjective}\}$. He then analyzed the case $G=\mathrm{SU}(2)\times F$, where F is a finite group, in [6, 1.3] and decided that it is necessary to restrict to a smaller class ([6, 1.4]): $\mathscr{L}_2=\{H\subset G : H\cap G^\circ = H^\circ \text{ and } H/H^\circ\simeq G/G^\circ\}$ so that there is a chance of an affirmative answer for the above question (Langlands expects this restricted class to be enough for his purpose in that $\mathscr{L}_2$ should contain all his conjectural groups ${}^{\lambda}H_{\pi}$'s; see also [2, Section 5]). Indeed for $G=\mathrm{SU}(2)\times F$ one can show the existence of $\{a_H\}_{H\in\mathscr{L}}$ for $\mathscr{L}=\mathscr{L}_2$. However, Langlans suspected ([6, discussions following (14)]) that in general the above question can not be solved exactly (for $\mathscr{L}=\mathscr{L}_2$). We confirmed this in [1] (see also Corollary3.3) by finding non-trivial linear relations. Again, such relations are relatively rare and it is natural to consider:

Problem$'$. Assume that G is connected. Identify all linear relations among $\{\mathscr{D}_H : H\subset G, H \text{ connected}\}$.

Again we will describe Jun Yu's solution to this problem in Section 4.

2. THE WORK OF LARSEN AND PINK

The most important result about the dimension datum is in the work of Larsen and Pink [8]. The key results there are:

Theorem 2.1. *If H_1 and H_2 are connected semisimple subgroups of G such that $\mathscr{D}_{H_1}=\mathscr{D}_{H_2}$, then H_1 is isomorphic to H_2.*

Theorem 2.2. *If $G=\mathrm{U}(n)$ and H_1 and H_2 are connected semisimple subgroups of G such that each H_i acts irreducibly on $\mathbb{C}^n$ and $\mathscr{D}_{H_1}=\mathscr{D}_{H_2}$, then H_1 is G-conjugate to H_2.*

These striking results are very inspiring and encouraging to the intended application in [6]. However, for that application it is desirable to have the semisimplicity hypothesis removed in those theorems. It seems that it was widely believed that indeed the semisimplicity hypothesis is unnecessary (private communications). However, we will show (Theorem 3.1) that this is not true.

2.3. **Translating the problem to root data.** Assume that G is connected. Let $(X,R,\check{X},\check{R})$ be the root datum of G. Since this gadget determines (and is determined by) G up to isomorphism, philosophically we can translate Problems 1.4 and 1.6$'$ to problems about $(X,R,\check{X},\check{R})$. Larsen and Pink established a formalism for doing so, which has been the foundation of all subsequent works. Below we review just enough of the Larsen-Pink formalism so that we can present Jun Yu's results in Section 4.

2.4. **Metrized root datum and metrized root system.** It will be useful to introduce an invariant Riemannian metric on G. This amounts to give an inner product m on $\check{X}_{\mathbb{R}} := \check{X} \otimes \mathbb{R}$ invariant under the Weyl group W. The 5-tuple $(X, R, \check{X}, \check{R}, m)$ then can be reduced to a triple (X, R, m). We call the triple (X, R, m) a metrized root datum.

We will not elaborate the definition of (X, R, m) being a metrized root datum here. Similarly, we are going to use the notion of "metrized $\mathbb{R}$-root datum" without any explanation more than that $(X_{\mathbb{R}}, R, m)$ is a metrized $\mathbb{R}$-root datum when (X, R, m) is a root datum. We also trust that the reader can figure out the meaning of semisimplicity of a metrized ($\mathbb{R}$-) root datum, and we call a semisimple metrized $\mathbb{R}$-root datum a *metrized root system*.

2.5. **The polynomial $F_{\Phi,\Gamma}$.** Assume H is connected and let T be a maximal torus of H. The natural map $T \to G^\natural$ is surjective onto the support of $\mu_H^\natural$, and the pull back of $\mu_H^\natural$ by this map is of the form $F_H \cdot \mu_T$, where F_H is a regular function on T by the Weyl integration formula. Notice that the ring of (complex-valued) regular function on T is simply the group algebra $\mathbb{C}[Y]$ of Y. We will not give Larsen-Pink's formula for $F_H \in \mathbb{C}[Y]$ here, but merely notice that it depends only on the set of roots $\Phi = R(H, T)$ and the finite group $\Gamma = N_G(T)/Z_G(T)$ as a subgroup of $\mathrm{Aut}(T) = \mathrm{Aut}(Y)$. Therefore it will be denoted by $F_{\Phi,\Gamma}$ also.

2.6. **Varying the tori.** For any torus T in G, let D_T be the linear span of $\mu_H^\natural$ for all connected H with maximal torus T. Then one can show that the subspaces $\{D_{T_i}\}_i$ are linearly independent (i.e. the sum $\sum D_{T_i}$ is direct) if the T_i's are pairwise non-conjugate in G. Therefore, when considering Problems 1.4 and 1.6$'$, about equalities/linear relations among dimension data, it suffices to work with one torus at a time.

2.7. **Conclusion.** Fix a torus T in G. Put $\Gamma = N_G(T)/Z_G(T)$ and $\mathscr{P} = \{R(H, T) : T \subset H \subset G\}$, where H ranges over all connected closed subgroups of G with maximal torus T. According to the above discussion, Problems 1.4 and 1.6$'$ amount to

Problem. Study the equalities/linear relations among $\{F_{\Phi,\Gamma} : \Phi \in \mathscr{P}\}$.

3. The first examples

The results in this section are from [1].

3.1. The following result gives the first known examples of $\mathscr{D}_{H_1} = \mathscr{D}_{H_2}$ with H_1 not isomorphic to H_2, and H_1, H_2 connected.

Theorem. *Let $m \geqslant 1$ be an integer. Put*

- $G = \mathrm{SU}(4m + 2)$,

- $H_1 = \mathrm{U}(2m+1)$, *embedded in* G *through* st $\oplus$ st*,
- $H_2 = \mathrm{Sp}(m) \times \mathrm{SO}(2m+2)$, *embedded in* G *in the only way.*

Then $\mathscr{D}_{H_1} = \mathscr{D}_{H_2}$.

We remark that connected, non-conjugate subgroups H_1, H_2 with $\mathscr{D}_{H_1} = \mathscr{D}_{H_2}$ are known to exist from the work of Larsen and Pink. However, the groups involved are of very large dimensions and not very easy to describe. In contrast, our example for $m = 1$ involves only groups of fairly small dimensions.

3.2. The following result gives the first known non-trivial linear relations (which is not an equality) among dimension data of connected subgroups.

Theorem. *Let* $m \geqslant 1$ *be an integer. Put*

- $G = \mathrm{SU}(4m)$,
- $H_1 = \mathrm{U}(2m)$, *embedded in* G *through* st $\oplus$ st*,
- $H_2 = \mathrm{Sp}(m-1) \times \mathrm{SO}(2m+2)$, *embedded in* G *in the only way,*
- $H_3 = \mathrm{Sp}(m) \times \mathrm{SO}(2m)$.

Then $2\mathscr{D}_{H_1} = \mathscr{D}_{H_2} + \mathscr{D}_{H_3}$.

Corollary 3.3. *Let* $m \geqslant 1$. *Let* H_1, H_2, H_3 *be as in Theorem 3.2, and let* G *be any connected compact Lie group containing* $\mathrm{SU}(4m)$. *Then the answer to Problem 1.6 is negative for any class* $\mathscr{L}$ *containing* $\{H_1, H_2, H_3\}$

3.4. **Application.** It is well-known, following the celebrated Sunada method ([4], [9], [10]), that non-conjugate subgroups with identical dimension data can be used to construct isospectral manifolds. These are Riemannian manifolds with identical Laplacian spectrum (counting multiplicities), in other words, counterexamples to the famous problem "Can you hear the shape of a drum?"

However, the following has been an outstanding problem for decades: *Can we have isospectral* M_1, M_2 *such that* M_1 *and* M_2 *are compact, connected and simply connected, but non-diffeomorphic?* In [1] we gave the first example to answer this question affirmatively.

Theorem. *Let* G, H_1, *and* H_2 *be as in Theorem 3.1. Then the compact homogeneous Riemannian manifolds* $M_1 := G/H_1$ *and* $M_2 := G/H_2$ *are isospectral, simply connected, and have different homotopy types.*

Indeed, M_1 and M_2 are isospectral by a theorem of Sutton [10], and it is easy to show $\pi_1(M_1) = \pi_1(M_2) = 1$, $\pi_2(M_1) \simeq \mathbb{Z}$, $\pi_2(M_2) \simeq \mathbb{Z}/2\mathbb{Z}$.

4. Classifying all equalities/linear relations

In this section, we will describe Jun Yu's solution [12] to Problem 2.7, which is equivalent to Problems 1.4 and 1.6′.

Let T be a torus in G as before. We may assume that T is contained in the maximal torus S of G such that $X^*(S) = X$. Then to specify T is to give a surjection $X \twoheadrightarrow Y := X^*(T)$. For any H with maximal torus T, our fixed Riemannian metric on G induces invariant Riemannian metric on H, and the corresponding inner product on $\check{Y}$ is simply the restriction $m|_{\check{Y}}$ of m on $\check{Y} \subset \check{X}$.

Lemma 4.1. *Let Y be any free $\mathbb{Z}$-module of finite rank and let m be an inner product on* $\mathrm{Hom}(Y, \mathbb{R})$. *Put*

$$\Phi(Y, m) := \{\alpha \in Y : \alpha \neq 0 \text{ and } 2m(\lambda, \alpha)/m(\alpha, \alpha) \in \mathbb{Z} \text{ for all } \lambda \in Y\}.$$

Then $(Y, \Phi(Y, m), m)$ is a (not necessarily reduced) metrized root datum. Moreover, for any metrized root datum of the form (Y, R, m), we have $R \subset \Phi(Y, m)$.

It follows from the above lemma that there exists a set Ψ which is minimal among sets with the following properties: (Y, Ψ, m) is a metrized root datum and $\Phi \subset \Psi$ for all $\Phi \in \mathscr{P}$, where $\mathscr{P}$ is defined in 2.7. Observe also we have $\Gamma \subset \mathrm{Aut}(Y, \Psi, m)$. Therefore, we see that Problem 2.7 is part of

Problem 4.2. Given a metrized root datum (Y, Ψ, m) and a finite subgroup $\Gamma \subset \mathrm{Aut}(Y, \Psi, m)$, study the equalities/linear relations among $\{F_{\Phi,\Gamma} : \Phi \subset \Psi\}$, where Φ ranges over all subsets of Ψ such that (Y, Φ, m) is a reduced metrized root datum.

We observe that Problem 4.2 is unchanged if we replace "metrized root datum" by "metrized $\mathbb{R}$-root datum". Moreover, when (Y, Ψ, m) is a metrized $\mathbb{R}$-root datum, we may decompose Y into the subspace Y_{ss} spanned by Ψ and its orthogonal complement Y_0. It can be shown easily that $F_{\Phi,\Gamma}$ lies in $\mathbb{C}[Y_{\mathrm{ss}}]$ and coincides with $F_{\Phi,\bar{\Gamma}} \in \mathbb{C}[Y_{\mathrm{ss}}] \subset \mathbb{C}[Y]$, where $\bar{\Gamma}$ is the image of $\gamma \mapsto \gamma|_{Y_{\mathrm{ss}}}$, $\Gamma \to \mathrm{Aut}(Y_{\mathrm{ss}}, \Psi, m)$. Since $(Y_{\mathrm{ss}}, \Psi, m)$ is semisimple, i.e., it is a metrized root system, we conclude that Problem 4.2 is equivalent to:

Problem 4.3. Study Problem 4.2 with "metrized root datum (Y, Ψ, m)" replaced by "metrized root system (Y, Ψ, m)".

Let $\Gamma \subset \Gamma'$ be finite subgroups of $\mathrm{Aut}(Y, \Psi, m)$. Then a relation $\sum c_\Phi F_{\Phi,\Gamma} = 0$ implies a relation $\sum c_\Phi F_{\Phi,\Gamma'} = 0$. Therefore, if we already have a solution to Problem 4.3 for (Y, Ψ, m) and Γ', then we only needs to examine the relations found there to solve Problem 4.3 for (Y, Ψ, m) and Γ. In this sense, we may reduce Problem 4.3 to the case where Γ is largest possible.

Problem 4.4. Study Problem 4.3 when $\Gamma = \mathrm{Aut}(Y, \Psi, m)$.

4.5. **The solution to Problem 4.4.** Jun Yu's solution [12] to Problem 4.4 takes the following shape. First, he gave a reduction theorem that reduces Problem 4.4 to the case where Ψ is simple. Next, for each simple (not necessarily reduced) root system Ψ, he gave an explicit description of all the equalities/linear relations.

When Ψ is simple of classical type (A_n, B_n, BC_n, C_n, or D_n), the equalities/relations are, very roughly speaking, generated by those responsible for Theorems 3.1 and 3.2.

When Ψ is simple of type F_4, there are two non-trivial equalities among dimension datum. There is no other non-trivial equality when Ψ is simple of exceptional type.

When Ψ is simple of exceptional type E_6, E_7, E_8, F_4, or G_2 respectively, the dimension of the space of linear relations among dimension datum is 2, 5, 10, 12, or 1, respectively.

5. Some consequences

There are quite a few reduction steps before we can start to use Jun Yu's classification results as described in 4.5. These steps are effective. Starting with a connected G, one can indeed figure out all the equalities/linear relations among the dimension data of connected subgroups using Jun Yu's theory by finite amount of computations. But it can be a long computation.

Here we will gather a few theoretical consequences from his work.

5.1. The Lie algebras of compact Lie groups form an additive category. Let K be the Grothendieck group of this additive category. Then K is a free abelian group with the set of all (isomorphism classes of) simple Lie algebras together with $\mathfrak{u}(1)$ as a basis. Let K' be the quotient of K by the subgroup generated by $\mathfrak{u}(2m+1) - \mathfrak{sp}(m) - \mathfrak{so}(2m+2)$ for all $m \geqslant 1$.

Theorem. *Let H_1, H_2 be closed subgroups of G such that $\mathscr{D}_{H_1} = \mathscr{D}_{H_2}$ Let $\mathfrak{h}_i'$ be the image of* $\operatorname{Lie} H_i$ *in K'. Then $\mathfrak{h}_1' = \mathfrak{h}_2'$.*

We remark that this result is *not* saying that the examples in Theorem 3.1 is responsible for all equalities among dimension data.

Theorem 5.2. *Let G be simple, not of type A_n, B_2, B_3, or G_2. Then there exist a list $H_1, \ldots, H_s$ of connected full-rank subgroups of G such that $s \geqslant 2$ and $\mathscr{D}_{H_1}, \ldots, \mathscr{D}_{H_s}$ are distinct and linearly dependent.*

Theorem 5.3. *Let $H_1, \ldots, H_s$ be closed connected subgroups of $U(n)$ such each H_i acts irreducibly on $\mathbb{C}^n$. Suppose that H_i is not G-conjugate to H_j for $i \neq j$, then $\mathscr{D}_{H_1}, \ldots, \mathscr{D}_{H_s}$ are linearly independent.*

This is a strengthening of Theorem 2.2.

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