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On the growth of Fourier coefficients of Siegel modular forms

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Recently W. Kohnen proposed to characterize cusp forms by the growth of their Fourier coefficients, more precisely he conjectured that (at least for large weights and level one) cusp forms can be characterized by the validity of the "Hecke bound" for the Fourier coefficients. One can translate this question into a question about the growth of eigenvalues of Hecke operators; Hecke eigenvalues allow to separate cusp forms from non cusp forms. We show how this basic strategy can be used in several ways to prove Kohnen's conjecture. The case of congruence subgroups will also be addressed, where several Fourier expansions come into play.

1 The Problem

1.1 Growth of Fourier coefficients in general

Let Γ be a congruence subgroup of $\Gamma^n := Sp(n, \mathbb{Z})$, typically of type $\Gamma_0(N)$ or a principal congruence subgroup $\Gamma(N)$. We denote by $M_k^n(\Gamma)$ the space of all Siegel modular forms of degree n, weight k for Γ. Any such modular form has a Fourier expansion

$$f(Z) = \sum_T a_f(T) e^{\frac{2\pi i}{N}\mathrm{trace}(TZ)}$$

where T runs over the set Λ_n of all half-integral symmetric positive-semidefinite matrices of size n and N satisfies $\Gamma(N) \subset \Gamma$.
If f is a cusp form, the the "Hecke bound" holds, i.e.

$$|a_f(T)| = \mathcal{O}(det(T)^{\frac{k}{2}}).$$

For better estimates see [8].
If f is noncuspidal, then we have much weaker estimates:

$$|a_f(T)| = \mathcal{O}(det(T)^k) \qquad (\forall\Gamma, \forall k),$$

$$|a_f(T)| = \mathcal{O}(det(T)^{k-\frac{n+1}{2}}) \qquad (\forall k \geq 2n+2), \forall\Gamma).$$

The latter result is contained in [13, Theorem D]; there are refined versions involving various minima of T; here we are only interested in the growth w.r.t. $det(T)$.

Remark: We point out here that for Siegel modular forms the relation between Fourier coefficients and Hecke eigenvalues is rather delicate. For instance, Hecke operators (away from the level, say for $\Gamma_0(N)$) can relate only Fourier coefficients $a_f(S)$ and $a_f(T)$ if S and T are in the same rational similitude class. For $n \geq 2$ there are infinitely many such similitude classes in Λ_n. We think that the notation of "Ramanujan conjecture" for Fourier coefficients of Siegel cusp forms as proposed in [20] should be avoided.

1.2 Kohnen's question

In 2010 Kohnen [14] proved that a modular form $f = \sum a_f(n)e^{2\pi inz} \in M_k^1(\Gamma_0(N))$ which satisfies the Hecke bound, is cuspidal, more precisely, he considered the weight 2 case with a sharper bound, but his proof carries over to higher weights. In his proof he used a convenient basis of the space of Eisenstein series, where the Fourier expansion is known explicitly and he showed that not too many cancellations can occur. His proof was generalized to Hilbert modular forms by Linowitz [17]. Kohnen then proposed to investigate the same kind of question for Siegel modular forms. To reformulate his question in a more general context, we introduce the following notation:

Definition: We say that a modular form $F \in M_k^n(\Gamma)$ has the $\mathcal{K}(\alpha)$-property in a cusp $g \in Sp(n, \mathbb{Z})$ if for all positive definite T we have

$$|a_F(T; g)| = \mathcal{O}(det(T)^{\alpha}), \tag{1}$$

for the Fourier expansion of F in the cusp g:

$$(F \mid_k g)(Z) = \sum_T a_F(T; g)e^{\frac{2\pi i}{N}trace(TZ)}.$$

Then we can ask:

Suppose that $F \in M_k^n(\Gamma)$ satisfies the $\mathcal{K}(\alpha)$- property in a cusp g; can we conclude that F is cuspidal ?.

Of course such a property will also depend on α; a weaker version would be to request $\mathcal{K}(\alpha)$ in all cusps and possibly some growth property for the lower rank Fourier coefficients as well. For a discussion of such cases we refer to [3].

For degree 2, $\Gamma = Sp(2,\mathbb{Z})$ and the Hecke bound (i.e. $\alpha = \frac{k}{2}$) this was confirmed independently by Kohnen and Martin [15], and Mizuno [19] using Fourier-Jacobi expansions and Imai's converse theorem (respectively).

Our aim is to explain:

Main Theorem: *Suppose that α and k satisfy*

$$k > max\{\alpha + n, \frac{3n-1}{2}\}.$$

Then any $F \in M_k^n(\Gamma_n)$ satisying $\mathcal{K}(\alpha)$ is cuspidal; in particular, if $k > 2n$, then any $F \in M_k^n(\Gamma_n)$ which satisfies the Hecke bound, is cuspidal.

Corollary: *For k, α as above, we have*

$$\limsup_{\det(T)\to\infty} \mid a_F(T)\det(T)^{-\alpha} \mid \to \infty$$

if $F \in M_k^n(\Gamma^n)$ is noncuspidal.

Our apprach is based on the following

Observation: *The subspace $\{F \in M_k^n(\Gamma_n) \mid F$ satisfies $\mathcal{K}(\alpha)\}$ is Hecke invariant. In particular, if there exists a nonzero, noncuspidal $F \in M_k^n(\Gamma_n)$ satisfying $\mathcal{K}(\alpha)$, then there exists a nonzero, noncuspidal Hecke eigenform $F \in M_k^n(\Gamma_n)$ satisfying $\mathcal{K}(\alpha)$.*

Remark: In our proof, we need only one rational similitude class of quadratic forms such that $\mathcal{K}(\alpha)$ holds for all T inside this similitude class (of course those Fourier coefficients should not all be zero for T in this similitude class!).

One may ask a modified question, for which our method has nothing to say:

Suppose that the Fourier coefficients of F satisfy (1) for all F-minimal T; is F then cuspidal ?

Here T is called F-minimal, if

$\det(T) = min\{\det(S) \mid a_F(S) \neq 0$, S is in the rational similitude class of T$\}$.

Note however, that this only makes sense if the existence of infinitely many inequivalent such F-minimal quadratic forms T is assured. This is not obvious at all (and not true for some theta series of low weight!).

2 Standard L-function

It is well-known that cuspidal and noncuspidal Hecke eigenforms can be characterized by poles of their standard L-functions; this goes back to M. Harris [12] and was also used to study nonvanishing of theta lifts [6].

Sketch of proof of the main theorem:
We assume that there is a noncuspidal nonzero Hecke eigenform $F \in M_k^n(\Gamma^n)$ satisfying $\mathcal{K}(\alpha)$; we fix a $T_0 \in \Lambda_n$ with $a_F(T_0) \neq 0$; we put

$$S := \{p \mid p \mid \det(2T_0)\}$$

and we consider the Dirichlet series

$$\sum_X a_F(X^t T_0 X) det(X)^{-s-k+1}.$$

Here X runs over the set

$$\{A \in \mathbb{Z}^{(n,n)} \mid \forall p \in S : p \nmid \det(A)\}/GL(n,\mathbb{Z}).$$

This series converges for $Re(s) > 2\alpha - k + n + 1$; by a fundamental result of Andrianov [1, 7] it equals

$$a_F(T_0) \cdot \Xi^S(s) \cdot L^S(F, st, s),$$

where $L^S(F, st, s)$ is the (degree $2n+1$) standard L-function attached to F and

$$\Xi^S(s) = \begin{cases} L^S(s+m, \chi_{T_0}) \prod_{i=0}^{m-1} \zeta^S(2s-2i)^{-1} & \text{if } \ n = 2m \\ \prod_{i=0}^{m} \zeta^S(2s-2i)^{-1} & \text{if } \ n = 2m+1. \end{cases}$$

On the other hand, there is $r < n$ with $\Phi^{n-r}(F) = f$ with f being non-zero cuspidal and by the Zharkovskaya-relation [1]

$$L^S(F, st, s) = \prod_{i=0}^{n-r-1} \zeta^S(s+k-n+i)\zeta^S(s-k+n-i)L^S(f, st, s). \qquad (2)$$

Here, by [21, 9] we know the absolute convergence of $L^S(f, st, s)$ for $Re(s) > 1 + \frac{r}{2}$. This leads to a contradiction if k is as in the statement of the main theorem.

Remark: This kind of proof also works for congruence subgroups $\Gamma_0(N)$ and half-integral weights, if we assume that the cuspidality is "violated at the cusp infinity", i.e. there exists $n-r>0$ such that $\Phi^{n-r}(F)$ is nonzero and cuspidal. For more general cases, see [3] and also the next sections.

3 An alternative approach by the Witt operator

Here we aim at

Theorem: *Suppose that N is squarefree and $F \in M_k^n(\Gamma_0(N))$ satisfies the $\mathcal{K}(\alpha)$- condition in at least one cusp. Then F is cuspidal, if $k > \frac{n}{2}+\alpha+1$.*

Note that the condition on k and n is weaker here. Details will be given in [4].

One first proves this directly for $n=1$, $\alpha < k-1$. Then one uses the fact that for $\Gamma_0(N)$ with N squarefree one can represent the relevant cusps in the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2,\mathbb{Z}) \times Sp(n-1,\mathbb{Z}) \subset Sp(n,\mathbb{Z})$$

using the standard diagonal embedding. It is then sufficient to study the image of F under the Witt operator:

$$F(\begin{pmatrix} \tau & 0 \\ 0 & \tau' \end{pmatrix}) = \sum_{i,j} g_i(\tau)\cdot h_j(\tau') \qquad (\tau \in \mathbb{H}, \tau' \in \mathbb{H}_{n-1}).$$

The degree one modular forms g_i will then satisfy a certain $\mathcal{K}(\alpha')$-condition in a cusp, and this then forces the g_i to be cuspidal.

The degree 1 case is proved more generally for principal congruence subgroups; note that the proof below does not require any properties of L-functions:

Proposition: *Let $F \in M_k^1(\Gamma(N))$ satisfy the $\mathcal{K}(\alpha)$ property in some cusp. Then F is cuspidal if $\alpha < k-1$.*

Proof(sketch): We assume the existence of a nonzero $F \in M_k^1(\Gamma(N)$, which satisfies $\mathcal{K}(\alpha)$ at some cusp. We may then assume that this cusp is just ∞.

As before, this implies the existence of a nonzero F with this property, which is at the same time an eigenform for all Hecke operator $T(p)$ for all primes $p \equiv 1 \bmod N$.
For all $\gamma \in SL(2,\mathbb{Z})$ we then have

$$(F \mid_k \gamma) \mid T(p) = (F \mid T(p) \mid_k \gamma.$$

If we apply this to the particular γ such that $(F \mid_k \gamma)(\infty) \neq 0$, we see (by looking at the constant term in the Fourier expansion), that the Hecke eigenvalue for $T(p)$ has to be

$$\lambda_p = p^{k-1} + 1.$$

On the other hand, starting from the smallest $n \neq 0$ with $a_F(n) \neq 0$, we get

$$a_F(np) = \lambda_p \cdot a_F(n),$$

and this implies

$$|\lambda_p| = \mathcal{O}(p^\alpha);$$

this gives a contradiction, if $\alpha < k-1$.

4 A purely local general approach

This is the most ambitious approach, aiming at principal congruence subgroups $\Gamma^n(N)$ of arbitrary level. There is a problem in generalizing the local method of the proposition above to arbitrary degree. One has to assure then that eigenvalues of T(p) for cusp forms of lower rank cannot become too small (this problem did not occur in degree 1). The reason for this difficulty is the Zharkovskaya-relation for the Hecke operator T(p):

$$\Phi \circ T^n(p) = (p^{k-n} + 1)T^{n-1}(p).$$

To avoid this problem, one may instead work (again for $p \equiv 1 \bmod N$) with a Hecke operator $\mathcal{T}_p$ associated with the double coset

$$\Gamma(N) \cdot \begin{pmatrix} D^{-1} & 0 \\ 0 & D \end{pmatrix} \cdot \Gamma(N),$$

where

$$D = diag(1, \ldots, 1, p).$$

For this Hecke operator, the Zharkowskaya-relation is of a different nature (a local version of the one given in (2)).
This approach will be worked out in detail in [5].

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