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The Burnside ring and the universal zeta function of finite dynamical systems

YOSHIDA, Tomoyuki (Hokkaido univ.)
yoshidat@math.sci.hokudai.ac.jp

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0. Introduction

0.1 Zeta functions of finite groups

There are some series associated to finite (or infinite) groups which remind us of zeta functions. We start with an example from elementary group theory ([Yo92]).

Let G be a finite group. Define two series $\{a_n\}$ and $\{b_n\}$ by

$$a_n := \sharp\{g \in G \mid g^n = 1\} = |\mathrm{Hom}(C_n, G)|.$$
$$b_n := \sharp\{g \in G \mid |g| = n\},$$

where $|g|$ denote the order of $g \in G$. We call a_n *Frobenius numbers* after Frobenius's theorem (1903):

$$a_n \equiv 0 \mod \gcd(n, |G|)$$

Note that $c_n := b_n/\varphi(n)$, where $\varphi(n)$ is the Euler function, is equal to the number of cyclic subgroups of order n. By the trivial formula $a_n = \sum_{d|n} b_d$, we have

$$\sum_{m=1}^{\infty} \frac{a_m}{m^z} = \zeta(z) \sum_{g\in G} \frac{1}{|g|^z} = \zeta(z) \sum_{n\geq 1} \frac{b_n}{n^z} \quad (1)$$

where $\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}$ is the Riemann zeta function. This equation can be presented by the following product formula:

$$Z_G(t) := \exp\left(\sum_{m=1}^{\infty} \frac{a_m}{m} t^m\right) = \prod_{n=1}^{\infty} \left(\frac{1}{1-t^n}\right)^{b_n/n} \quad (2)$$

In particular, when $G = C_N$, a cyclic group of order N, we have

$$a_n = (N, n), \quad b_n = \begin{cases} \varphi(n) & n|N \\ 0 & \text{else} \end{cases}$$

Thus

$$Z_{C_N}(t) = \exp\left(\sum_{n=1}^{\infty} \frac{(N,n)}{n} t^n\right) = \prod_{n|N} \left(\frac{1}{1-t^n}\right)^{\varphi(n)/n} \quad (3)$$

0.2 Absolute zeta function

"Absolute mathematics" means mathematics over the field $\mathbb{F}_1$ with one element (see [KOW03], [KK10]). This curious idea goes back to Tits (1956). He stated that a Weyl group is a simple algebraic group with corresponding Dynkin diagram, for example, $GL(n, \mathbb{F}_1) = S_{n-1}$.

During the last twenty years, this imaginary field has been studied mainly in algebra, especially in algebraic number theory and algebraic geometry. At present time, Kurokawa and Koyama's book [KK10] is a unique and nice literature on absolute mathematics.

In this book, we find many interesting statements.

(Tits 1956) *The N-dimensional projective space $\mathbb{P}^{N-1}(\mathbb{F}_1)$ over $\mathbb{F}_1$ is an N-point set.* Note that

$|\mathbb{P}^{N-1}(\mathbb{F}_q)| = 1+q+q^2+\cdots+q^{N-1} \to N \quad (q \to 1)$

(Manin 1993) *The zeta function of $\mathbb{P}^{N-1}(\mathbb{F}_1)$ should be*

$$\zeta(s, \mathbb{P}^N(\mathbb{F}_1) = s(s-1)(s-2)\cdots(s-N)$$

(Soulé 1999) *An extension field of $\mathbb{F}_1$ of degree N is defined to be*

$$F_{1^N} := \{0\} \cup \mu_N$$

where μ_N is the group of N-th roots of unity. Thus F_{1^N} is a multiplicative monoid with zero element.

In absolute mathematics, an $\mathbb{F}_1$-algebra is defined to be a multiplicative monoid with zero element([KK10]).

0.3 Absolute Weil zeta function

The Weil zeta function of an $\mathbb{F}_q$-algebra A is defined by

$$Z_A^{\mathrm{Weil}}(t) := \exp\left(\frac{|\mathrm{Hom}_{\mathbb{F}_q}(A, \mathbb{F}_{q^m})|}{m}t^m\right)$$

Thus the *absolute Weil zeta function* of an $\mathbb{F}_1$-algebra should be defined by formally replacing $\mathbb{F}_q$ by $\mathbb{F}_1$.

In particular, the absolute zeta function of $A = \mathbb{F}_{1^N}$ is given by

$$Z_{\mathbb{F}_{1^N}}^{\mathrm{Weil}}(t) = \exp\left(\sum_{n=1}^{\infty}\frac{(N,n)}{n}t^n\right)$$

because an $\mathbb{F}_1$-algebra homomorphism from $\mathbb{F}_{1^N} = \{0\} \cup \mu_N$ to $\mathbb{F}_{1^m} = \{0\} \cup \mu_m$ is uniquely given by a group homomorphism from μ_m to μ_N, that is,

$$Z_{\mathbb{F}_{1^N}}^{\mathrm{Weil}}(t) = Z_{C_N}(t)$$

As a consequence, the absolute Weil zeta function of the $\mathbb{F}_1$-algebra $\mathbb{F}_{1^N}$ is nothing but the "zeta function" of a cyclic group defined by (3).

0.4 Dynamical zeta functions

Let X is a dynamical system, that is, X is a finite set with permutation $\sigma : X \to X$. Then the **Artin-Mazur zeta function** of X is defined by

$$Z_X^{\mathrm{AM}}(t) := \exp\left(\sum_{m=1}^{\infty}\frac{|\mathrm{Fix}_X(\sigma^m)|}{m}t^m\right)$$

Let $\mathbb{R} \times X \to X; (t,x) \mapsto f^t(x)$ be a flow (or $\mathbb{R}$-dynamical system), and so $f^0 = \mathrm{id}_X$, $f^{s+t} = f^s \circ f^t$. Then the Ruelle zeta function is defined by

$$\zeta_X(s) = \prod_{\gamma}\left(1 - e^{-sT(\gamma)}\right)^{-1}$$

where γ runs over periodic orbits and where $T(\gamma 9$ is the period of γ. The Artin-Mazur zeta functions is a special case of the Ruelle zeta functions.

0.5 The universal zeta function

The **universal zeta function** (**UZF**) of a category $\mathscr{C}$ is defined by

$$Z_{\mathscr{C}}(t) := \mathscr{C}(t) := \sum_{N \in \mathscr{C}/\cong}\frac{1}{|\mathrm{Aut}(N)|}t^N$$

Here the formal summation on the right hand side is taken over all isomorphism classes of objects of $\mathscr{C}$. The symbols t^N denotes a variable associated to the object $N \in \mathscr{C}$. Of course, every automorphism group $\mathrm{Aut}(N)$ must be a finite group. The author called such a series as an *exponential function* of $\mathscr{C}$ in the paper [Yo01]

Assume that $\mathscr{C}$ has any finite coproduct. In this case, we usually assume that the symbols t^N's satisfy the relations

$$t^{M+N} = t^M \cdot t^N, t^{\emptyset} = 1,\ t^N = t^{N'} \text{ if } N \cong N'$$

Thus $Z_{\mathscr{C}}(t)$ belongs to a complete semigroup algebra $\mathbb{Q}[[\mathscr{C}^{\mathrm{op}}/\cong]]$. Furthermore, the Krull-Schmidt property for $\mathscr{E}$ induces the exponential formula

$$\mathscr{E} = \exp(\mathscr{I}(t))$$

where $\mathscr{I}$ is the subcategory of connected objects.

0.6 Zeta properties

As is well-known, the Riemann zeta function $\zeta(z)$ has has some remarkable properties called the zeta properties (of course, Riemann hypothesis is still open).

(M) $\zeta(z)$ is a meromorphic function on $\mathbb{C}$ with simple pole $z = 1$.
(FE) $\zeta(1-z) = 2^{1-z}\pi^{-z}\cos(\pi z/2)\Gamma(z)\zeta(z)$.
(EP) Euler product: $\zeta(z) = \prod_p (1-p^{-z})^{-1}$
(SV) $\zeta(-n) = \dfrac{-B_{n+1}}{n+1}$, $\zeta(-2n) = 0$, $n \geq 1$.
(RH) Non-trivial zeros of $\zeta(z)$ lie on $\Re(z) = 1/2$.
(PF) $\zeta(z) = \dfrac{-\pi^{z/2}}{z(1-z)\Gamma(z/2)} \prod_\rho \left(1 - \dfrac{z}{\rho}\right)$ (ρ runs over nontrivial zeros).

0.7 Zeta properties for UZF

We are interested in a "category" whose universal zeta function (UZF) satisfy "zeta properties". However, a UZF is not a series with single variable despite its appearance , but it is a series with infinitely many variables in general. Zeta properties for UZF might be written by the language of category theory. Perhaps, such a categorical zeta property for UZF gives , for example, Riemann hypothesis for a classical zeta function by applying a suitable functor.

Example. A UZF give some classical zeta functions by specializations. Let DS be the category of finite DS's. Then the specialization $t^N := |\mathrm{Hom}(N, X)|u^{|N|}$ gives the Artin-Mazur zeta function:

$$Z_{\mathrm{DS}}(t) = Z_X^{\mathrm{AM}}(u)$$

The main purpose of this study is to find categorical zeta properties for categorical Artin-Mazur zeta function $Z_{\mathrm{DS}}(t)$

1 Dynamical systems and their zeta

1.1 Almost finite dynamical systems (DS)

A **dynamical system (DS)** (X, σ) or simply X is a set equipped with a permutation $\sigma \in \mathrm{Sym}(X)$. Thus a DS is algebraically nothing but a C-set, where $C = \langle\sigma\rangle$ is a infinite cyclic group. Such a set is often called a **cyclic set**([DS89]).

Any DS is a disjoint union of **orbits**. An orbit is **transitive** (or often **connected**) as a C-set.

Let $C^n = \langle\sigma^n\rangle \leq C$. Then $C(n) := C/C^n$ is a dynamical system of size n. Let $C^\infty := 1$, the trivial subgroup of C. Then $C(\infty) := C/C^\infty$ is an infinite and transitive C-set and is called to be **free**. Conversely, any transitive DS of size $n \leq \infty$ is isomorphic to $C(n)$.

Now, a DS X is called to be **almost finite** ([DS89])if

(a) X has only a finite number of orbits of given length n for any $n < \infty$.
(b) X has no free orbits.

The condition (a) is equivalent to

(a') $N_m := \mathrm{Fix}_X(\sigma^m)| < \infty$ for any $m = 1, 2, \cdots$.

A DS satisfying (a) or (a') is called to be **essentially finite**.

Furthermore, the condition (b) is equivalent to

(b') Any element of X is periodic, that is, it is contained in an orbit of finite length, or equivalently, for any x, their exists $m \geq 1$ such that $\sigma^m x = x$.

For any infinite series of non-negative integers $\boldsymbol{b} = (b_1, b_2, \cdots)$,

$$X(\boldsymbol{b}) := b_1 C(1) + b_2 C(2) + \cdots$$

is almost finite. Here $bnC(n)$ is a disjoint union of n copies of DS isomorphic to $C(n)$.

Conversely, any almost finite DS X is isomorphic to $X(\boldsymbol{b})$ for a uniquely determined $\boldsymbol{b}$.

Let X^{per} (resp. X^{aper}) be the set of periodic (resp. aperiodic) of a DS X. Then any DS X is the disjoint union of its periodic part X^{per} and its aperiodic part X^{aper}. Thus for any essentially finite DS X, we have

$$X \cong X(\boldsymbol{b}) + b_\infty C(\infty), \tag{4}$$

for some integral vector $\boldsymbol{b}$ and cardinal number b_∞.

1.2 The Artin-Mazur zeta functions

Let (X, σ) be an essentially finite DS. Since

$$N_m := |\mathrm{Fix}_X(\sigma^m)| = \sharp\{x \in X \mid \sigma^m x = x\} < \infty,$$

the **Artin-Mazur zeta function (AMZ)**

$$Z_X^{\text{AM}}(u) := \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} u^m\right)$$

is well-defined.

Lemma. *For AMZ's of essentially finite DS's, the following hold*:

(i) $Z_X^{\text{AM}}(u) = Z_{X^{\text{per}}}^{\text{AM}}(u)$, $Z_{X^{\text{aper}}}^{\text{AM}}(u) = 1$.
(ii) $Z_\emptyset^{\text{AM}}(u) = 1$, $Z_{X+Y}^{\text{AM}}(u) = Z_X^{\text{AM}}(u) \cdot Z_Y^{\text{AM}}(u)$.
(iii) $Z_{C(n)}^{\text{AM}}(u) = \dfrac{1}{1-u^n}$.
(iv) $Z_{X(\boldsymbol{b})}^{\text{AM}}(u) = \displaystyle\prod_{n=1}^{\infty} \left(\frac{1}{1-t^n}\right)^{b_n}$.

PROOF. (i) and (ii) are trivial. (iii) For $C(n) = \langle\sigma\rangle / \langle\sigma^n\rangle$,

$$N_m = |\mathrm{Fix}_{C(n)}(\sigma^m)| = \begin{cases} n & \text{if } n|m \\ 0 & \text{else} \end{cases}$$

Thus

$$Z_{C(n)}^{\text{AM}}(u) = \exp \sum_{m \equiv 0(n)} \frac{n}{m} u^m = \exp \sum_{k \geq 1} \frac{1}{k} u^{kn} = \frac{1}{1-u^n}$$

(iv) follows from (4).

1.3 Zeta properties of AMZ of finite DS

AMZ of a finite DS satisfies some zeta properties and they are proved easily.

Theorem. *Let (X, σ) be a finite DS. Then the following hold*:
(EP) $Z_X^{AM}(u) = \displaystyle\prod_P \left(\frac{1}{1-u^{|P|}}\right)$,
where P runs over periodic orbits. In particular, AMZ is a rational function.

(DE) $Z_X^{AM}(u) = \det(I_X - A_\sigma u)^{-1}$,
where $A_\sigma = (\delta_{x,\sigma y})_{x,y \in X}$ is the permutation matrix associated to σ.

(FE) $Z_X^{AM}(1/u) = (-u)^{|X|} \det(\sigma) Z_X^{AM}(u)$.

(RH) *$Z_X^{AM}(u)$ has no zero on $\mathbb{C}$. Its poles lie on $|u| = 1$. Furthermore, $u = e^{-s}$ is a pole if and only if $\Re(s) = 0$.*

The theory of AMZ of almost finite but not finite DS's becomes extremely difficult. The zeta properties except for (EP) do not hold in general. There are some examples of well-known DS which show the difficulty.

1.4 Ihara zeta functions

Let (V, E) be a finite simple graph and $X^{\text{I}}(V, E)$ the set of walks with no backtrak:

$$X^{\text{I}}(V, E) := \left\{(x_i) \in V^{\mathbb{Z}} \mid (x_i, x_{i+1}) \in E, x_{i+2} \neq x_i\right\}$$

makes an essentially finite DS together with right shift operator:

$$\sigma : (x_i)_i \mapsto (x_{i-1})_i$$

Then the **Ihara zeta function:** is defined by

$$Z_{(V,E)}^{\text{I}}(u) = Z_{X^{\text{I}}(V,E)}^{\text{AM}}(u) = \prod_P (1 - u^{|P|})^{-1}$$

where P runs over prime walks with no backtrack. See [GIL08]

If $G = (V, E)$ is $(q+1)$-regular, then

$$Z_G^{\mathrm{I}}(u) = (1-u^2)^{\chi(G)} \det(I - uA + qu^2)^{-1},$$

where $\chi(G) := |E| - |V| + 1$ and A is the incidence matrix.

1.5 Symbolic dynamical systems

A **symbolic DS** on alphabet Q with $|Q| = q$ is defined by

$$Q^{\mathbf{Z}} := \{(x_i)_{i\in\mathbf{Z}} \mid x_i \in Q\}$$

with shift operator $\sigma : (x_i) \mapsto (x_{i-1})$. It is essentially finite because

$$x = (x_i) \in \mathrm{Fix}(\sigma^m) \iff x_{i+m} = x_i (\forall i \in \mathbb{Z})$$
$$\therefore N_m = |\mathrm{Fix}(\sigma^m)| = q^m < \infty$$

Thus AMZ has the following form:

$$Z_{Q^{\mathbf{Z}}}^{\mathrm{AM}}(u) = \exp \sum_{m=1}^{\infty} \frac{q^m}{m} u^m = \frac{1}{1-qu}$$

$Q^{\mathbf{Z}}$ is essentially finite and furthermore we have

$$(Q^{\mathbf{Z}})^{\mathrm{per}} \cong \coprod_{n=1}^{\infty} M(q,n)C(n)$$
$$\therefore Z_{Q^{\mathbf{Z}}}^{\mathrm{AM}}(u) = \frac{1}{1-qu} = \prod_{n=1}^{\infty} \left(\frac{1}{1-t^n}\right)^{M(q,n)}$$

where $M(q,n)$ is the **Necklace polynomial**:

$$M(q,n) := \frac{1}{n} \sum_{k|n} \mu\left(\frac{n}{k}\right) q^k \qquad (5)$$

1.6 Weil zeta functions as AMZ

The **Weil zeta function** (or **congruence zeta function**) of a variety X over a finite field $\mathbb{F}_p$ is defined by

$$\zeta(X, u) = \exp \sum_{m=1}^{\infty} \frac{|X(\mathbb{F}_{p^m})|}{m} u^m$$

where $X(\mathbb{F}_{p^m})$ is the (finite) set of $\mathbb{F}_{p^m}$-rational points.

Let $F : X \to X$ be the Frobenius automorphism (induced by $x \mapsto x^p$). Then (X, F) is essentially finite DS. Since $\mathrm{Fix}_X(F^m) = X(\mathbb{F}_{p^m})$, the Weil zeta function is equal to the AMZ of this DS:

$$\zeta(X, u) = Z_{(X,F)}^{\mathrm{AM}}(u)$$

Zeta properties for Weil zeta functions called Weil conjectures were proved mainly by Grothendieck and Deligne.

The Weil zeta function is essentially same as $Z_A^{\mathrm{Weil}}(t)$ of $\mathbb{F}_p$-algebra A appeared in section 0.3.

2 The Burnside ring $\Omega(C)$

2.1 The category of dynamical systems

Let $C = \langle\sigma\rangle$ be an infinite cyclic group. Since a finite DS is nothing but a finite C-set, we denote by $\mathbf{set}^C$ the category of finite DS's. Furthermore, we denote by $\mathbf{afset}^C$ the category of almost finite dDS's. These categories are like to $\mathbf{set}^G$, the category of finite G-sets, where G is a finite group.

$X \times Y$, $X + Y$ denote a direct product and a disjoint union of two DS', respectively. In particular, X^n (resp. nX) denotes the direct product (resp. disjoint union) of n-copies of X.

$\mathrm{Map}_C(X, Y)$ denotes the set of C-maps between two DS's X, Y. Note that for $m = 1, 2, \cdots$,

$$\mathrm{Map}_C(C(m), X) \cong \mathrm{Fix}_X(\sigma^m); f \mapsto f(C^m)$$
$$\mathrm{Map}_C(C(\infty), X) \cong X; f \mapsto f(C^\infty)$$
$$\mathrm{Map}_C(C(m), C(\infty)) = \emptyset$$

Let Y^X be the set of maps between from X to Y, so that Y^X is a DS with C-action defined by

$${}^{\sigma}f : X \to Y; x \mapsto \sigma f(\sigma^{-1}x)$$

If X is finite and Y is almost finite, then Y^X is almost finite. If Y is finite and if X is essentially finite, then Y^X is essentially finite iff X has only finite number of aperiodic orbits. In particular, Y is a finite set with trivial C-action, then $Y^{C(\infty)}$ is a symbolic DS.

2.2 The Burnside ring

The **Burnside ring** $\Omega(G)$ of a pro-finite group G is the Grothendieck ring of $\mathbf{set}^G$, the category of finite G-sets, that is, the abelian group generated by the symbols $[X]$, where X is any finite G-set, with relation

$$[X] = [X'] \quad \text{if } X \cong X' \text{ and}$$
$$[X+Y] = [X] + [Y]$$

where $X+Y$ denotes the disjoint union . The multiplication on $\Omega(G)$ is defined by $[X]\cdot[Y] = [X \times Y]$.

For each subgroups S, the map

$$\varphi_S : \Omega(G) \longrightarrow \mathbb{Z}; [X] \mapsto |X^S|$$

where $X^S := \{x \in X \mid Sx = x\}$, defines a homomorphism called a **Burnside homomorphism**:

$$\varphi = \prod_{(S)} \varphi_S : \Omega(G) \to \mathrm{gh}(G) := \prod_{(S)} \mathbb{Z}$$

where (S) runs over all conjugacy classes of subgroups of G. The Burnside homomorphism is an injective ring homomorphism, and so $\Omega(G)$ is viewed as a subring of the **ghost ring** $\mathrm{Gh}(G)$.

The **complete Burnside ring** $\widehat{\Omega}(G)$ is the closure of $\Omega(G)$ in the product space $\mathrm{gh}(G)$, where

2.3 The complete Burnside ring $\widehat{\Omega}(C)$

The complete Burnside ring $\widehat{\Omega}(C)$ of an infinite cyclic group $C = \langle\sigma\rangle$ is the Grothendieck ring of almost finite DS's. Thus its element is presented by an infinite sum

$$X(\boldsymbol{b}) := \sum_{n=1}^{\infty} b_n[C(n)]$$

for some integral vector $\boldsymbol{b} = (b_1, b_2, \cdots) \in \mathbb{Z}^{\mathbb{N}}$. The map $\boldsymbol{b} \mapsto X(\boldsymbol{b})$ induces a bijection

$$X : \mathbb{Z}^{\mathbb{N}} \xrightarrow{\cong} \widehat{\Omega}(C); \boldsymbol{b} \mapsto X(\boldsymbol{b})$$

Since $\widehat{\Omega}(C)$ is a commutative ring, by pulling back along X, we have that $\mathbb{Z}^{\mathbb{N}}$ becomes a commutative ring which is called a **Necklace algebra** and is denoted by $\mathrm{Nr}(\mathbb{Z})$.

The multiplication is defined by

$$[C(m)] \cdot [C(n)] = (m,n)[C([m,n])]$$

where (m,n) and $[m,n]$ denote the gcd and lcm, respectively. This formula gives the multiplication formula in the Necklace algebra $\mathbb{Z}^{\mathbb{N}}$ as follows:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{c},\ c_l = \sum_{[m,n]=l} (m,n) a_m b_n$$

where the summation is taken over non-negative integers l such that $[m,n] = l$.

The k-th component of the (complete) Burnside homomorphism $\widehat{\varphi} : \widehat{\Omega}(C) \to \mathrm{gh}(G)$ is given by

$$\widehat{\varphi}_k := \varphi_{C^k} : [C(n)] \mapsto \begin{cases} n & \text{if } n|k \\ 0 & \text{if } n \not| k \end{cases}$$

Thus we have

$$\widehat{\varphi}_k(X(\boldsymbol{b})) = d_k := \sum_{i|k} i b_i$$

This relation between (d_k) and (b_i) is equivalent to the following cyclotomic identity.

$$\prod_{n=1}^{\infty} \left(\frac{1}{1-t^n}\right)^{b_n} = \exp \sum_{n=1}^{\infty} \frac{d_n}{n} t^n \left(= Z^{\mathrm{AM}}_{X(\boldsymbol{b})}(t)\right)$$

Note that two almost finite DS's X and Y are isomorphic if and only if $\widehat{\varphi}(X) = \widehat{\varphi}(Y)$.

2.4 Dress-Siebeneicher's theory

In their paper in [DS89], Dress and Siebeneicher stated an important and surprising result which connects the Burnside ring $\widehat{\Omega}(C)$ of infinite cyclic group C, λ-rings and Witt vectors Their result is presented by the following diagram:

$$\begin{tikzcd}
 & \mathrm{Nr}(\mathbb{Z}) = \mathbb{Z}^{\mathbb{N}} \arrow[d, "X", "\cong"'] & \\
W(\mathbb{Z}) \arrow[r, "\tau", "\cong"'] \arrow[d, "\Phi"] & \widehat{\Omega}(C) \arrow[r, "s_t", "\cong"'] \arrow[d, "\widehat{\varphi}"] & \Lambda(\mathbb{Z}) \arrow[d, "L"] \\
\mathbb{Z}^{\mathbb{N}} \arrow[r, equal] & \mathrm{gh}(C) \arrow[r, "\cong"] & t\mathbb{Z}[[t]]
\end{tikzcd}$$

Here, we briefly explain the rings and maps appeared in the diagram([DS89], [Ya10], [Ha09]).

(1-a) $\mathrm{Nr}(\mathbb{Z}) := \mathbb{Z}^{\mathbb{N}}$: Necklace algebra. Its element is an infinite vector $\boldsymbol{b} = (b_n)_{n=1}^{\infty}$ of integers.

(1-b) $W(\mathbb{Z}) := \mathbb{Z}^{\mathbb{N}}$: the ring of universal Witt vectors. Its element is a sequence $\boldsymbol{q} = (q_n)_{n=1}^{\infty}$ of integers.

(1-c) $\widehat{\Omega}(C)$: the complete Burnside ring of an infinite cyclic group C. Its element is an infinite sum $\sum_{n=1}^{\infty} b_n[C(n)] =: X(\boldsymbol{b})$.

(1-d) $\Lambda(\mathbb{Z}) := 1+t\mathbb{Z}[[t]]$: the universal λ-ring. The addition is defined by the multiplication of formal power series.

(2-a) $X : \boldsymbol{b} \mapsto X(\boldsymbol{b}) = \sum_{n=1}^{\infty} b_n[C(n)]$

(2-b) $\tau : \boldsymbol{q} \mapsto \sum \mathrm{ind}_n q_n^{(C)}$, where $q^{(C)}$ is the periodic part of the (virtual) symbolic DS, and $\mathrm{ind}_n : [C(m)] \mapsto [C(mnr)]$.

(2-c) $s_t : [X] \mapsto 1+\sum \varphi_n(S^n(X))t^n$, where $S^n(X)$ is the n-th symmetric power.

(2-d) $\Phi : (q_n) \mapsto \left(\sum_{k|n} k q_k^{n/k}\right)_n$

(2-e) $\widehat{\varphi}$: the complete Burnside homomorphism.

(2-f) $L : a(t) \mapsto t\frac{d}{dt}\log a(t)$

The maps $\Phi, \widehat{\varphi}, L$ are injective ring homomorphisms. τ, s_t are ring isomorphisms.

Assume that $\boldsymbol{b} \in \mathrm{Nr}(\mathbb{Z})$, $\boldsymbol{q} = (q_n) \in W(\mathbb{Z})$, $a(t) = 1 + \sum a_n t^n \in \Lambda(\mathbb{Z})$, $\boldsymbol{d} = (d_n) \in \mathrm{gh}(\mathbb{Z})$ are corresponding each other by these maps:

$$\tau(\boldsymbol{q}) = X(\boldsymbol{b}), s_t(X(\boldsymbol{b})) = a(t), \widehat{\varphi}(X(\boldsymbol{b})) = \boldsymbol{d}$$
$$(q_n) \leftrightarrow (b_n) \leftrightarrow (a_n) \leftrightarrow (d_n)$$

Then the above diagram implies the following identities which remind us of dynamical zeta functions:

$$\boxed{\begin{aligned}\prod_{n=1}^{\infty} \frac{1}{1-q_n t^n} &= \prod_{n=1}^{\infty}\left(\frac{1}{1-t^n}\right)^{b_n} \\ &= 1+\sum a_n t^n = \exp\sum_{n=1}^{\infty}\frac{d_n}{n}t^n\end{aligned}}$$

3 The universal zeta functions

3.1 The UZF of categories and functors

Let $\mathscr{C}$ be a category. We assume that

(i) $\mathscr{C}$ is **locally finite**, that is, $|\mathrm{Hom}(X,Y)| < \infty$.
(ii) The isomorphism classes $\mathscr{C}/\cong$ of objects is a small set.

The **universal zeta function** (UZF) or the **exponential function** ([Yo01]) of a category $\mathscr{E}$ is defined by

$$\boxed{Z_{\mathscr{C}}(t) := \mathscr{C}(t) := \sum_{N\in\mathscr{C}/\cong} \frac{1}{|\mathrm{Aut}(N)|}t^N}$$

Here the summation is taken over a complete representatives of isomorphism classes of objects. Furthermore, t^N is a variable associated to an object N such that $t^N = t^{N'}$ if $N \cong N'$.

Let $\mathscr{C}^C$ be the category of dynamical systems whose underlying set is an object of $\mathscr{C}$. Then

$$\mathscr{C}^C(t) = \sum_{N\in\mathscr{C}/\cong} t^N$$

is the **geometric series type UFZ** of $\mathscr{C}$.

$$(\mathscr{C}^C)^{\mathrm{op}}(t) = \sum_{N\in\mathscr{C}/\cong} N^t$$

is the **Dirichlet type UFZ** of $\mathscr{C}$.

The UZF of a functor $F : \mathscr{C} \to \mathscr{S}$ is defined by

$$\boxed{F(t) := \sum_{A\in\mathscr{C}/\cong} \frac{1}{|\mathrm{Aut}(A)|}t^{F(A)}}$$

Assume that F is faithful and that for any $N \in \mathscr{S}$, there exists only finite number of isomorphism classes of $A \in \mathscr{C}$ such that $F(A) \cong N$. For an $N \in \mathscr{S}$, a **$\mathscr{C}$-structure** on N is defined to be a pair (A,α) of $A \in \mathscr{C}$ and an isomorphisms $\alpha : F(A) \cong N$. Two $\mathscr{C}$-structure (A,α) and (B,β) on N is called to be isomorphic if there exists an automorphism $f : A \cong B$ such that $\beta \circ F(f) = \alpha$.

Let denote by $\mathrm{Str}(\mathscr{C}/N)/\cong$ the (finite) set of isomorphism classes of $\mathscr{C}$-structures on N. Then

$$\boxed{F(t) = \sum_{N \in \mathscr{S}/\cong} \frac{|\mathrm{Str}(\mathscr{C}/\mathbb{N})/\cong|}{|\mathrm{Aut}(N)|} t^N}$$

3.2 The exponential formula

Assume that a locally finite category $\mathscr{E}$ has any finite coproducts. In this case, we can think that the UZF $\mathscr{E}(t) = Z_{\mathscr{E}}(t)$ belongs to the complete semigroup algebra $\mathbb{Q}[[\mathscr{E}^{\mathrm{op}}/\cong]]$, and so we introduce the following rule:

$$t^{M+N} = t^M \cdot t^N, t^{\emptyset} = 1$$

Assume further that $\mathscr{E}$ satisfies the strict **Krull-Schmidt property**, that is, that any object has a unique decomposition into a coproduct of connected objects. Then the **exponential formula** holds:

$$\boxed{\mathscr{E} = \exp(\mathscr{I}(t))}$$

where $\mathscr{I} = \mathrm{Con}(\mathscr{E})$ is the subcategory of connected objects. Under some technical conditions, the exponential formula implies the KS-property, that is, KS-property is a **categorification** of exponential formulas.

3.3 Wohlfahrt formula

Let G be a finitely generated group. Then the exponential formula has the form:

$$\sum_{X \in \mathbf{set}^G/\cong} \frac{t^X}{|\mathrm{Aut}(X)|} = \exp\left(\sum_{H \leq G} \frac{t^{G/H}}{(G:H)}\right)$$

(Here we used that $\mathrm{Aut}(G/H) \cong N_G(H)/H$ and that the number of subgroups conjugate to H is equal to $(G : N_G(H))$.

Applying this formula to the forgetful functor, we have the Wohlfahrt formula:

$$\boxed{\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(G, S_n)|}{n!} t^n = \exp\left(\sum_{H \leq G} \frac{t^{(G:H)}}{(G:H)}\right)}$$

3.4 The UZF of finite dynamical systems

Since the category of finite dynamical systems satisfies the strict Krull-Schmidt property, its UZF satisfies the exponential formula. The automorphism group of $C(n)$ is a cyclic group of order n. Furthermore, by the KS-property,

$$\mathrm{Aut}(X(b)) \cong \prod_{n \geq 1} \mathrm{Aut}(b_n C(n)) \cong \prod_{n \geq 1} C_n \wr S_{b_n}$$

Thus the universal zeta function of finite DS's is given by

$$\boxed{Z_{\mathrm{DS}}(t) := \sum_N \frac{t^N}{|\mathrm{Aut}(N)|} = \exp \sum_{m=1}^{\infty} \frac{t^{C(m)}}{m}}$$

Let X be an essentially finite DS. Then by the specialization $t^N \leftarrow |\mathrm{Hom}(N, X)| u^{|N|}$, we have the Artin-Mazur zeta function

$$Z_X^{\mathrm{AM}}(u) = \exp\left(\sum_{m^1}^{\infty} \frac{|\mathrm{Fix}_X(\sigma^m)|}{m} u^m\right)$$

Here, note that $\mathrm{Hom}(C(m), X) = \mathrm{Fix}_X(\sigma^m)$. Thus we have a group homomorphism

$$Z^{\mathrm{AM}} : \widehat{\Omega}(C) \to \mathbb{Q}(u)^{\times}; [X] \mapsto Z_X^{\mathrm{AM}}(u)$$

4 Problems

4.1 Zeta properties for UZF

Let $Z_{\mathrm{DS}}(t)$ be the universal zeta function of finite dynamical systems. It is mapped by the Burnside homomorphism φ_k to the following series:

$$\begin{aligned}\varphi_k(Z_{\mathrm{DS}}(t)) &= \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\varphi_k(t^{C(n)})\right)\\ &= \exp\left(\sum_{n=1}^{\infty}\frac{\langle C(k)\times C(n),t\rangle}{n}\right)\\ &= \exp\left(\sum_{n=1}^{\infty}\frac{\langle C([k,n]),t\rangle^{(k,n)}}{n}\right)\\ &= \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}(t_{[k,n]})^{(k,n)}\right)\end{aligned}$$

Thus $Z_{\mathrm{DS}}(t)$ can be viewed as a series with infinite number of variables.

Note that two almost finite DS's X and Y are isomorphic if and only if they have the same AMZ. The theory of almost finite DS is extremely difficult. whereas it is rather strange that almost finite DS's are controlled by only finite DS's.

Question. Are there any zeta properties for UZF of finite DS's which induce those for usual zeta properties? For example, is there a universal functional equation

$$Z_{\mathrm{DS}}(1/t) = \gamma(t)Z_{\mathrm{DS}}(t) \quad ??$$

where the gamma factor $\gamma(t)$ is a group homomorphism

$$\gamma(t) : \widehat{\Omega}(C) \mapsto \mathbb{Q}(t)^{\times}$$

If such a formula holds, then applying functors we have a usual function equation. However, there is no possibility in this form, because

$$\varphi_1(Z_{\mathrm{DS}}(t)) = \exp\left(\sum_{m}\frac{1}{n}t_n\right)$$

does not satisfy any functional equation.

Remark. There is a categorical exponential formula for UZF

$$Z_{\mathrm{DS}}(t) = \sum_{N}\frac{t^N}{|\mathrm{Aut}(N)|} = \exp\sum_{m=1}^{\infty}\frac{t^{C(m)}}{m}$$

It induces many formula, e.g. the Wohlfahrt formula for one-variable series. Unfortunately, exponential formulas are not called one of zeta properties.

4.2 AMZ of non-invertible DS's

A rooted forest F can be viewed as a non-invertible dynamical system (F,σ), where σ maps any vertex to its unique child, such that any periodic point is a fixed point of σ. Let RF be the category of rooted forests. Let $\mathbf{set}^{\mathrm{N}_0}$ be the category of non-invertible dynamical systems, whose object is a pair (X,σ), where X is a set with self-map $\sigma : X \to X$.

Then the category RF of rooted forests is equivalent to the indexed category of the category RT of rooted trees. RF $\cong$ $\mathbf{set}$(RT), that is, RF is a strict Krull-Schmidt category whose connected objects are rooted trees.

$$\begin{tikzcd} \mathbf{set}^{\mathrm{N}_0} \arrow[r, "\text{forest}"] \arrow[d] \arrow[dr, phantom, "\text{Per PB}"] & \mathbf{RF} \arrow[d, "\text{root}"] \\ \mathbf{set}^{\mathbb{Z}} \arrow[r, "\text{forget}"] & \mathbf{set} \end{tikzcd}$$

The pullback diagram in the 2-category CAT, the category of categories, For a non-invertible finite dynamical system (X,σ),
$\mathrm{Per}(X) := \{x \in X \mid \sigma^m x = x \text{ for a } m \geq 1\}$,
$\mathrm{Root}(F)$: the set of roots.

Then we have

$$\sum_{(X,\sigma)}\frac{t^{F(X,\sigma)}}{|\mathrm{Aut}(X,\sigma)|} = \left(1-\sum_{T}\frac{t^T}{|\mathrm{Aut}(T)|}\right)^{-1},$$

where T runs over rooted trees, (X,σ) non-invertible finite DS's.
Specialization: $t^T \mapsto t^{|V(T)|}$ gives

$$\sum_{n=0}^{\infty}\frac{n^n}{n!}t^n = \left(1-\sum_{n=1}^{\infty}\frac{n^{n-1}}{n!}t^n\right)^{-1}$$

Problem. Build the theory of non-invertible finite dynamical systems (NIFDS).

Let $D = \langle\sigma\rangle = \{1, \sigma, \sigma^2, \cdots\}$ be a semigroup generating only one element σ. Then a NIFDS is nothing but finite D-set. A NIFDS X is **irreducible** if $X = A \cup B$ implies $A = X$ or $B = X$. Any irreducible NIFDS has the form $D(h, n) = D/(\sigma^h = \sigma^{h+n})$.

Let $G_0(C)$ be a abelian group generated by $[X]$, where X is satisfying the relation:

(i) $[X] = [Y]$ iff $|\mathrm{Hom}(D[h, n)$ for any $h \geq 0$and $n \geq 1$.
(ii) $X[cupY] + [X \cap Y] = [X] + [Y]$, $[\emptyset] = 0$

We call $G_0(C)$ the Burnside ring of a semigroup D. The Burnside homomorphism is defined by

$$\varphi : G_0(D) \to \mathrm{gh}(\mathbb{Z}) = \prod_{(h,n)} \mathbb{Z}$$
$$; [X] \mapsto \mathrm{Hom}(D(h, n), X]$$

By this way, we have the theory of Burnside rings.

4.3 Rational DS's ?

Remember the equation (2). In this case, $\boldsymbol{b} = (b_n/b)$ is not corresponding to any dynamical system. Since b_n is the number of elements of G of order n, b_n/n is not an integer in general. But there exists a "rational DS":

$$X(\boldsymbol{b}) = \sum_{g \in G} \frac{1}{|g|} C(n)$$

In particular, we can not treat absolute Weil zeta function as AMZ of DS. Is there any idea by which we involve it in the theory of dynamical zeta functions.

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