

Poly-Cauchy numbers and poly-Bernoulli numbers

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1 Introduction

The *Cauchy numbers* of the first kind, denoted by c_n ([5]), are defined by the integral of the falling factorial:

$$c_n = \int_0^1 x(x-1)\dots(x-n+1)dx.$$

The generating function of the Cauchy numbers of the first kind c_n is given by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}$$

([23]).

Cauchy numbers are not so famous, though they seem to have similar properties to those of the *Bernoulli numbers*. The classical Bernoulli numbers B_n are defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad \left(B_1 = -\frac{1}{2} \right).$$

Before the terminology of Cauchy numbers appeared in Comtet's book ([5]), the concept of the Cauchy numbers was first introduced by Nörlund ([24, pp.146–147]) in 1924. Here, the higher order Bernoulli numbers $B_n^{(r)}$ are defined by

$$\left(\frac{x}{e^x - 1} \right)^r = \sum_{n=0}^{\infty} B_n^{(r)} \frac{x^n}{n!} \quad (|x| < 2\pi)$$

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or

$$\left(\frac{\ln(1+x)}{x}\right)^r = r \sum_{n=0}^{\infty} \frac{B_n^{(r+n)} x^n}{r+n} \frac{1}{n!} \quad (|x| < 1).$$

See also [8, p.257,p.259]. Then

$$B_n^{(n)} = \int_0^1 (x-1)(x-2)\cdots(x-n)dx$$

or

$$B_{n+1}^{(n)} = -n \int_0^1 x(x-1)\cdots(x-n)dx.$$

Hence, $c_n = -B_n^{(n-1)}/(n-1)$. Ch. Jordan studied the Bernoulli numbers of the second kind b_n ([13, p.131]), defined by

$$b_n = \psi_{n+1}(1) - \psi_{n+1}(0) = \int_0^1 \binom{x}{n} dx.$$

Hence, $b_n = c_n/n!$. In 1961 Carlitz ([4]) introduced the numbers β_n , defined by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} \beta_n \frac{x^n}{n!}$$

Namely, $\beta_n = c_n$.

Cauchy numbers and Bernoulli numbers are much related to the Stirling numbers of the first kind and of the second kind. The (unsigned) Stirling numbers of the first kind $\left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right]$ arise as coefficients of the rising factorial

$$x(x+1)\cdots(x+n-1) = \sum_{m=0}^n \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] x^m.$$

The Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\}$ are determined by

$$\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\} = \frac{1}{m!} \sum_{j=0}^m (-1)^j \binom{m}{j} (m-j)^n.$$

There are many identities about the Bernoulli numbers. They are much related to the (unsigned) Stirling numbers of the first kind $\left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right]$ and the

Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\}$. Some of them are

$$\frac{1}{n!} \sum_{m=0}^n (-1)^m \left[\begin{smallmatrix} n+1 \\ m+1 \end{smallmatrix} \right] B_m = \frac{1}{n+1},$$

$$B_n = (-1)^n \sum_{m=0}^n \left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\} \frac{(-1)^m m!}{m+1}.$$

The corresponding identities of the classical Cauchy numbers are

$$\sum_{m=0}^n \left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\} c_m = \frac{1}{n+1},$$

$$c_n = (-1)^n \sum_{m=0}^n \left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right] \frac{(-1)^m}{m+1}.$$

2 Polylogarithms

The k -th *polylogarithm* function is defined by

$$\text{Li}_k(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^k}.$$

The k -th *polylogarithm factorial* function is defined by

$$\text{Lif}_k(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!(m+1)^k}.$$

For $k \geq 2$

$$x \frac{d}{dx} \text{Li}_k(x) = \text{Li}_{k-1}(x),$$

so

$$\text{Li}_k(x) = \int_0^x \frac{\text{Li}_{k-1}(t)}{t} dt;$$

on the other hand,

$$\frac{d}{dx} (x \text{Lif}_k(x)) = \text{Lif}_{k-1}(x),$$

so

$$\text{Lif}_k(x) = \frac{1}{x} \int_0^x \text{Lif}_{k-1}(t) dt.$$

In special, for $k = 0, 1$ we have

$$\text{Li}_0(x) = \frac{x}{1-x}, \quad \text{Li}_1(x) = -\ln(1-x)$$

and

$$\text{Lif}_0(x) = e^x, \quad \text{Lif}_1(x) = (e^x - 1)/x.$$

For $k = -r$ we have

$$\text{Li}_{-r}(x) = \frac{1}{(1-x)^{r+1}} \sum_{j=0}^r \left\langle \begin{matrix} r \\ j \end{matrix} \right\rangle x^{r-j} \quad (r = 0, 1, 2, \dots)$$

([3]), where

$$\left\langle \begin{matrix} r \\ j \end{matrix} \right\rangle = \sum_{l=0}^{j+1} (-1)^l \binom{r+1}{l} (j-l+1)^r$$

are the Eulerian numbers.

On the other hand, for $k = -r$ we have

$$\text{Lif}_{-r}(x) = e^x \sum_{j=0}^r \left\{ \begin{matrix} r+1 \\ j+1 \end{matrix} \right\} x^j \quad (r = 0, 1, 2, \dots).$$

We have the record for the first some values r .

$$\begin{aligned} \text{Lif}_0(x) &= e^x, \\ \text{Lif}_{-1}(x) &= (1+x)e^x, \\ \text{Lif}_{-2}(x) &= (1+3x+x^2)e^x, \\ \text{Lif}_{-3}(x) &= (1+7x+6x^2+x^3)e^x, \\ \text{Lif}_{-4}(x) &= (1+15x+25x^2+10x^3+x^4)e^x, \\ \text{Lif}_{-5}(x) &= (1+31x+90x^2+65x^3+15x^4+x^5)e^x. \end{aligned}$$

In 1997 M. Kaneko ([18]) introduced the *poly-Bernoulli numbers* $B_n^{(k)}$ by

$$\frac{\text{Li}_k(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}.$$

When $k = 1$, $B_n^{(1)}$ is the classical Bernoulli number with $B_1^{(1)} = 1/2$.

Recently, we [19] introduced the *poly-Cauchy numbers* $c_n^{(k)}$ by

$$\text{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}.$$

When $k=1$, $c_n^{(1)} = c_n$ is the classical Cauchy number.

Poly-Cauchy numbers of the first kind $c_n^{(k)}$ may be defined by

$$c_n^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k} (x_1 x_2 \cdots x_k) (x_1 x_2 \cdots x_k - 1) \cdots (x_1 x_2 \cdots x_k - n + 1) dx_1 dx_2 \cdots dx_k.$$

In addition, *poly-Cauchy numbers of the second kind* $\hat{c}_n^{(k)}$ are defined by

$$\hat{c}_n^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k} (-x_1 x_2 \cdots x_k) (-x_1 x_2 \cdots x_k - 1) \cdots (-x_1 x_2 \cdots x_k - n + 1) dx_1 dx_2 \cdots dx_k.$$

The generating function of the poly-Bernoulli numbers are written in terms of iterated integrals:

$$\frac{e^x}{e^x - 1} \underbrace{\int_0^x \frac{1}{e^x - 1} \cdots \int_0^x \frac{1}{e^x - 1}}_{k-1} \times x \underbrace{dx \cdots dx}_{k-1} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}.$$

An explicit formula for $B_n^{(k)}$ is given by

$$B_n^{(k)} = (-1)^n \sum_{m=0}^n \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \frac{(-1)^m m!}{(m+1)^k} \quad (n \geq 0, k \geq 1). \quad (1)$$

The generating function of the poly-Cauchy numbers can be also written in the form of iterated integrals:

$$\begin{aligned} \frac{1}{\ln(1+x)} \underbrace{\int_0^x \frac{1}{(1+x)\ln(1+x)} \cdots \int_0^x \frac{1}{(1+x)\ln(1+x)}}_{k-1} \times x \underbrace{dx \cdots dx}_{k-1} \\ = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}. \end{aligned}$$

An explicit formula for $c_n^{(k)}$ is given by

$$c_n^{(k)} = (-1)^n \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-1)^m}{(m+1)^k}. \quad (2)$$

There are some relations between poly-Cauchy numbers and poly-Bernoulli numbers.

Theorem 1. For $n \geq 1$ we have

$$B_n^{(k)} = \sum_{l=1}^n \sum_{m=1}^n m! \begin{Bmatrix} n \\ m \end{Bmatrix} \begin{Bmatrix} m-1 \\ l-1 \end{Bmatrix} c_l^{(k)},$$

$$c_n^{(k)} = \sum_{l=1}^n \sum_{m=1}^n \frac{(-1)^{n-m}}{m!} \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ l \end{bmatrix} B_l^{(k)}.$$

3 Duality theorem

It is known that the duality theorem holds for poly-Bernoulli numbers ([18]). Namely,

$$B_n^{(-k)} = B_k^{(-n)} \quad (n, k \geq 0).$$

It is due to the symmetric formula:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_n^{(-k)} \frac{x^n y^k}{n! k!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}.$$

It follows that

$$B_n^{(-k)} = \sum_{m=0}^n (-1)^{m+n} m! \begin{Bmatrix} n \\ m \end{Bmatrix} (m+1)^k,$$

$$B_n^{(-k)} = \sum_{j=0}^k (j!)^2 \begin{Bmatrix} n+1 \\ j+1 \end{Bmatrix} \begin{Bmatrix} k+1 \\ j+1 \end{Bmatrix}.$$

However, the duality theorem does not hold for poly-Cauchy numbers. In fact, we have

Proposition 1.

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_n^{(-k)} \frac{x^n y^k}{n! k!} = e^y (1+x)^{e^y},$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \hat{c}_n^{(-k)} \frac{x^n y^k}{n! k!} = \frac{e^y}{(1+x)^{e^y}}.$$

By using Proposition 1 we have explicit expressions of the poly-Cauchy numbers with negative indices.

Theorem 2 ([16]).

$$c_n^{(-k)} = \sum_{j=0}^k (-1)^{n+j} j! \left(\begin{bmatrix} n \\ j \end{bmatrix} - n \begin{bmatrix} n-1 \\ j \end{bmatrix} \right) \left\{ \begin{matrix} k+1 \\ j+1 \end{matrix} \right\},$$

$$\hat{c}_n^{(-k)} = \sum_{j=0}^k (-1)^n j! \begin{bmatrix} n+1 \\ j+1 \end{bmatrix} \left\{ \begin{matrix} k+1 \\ j+1 \end{matrix} \right\}.$$

Moreover, using Theorem 2 with (2) we have the following congruence results.

Theorem 3. For any positive integer k , $c_n^{(-k)} \equiv c_n^{(-k-4)} \pmod{10}$ and $\hat{c}_n^{(-k)} \equiv \hat{c}_n^{(-k-4)} \pmod{10}$. In special, when $n = 1$, for $k \geq 1$, $c_1^{(-k-4)} \equiv c_1^{(-k)} \pmod{30}$ and $\hat{c}_1^{(-k-4)} \equiv \hat{c}_1^{(-k)} \pmod{30}$.

Theorem 4. For $k \geq 1$ we have

$$c_n^{(-k)} \equiv \begin{cases} 0 \pmod{2} & \text{if } n = 1 \text{ or } n \geq 4; \\ 1 \pmod{2} & \text{if } n = 2, 3, \end{cases}$$

$$\hat{c}_n^{(-k)} \equiv \begin{cases} 0 \pmod{2} & \text{if } n = 1 \text{ or } n \geq 4; \\ 1 \pmod{2} & \text{if } n = 2, 3. \end{cases}$$

4 Sums of products

Sums of products of Bernoulli numbers

$$\sum_{\substack{i_1+\dots+i_m=n \\ i_1, \dots, i_m \geq 0}} \frac{n!}{i_1! \cdots i_m!} B_{i_1} \cdots B_{i_m} \quad (m \geq 1, n \geq 0)$$

have been considered by many authors (see, e.g. [1, 2, 6]). When $m = 2$, one has the famous Euler's identity:

$$\sum_{i=0}^n \binom{n}{i} B_i B_{n-i} = -n B_{n-1} - (n-1) B_n \quad (n \geq 1). \quad (3)$$

Kamano ([14]) considered the sums of products of Bernoulli numbers, including poly-Bernoulli numbers

$$S_m^{(k)}(n) := \sum_{\substack{i_1+\dots+i_m=n \\ i_1, \dots, i_m \geq 0}} \frac{n!}{i_1! \cdots i_m!} B_{i_1} \cdots B_{i_{m-1}} B_{i_m}^{(k)} \quad (m \geq 1, n \geq 0).$$

Then, $S_m^{(k)}(n)$ satisfies the following relation:

Proposition 2.

$$\begin{aligned} \sum_{l=0}^m (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} S_{m+1}^{(k-l)}(n) \\ = \begin{cases} \frac{n!}{(n-m)!} \sum_{r=0}^m \begin{bmatrix} m \\ r \end{bmatrix} B_{n-m+r}^{(k)} & (n \geq m), \\ 0 & (0 \leq n \leq m-1). \end{cases} \end{aligned}$$

Kamano also showed the explicit formulae $S_m^{(k)}(n)$ for $m = 2, 3$. For example, when $m = 2$ we have

Proposition 3. For $k \geq 1$ and $n \geq 0$,

$$\begin{aligned} S_2^{(0)}(n) &= B_n^{(1)}, \\ S_2^{(k)}(n) &= B_n^{(1)} - n \sum_{j=1}^k B_n^{(j)}, \\ S_2^{(-k)}(n) &= B_n^{(1)} + n \sum_{j=0}^{k-1} B_n^{(-j)}. \end{aligned}$$

It seemed to be difficult to give an explicit formula for $S_m^{(k)}(n)$ for $m \geq 4$, but recently a general formula for all $m \geq 1$ is given.

Theorem 5 ([20]). *For $m \geq 1$, $n \geq 0$ and $k \geq 1$, we have*

$$\begin{aligned}
S_{m+1}^{(0)}(n) &= S_m^{(1)}(n), \\
S_{m+1}^{(k)}(n) &= \sum_{r=0}^{m-1} (-1)^r r! \binom{n}{r} \sum_{i=0}^r \binom{r}{i} \frac{(-1)^i}{(i+1)^k} S_{m+1-r}^{(0)}(n-r) \\
&\quad + (-1)^m \binom{n}{m} \sum_{\substack{j_1+\dots+j_m \leq k-1 \\ j_1, \dots, j_m \geq 0}} \frac{1}{2^{j_2} \dots m^{j_m}} \sum_{\nu=1}^m \begin{bmatrix} m \\ \nu \end{bmatrix} B_{n-m+\nu}^{(1+j_1)}, \\
S_{m+1}^{(-k)}(n) &= \sum_{r=0}^{m-1} (-1)^r r! \binom{n}{r} \sum_{i=0}^r \binom{r}{i} (-1)^i (i+1)^k S_{m+1-r}^{(0)}(n-r) \\
&\quad + \binom{n}{m} \sum_{\substack{j_1+\dots+j_m \leq k \\ j_1, \dots, j_m \geq 1}} 2^{j_2} \dots m^{j_m} \sum_{\nu=1}^m \begin{bmatrix} m \\ \nu \end{bmatrix} B_{n-m+\nu}^{(1-j_1)}.
\end{aligned}$$

Sums of products of Cauchy numbers

$$\sum_{\substack{i_1+\dots+i_m=n \\ i_1, \dots, i_m \geq 0}} \frac{n!}{i_1! \dots i_m!} c_{i_1} \dots c_{i_m} \quad (m \geq 1, n \geq 0)$$

were studied by Zhao ([25]). Consider the sums of products of Cauchy numbers, including poly-Cauchy numbers

$$T_m^{(k)}(n) := \sum_{\substack{i_1+\dots+i_m=n \\ i_1, \dots, i_m \geq 0}} \frac{n!}{i_1! \dots i_m!} c_{i_1} \dots c_{i_{m-1}} c_{i_m}^{(k)} \quad (m \geq 1, n \geq 0).$$

Then, $T_m^{(k)}(n)$ satisfies the following relation:

Proposition 4 ([22]).

$$\begin{aligned}
&\sum_{l=0}^m (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} T_{m+1}^{(k-l)}(n) \\
&= \begin{cases} \sum_{l=0}^m \sum_{i=0}^{n-m} \frac{n!}{i!} \binom{l}{n-m-i} \begin{Bmatrix} m \\ l \end{Bmatrix} c_{l+i}^{(k)} & (n \geq m); \\ 0 & (0 \leq n \leq m-1). \end{cases}
\end{aligned}$$

When $m = 2$, we have the following explicit formulae.

Proposition 5 ([22]). *For $n \geq 0$ and $k \geq 1$ we have*

$$\begin{aligned} T_2^{(0)}(n) &= c_n + nc_{n-1}, \\ T_2^{(k)}(n) &= T_2^{(0)}(n) - n \sum_{j=1}^k (c_n^{(j)} + (n-1)c_{n-1}^{(j)}), \\ T_2^{(-k)}(n) &= T_2^{(0)}(n) + n \sum_{j=0}^{k-1} (c_n^{(-j)} + (n-1)c_{n-1}^{(-j)}). \end{aligned}$$

Putting $k = 1$ in the second identity, we have

Corollary 1 ([25]).

$$\sum_{i=0}^n \binom{n}{i} c_i c_{n-i} = -n(n-2)c_{n-1} - (n-1)c_n \quad (n \geq 0).$$

This is an analogue of Euler's identity (3).

In general, we can obtain the following explicit expression of $T_m^{(k)}(n)$ for any general $m \geq 2$.

Theorem 6. *For $n \geq 0$ and $k > 0$ we have*

$$\begin{aligned} T_m^{(0)}(n) &= T_{m-1}^{(1)}(n) + nT_{m-1}^{(1)}(n-1), \\ T_m^{(k)}(n) &= \sum_{r=0}^{m-2} (-1)^r \binom{n}{r} \sum_{i=0}^r \binom{r}{i} \frac{(-1)^i}{(i+1)^k} T_{m-r}^{(0)}(n-r) \\ &\quad + \frac{(-1)^{m-1}n!}{(n-m+1)!} \sum_{\substack{j_1+j_2+\dots+j_{m-1}=k+m-2 \\ j_1, j_2, \dots, j_{m-1} \geq 1}} 2^{-j_2} 3^{-j_3} \dots (m-1)^{-j_{m-1}} \sum_{j=1}^{j_1} \sum_{\kappa=0}^{m-1} P_{m,\kappa}(n) c_{n-\kappa}^{(j)}, \\ T_m^{(-k)}(n) &= \sum_{r=0}^{m-2} (-1)^r \binom{n}{r} \sum_{i=0}^r \binom{r}{i} (-1)^i (i+1)^k T_{m-r}^{(0)}(n-r) \\ &\quad + \frac{n!}{(n-m+1)!} \sum_{\substack{j_1+j_2+\dots+j_{m-1}=k-m+1 \\ j_1, j_2, \dots, j_{m-1} \geq 0}} 2^{j_2} 3^{j_3} \dots (m-1)^{j_{m-1}} \sum_{j=0}^{j_1} \sum_{\kappa=0}^{m-1} P_{m,\kappa}(n) c_{n-\kappa}^{(-j)}. \end{aligned}$$

where

$$P_{m,\kappa}(n) = \sum_{t=0}^{\kappa} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \binom{m-t-1}{m-\kappa-1} \frac{(n-m+1)!}{(n-m-\kappa+t+1)!}$$

$(\kappa = 0, 1, \dots, m-2)$

and

$$P_{m,m-1}(n) = \sum_{t=0}^{m-2} \left\{ \begin{matrix} m-1 \\ m-t-1 \end{matrix} \right\} \frac{(n-m+1)!}{(n-2m+t+2)!}$$

$$= (n-m+1)^{m-1}.$$

5 Hypergeometric Bernoulli numbers and hypergeometric Cauchy numbers

Hypergeometric Bernoulli numbers $B_{N,n}$ ($N \geq 1$, $n \geq 0$) ([7, 9, 10, 11, 12]) are defined by

$$\frac{1}{{}_1F_1(1; N+1; x)} = \frac{x^N/N!}{e^x - \sum_{n=0}^{N-1} x^n/n!} = \sum_{n=0}^{\infty} B_{N,n} \frac{x^n}{n!},$$

where ${}_1F_1(a; b; z)$ is the confluent hypergeometric function defined by

$${}_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!}$$

with the Pochhammer symbol $(x)_n = x(x+1)\dots(x+n-1)$ ($n \geq 1$) and $(x)_0 = 1$. When $N = 1$, $B_{1,n} = B_n$ are classical Bernoulli numbers.

Hypergeometric Cauchy numbers $c_{N,n}$ ($N \geq 1$, $n \geq 0$) ([21]) are defined by

$$\frac{1}{{}_2F_1(1, N; N+1; -x)} = \frac{(-1)^{N-1} x^N/N}{\ln(1+x) - \sum_{n=1}^{N-1} (-1)^{n-1} x^n/n} = \sum_{n=0}^{\infty} c_{N,n} \frac{x^n}{n!},$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function defined by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!}.$$

When $N = 1$, $c_{1,n} = c_n$ are classical Cauchy numbers.

We record of the first few values of $c_{N,n}$:

$$c_{N,0} = 1,$$

$$c_{N,1} = \frac{N}{N+1},$$

$$c_{N,2} = -\frac{2N}{(N+1)^2(N+2)},$$

$$c_{N,3} = \frac{6N(N^2 + N + 2)}{(N+1)^3(N+2)(N+3)}$$

$$c_{N,4} = -\frac{4!N(N^5 + 5N^4 + 14N^3 + 24N^2 + 20N + 12)}{(N+1)^4(N+2)^2(N+3)(N+4)},$$

$$c_{N,5} = \frac{5!N(N^7 + 8N^6 + 35N^5 + 96N^4 + 160N^3 + 184N^2 + 116N + 48)}{(N+1)^5(N+2)^2(N+3)(N+4)(N+5)}.$$

The sums of products of hypergeometric Bernoulli numbers were studied by Kamano ([15]) and those of hypergeometric Cauchy numbers are also studied in [21].

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