数理解析研究所講究録
第 1874 巻 2014 年 161-173



# Hypergeometric type generating functions of several variables associated with the Lerch zeta-function (summarized version) *

Masanori KATSURADA
Department of Mathematics, Hiyoshi Campus, Keio Univeristy
(慶應義義塾大学・経済学部・数学教室・桂田 昌紀)

**Abstract**

This is a summarized version of the forthcoming paper [10].

Let $s$, $z$ and $(z_0, \boldsymbol{z}) = (z_0, z_1, \ldots, z_n)$ be complex variables, and $\zeta(s, z, \lambda)$ denote the Lerch zeta-function defined by (1.1) below. We introduce in the present article a class of generating functions and their confluent analogues, denoted by $\mathcal{Z}_{a,\lambda}^{(n)}\left(\begin{smallmatrix} s,\boldsymbol{\beta} \\ \gamma \end{smallmatrix}; z_0, \boldsymbol{z}\right)$ and $\widehat{\mathcal{Z}}_{a,\lambda}^{(n)}\left(\begin{smallmatrix} s,\boldsymbol{\beta}_{n-1} \\ \gamma \end{smallmatrix}; z_0, \boldsymbol{z}\right)$ respectively (see (2.1) and (2.3)), in the forms of the fourth Lauricella hypergeometric type (of several variables) associated with $\zeta(s, z, \lambda)$. It is shown that complete asymptotic expansions of $\mathcal{Z}_{a,\lambda}^{(n)}\left(\begin{smallmatrix} s,\boldsymbol{\beta} \\ \gamma \end{smallmatrix}; z_0, \boldsymbol{z}\right)$ exit when $z_0 \to 0$ (Theorem 1) as well as when $z_0 \to \infty$ (Theorem 2) through the sectorial region $|\arg z - \theta_0| < \pi/2$ with any fixed angle $\theta_0 \in [-\pi/2, \pi/2]$, while other $z_j$'s move through the same sector satisfying the conditions $z_j \ll z_0$ $(j = 1, \ldots, n)$. Similar asymptotic results also hold for $\widehat{\mathcal{Z}}_{a,\lambda}^{(n)}\left(\begin{smallmatrix} s,\boldsymbol{\beta} \\ \gamma \end{smallmatrix}; z_0, \boldsymbol{z}\right)$ (Theorems 3 and 4) through the confluence operation in (2.3). Our main formulae (3.1) and (3.4) (resp. (3.7) and (3.10)) first assert that $\mathcal{Z}_{a,\lambda}^{(n)}\left(\begin{smallmatrix} s,\boldsymbol{\beta} \\ \gamma \end{smallmatrix}; z_0, \boldsymbol{z}\right)$ (resp. $\widehat{\mathcal{Z}}_{a,\lambda}^{(n)}\left(\begin{smallmatrix} s,\boldsymbol{\beta}_{n-1} \\ \gamma \end{smallmatrix}; z_0, \boldsymbol{z}\right)$) can be continued to a meromorphic function of $s$ over the whole $s$-plane, to the whole poly-sector $|\arg z_j| < \pi$ $(j = 0, 1, \ldots, n)$, and for all $(\boldsymbol{\beta}, \gamma) \in \mathbb{C}^n \times (\mathbb{C} \setminus \{0, -1, \ldots\})$ (resp. for all $(\boldsymbol{\beta}_{n-1}, \gamma) \in \mathbb{C}^{n-1} \times (\mathbb{C} \setminus \{0, -1, \ldots\})$). We can further apply (3.1) and (3.4) to deduce complete asymptotic expansions of $(\partial/\partial s)^m \mathcal{Z}_{a,\lambda}^{(n)}\left(\begin{smallmatrix} s,\boldsymbol{\beta} \\ \gamma \end{smallmatrix}; z_0, \boldsymbol{z}\right)$ $(m = 1, 2, \ldots)$ at any integer arguments $s = l \in \mathbb{Z}$ when $(z_0, \boldsymbol{z})$ becomes small (Corollary 6) and large (Corollary 8) under the same settings as in Theoerms 1 and 2. Furthermore, several applications of Theorems 1–4 in the cases of $n = 1$ and 2 are finally presented.

## Introduction

Throughout this article, $s = \sigma + \sqrt{-1}t$, $z$ and $(z_0, \boldsymbol{z}) = (z_0, z_1, \ldots, z_n)$ are complex variables with $|\arg z| < \pi$ and $|\arg z_j| < \pi$ $(j = 0, 1, \ldots, n)$, and $a$ and $\lambda$ real parameters with $a > 0$. We hereafter set $e(\lambda) = e^{2\pi\lambda\sqrt{-1}}$, use the vectorial notation $\boldsymbol{x} = (x_1, \ldots, x_m)$ with the abbreviation

$$\langle \boldsymbol{x} \rangle = x_1 + \cdots + x_m$$

for any $m \geq 1$ and any complex $x_i$ $(i = 1, \ldots, m)$, and further write $\boldsymbol{x}_{m-1} = (x_1, \ldots, x_{m-1})$ and

$$\frac{\boldsymbol{x}}{y} = \left(\frac{x_1}{x}, \ldots, \frac{x_m}{y}\right)$$

for any $y \neq 0$. The Lerch zeta-function $\zeta(s, z, \lambda)$ is defined by the Dirichlet series

$$\zeta(s, z, \lambda) = \sum_{l=0}^{\infty} e(\lambda l)(l + z)^{-s} \qquad (\sigma = \operatorname{Re} s > 1), \tag{1.1}$$

and its meromorphic continuation over the whole $s$-plane; this is an entire function when $\lambda \in \mathbb{R} \setminus \mathbb{Z}$, while if $\lambda \in \mathbb{Z}$ it reduces to the Hurwitz zeta-function $\zeta(s, a)$, and so $\zeta(s) =$

---

2010 *Mathematics Subject Classification.* Primary 11E45; Secondary 33C65.

*$^*$Key words and phrases.* Lerch zeta-function, Lauricella hypergeometric function, Mellin-Barnes integral, asymptotic expansion.

$\zeta(s,1)$ is the Riemann zeta-function. We remark here that the notation (1.1) differs from the original $\phi(z,\lambda,s)$ due to Lerch [13], in order to retain notational consistency with other terminology.

It is the principal aim of the present article to introduce a class of generating functions and their confluent analogues, denoted by $\mathcal{Z}_{a,\lambda}^{(n)}\left({s,\boldsymbol{\beta} \atop \gamma};z_0,\boldsymbol{z}\right)$ and $\widehat{\mathcal{Z}}_{a,\lambda}^{(n)}\left({s,\boldsymbol{\beta}_{n-1} \atop \gamma};z_0,\boldsymbol{z}\right)$ respectively (see (2.1) and (2.4) below), in the forms of the (fourth) Lauricella hypergeometric type (of several variables) associated with $\zeta(s,z,\lambda)$. We shall first show that complete asymptotic expansions of $\mathcal{Z}_{a,\lambda}^{(n)}\left({s,\boldsymbol{\beta} \atop \gamma};z_0,\boldsymbol{z}\right)$ and $\widehat{\mathcal{Z}}_{a,\lambda}^{(n)}\left({s,\boldsymbol{\beta} \atop \gamma};z_0,\boldsymbol{z}\right)$ exist when $(z_0,\boldsymbol{z})$ becomes small (Theorems 1 and 3) and large (Theorems 2 and 4) under certain settings on the movement of $(z_0,\boldsymbol{z})$. Several applications of Theorems 1–4 will further be presented. Before stating our main results, some necessary notations and terminology will be prepared.

Let $\Gamma(s)$ be the gamma function, $(s)_k=\Gamma(s+k)/\Gamma(s)$ for any $k\in\mathbb{Z}$ the shifted factorial of $s$, and write

$$\Gamma\binom{\boldsymbol{\mu}}{\boldsymbol{\nu}}=\Gamma\binom{\mu_1,\ldots,\mu_h}{\nu_1,\ldots,\nu_k}=\frac{\prod_{i=1}^h\Gamma(\mu_i)}{\prod_{j=1}^k\Gamma(\nu_j)}$$

for complex vectors $\boldsymbol{\mu}=(\mu_1,\ldots,\mu_h)$ and $\boldsymbol{\nu}=(\nu_1,\ldots,\nu_k)$. In the sequel the sets of non-negative and non-positive integers are respectively denoted by $\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and $-\mathbb{N}_0=\{-k\mid k\in\mathbb{N}_0\}$. The (fourth) Lauricella hypergeometric function of $m$-variables $x_i$ $(i=1,\ldots,m)$ is defined by the $m$-ple power series

$$F_D^{(m)}\left({\alpha,\beta_1,\ldots,\beta_m \atop \gamma};x_1,\ldots,x_m\right)=\sum_{k_1,\ldots,k_m=0}^{\infty}\frac{(\alpha)_{k_1+\cdots+k_m}(\beta_1)_{k_1}\cdots(\beta_m)_{k_m}}{(\gamma)_{k_1+\cdots+k_m}k_1!\cdots k_m!}x_1^{k_1}\cdots x_m^{k_m} \tag{1.2}$$

for complex parameters $\alpha$, $\beta_i$ $(i=1,\ldots,m)$ and $\gamma\neq-k$ $(k\in\mathbb{N}_0)$, where the series converges absolutely in the poly-disk $|x_i|<1$ $(i=1,\ldots,m)$; this is continued to a one-valued holomorphic function of $(\alpha,\boldsymbol{\beta},\gamma,\boldsymbol{x})$ for all $(\alpha,\boldsymbol{\beta},\gamma)\in\mathbb{C}^{m+1}\times\{\mathbb{C}\setminus(-\mathbb{N}_0)\}$, and $\boldsymbol{x}$ in the poly-sector $|\arg(1-x_i)-\varphi_0|<\pi/2$ $(i=1,\ldots,m)$ for any angle fixed with $\varphi_0\in[-\pi/2,\pi/2]$ (cf. [1]). Note that (1.2) reduces when $m=1$ to Gauß' hypergeomtric function ${}_2F_1\left({\alpha,\beta \atop \gamma};x\right)$, and when $m=2$ to (the first) Appell's hyepergeometric function $F_1\left({\alpha,\beta_1,\beta_2 \atop \gamma};x_1,x_2\right)$. The abbreviations

$$(\boldsymbol{\beta})_{\boldsymbol{k}}=(\beta_1)_{k_1}\cdots(\beta_m)_{k_m},\qquad \boldsymbol{k}!=k_1!\cdots k_m!,$$
$$\boldsymbol{x}^{\boldsymbol{k}}=x_1^{k_1}\cdots x_m^{k_m}$$

for $\boldsymbol{k}=(k_1,\ldots,k_m)$ and $\boldsymbol{x}=(x_1,\ldots,x_m)$ allow to rewrite (1.2) in a more concise form

$$F_D^{(m)}\left({\alpha,\boldsymbol{\beta} \atop \gamma};\boldsymbol{x}\right)=\sum_{\boldsymbol{k}\geq\boldsymbol{0}}\frac{(\alpha)_{\langle\boldsymbol{k}\rangle}(\boldsymbol{\beta})_{\boldsymbol{k}}}{(\gamma)_{\langle\boldsymbol{k}\rangle}\boldsymbol{k}!}\boldsymbol{x}^{\boldsymbol{k}},$$

where (and hereafter) the summation condition $\boldsymbol{k}\geq\boldsymbol{h}$ means that the sum runs over all indices $\boldsymbol{k}$ with $k_j\geq h_j$ $(j=1,\ldots,n)$. Furthermore, a new class of $m$-variable hypergeometric functions $\widehat{F}_D^{(m)}\left({\alpha,\boldsymbol{\beta}_{m-1} \atop \gamma};\boldsymbol{x}\right)$ is obtained from $F_D^{(m)}\left({\alpha,\boldsymbol{\beta} \atop \gamma};\boldsymbol{x}\right)$ through the confluence operation

$$F_D^{(m)}\left({\alpha,\boldsymbol{\beta}_{m-1},\beta_n \atop \gamma};\boldsymbol{x}_{m-1},\frac{x_m}{\beta_m}\right)\xrightarrow[(\beta_m\to+\infty)]{}\widehat{F}_D^{(m)}\left({\alpha,\boldsymbol{\beta}_{m-1} \atop \gamma};\boldsymbol{x}\right)=\sum_{\boldsymbol{k}\geq\boldsymbol{0}}\frac{(\alpha)_{\langle\boldsymbol{k}\rangle}(\boldsymbol{\beta}_{m-1})_{\boldsymbol{k}_{m-1}}}{(\gamma)_{\langle\boldsymbol{k}\rangle}\boldsymbol{k}!}\boldsymbol{x}^{\boldsymbol{k}}. \tag{1.3}$$

Note that the case $m = 1$ of (1.3) gives Kummer's hypergeometric function

$$\widehat{F}_D^{(1)}\left({\alpha \atop \gamma}; x\right) = {}_1F_1\left({\alpha \atop \gamma}; x\right) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\gamma)_k k!} x^k$$

for $|x| < +\infty$, while $m = 2$ the confluent form of $F_1\left({\alpha,\beta_1,\beta_2 \atop \gamma}; x_1, x_2\right)$, defined by

$$\widehat{F}_D^{(2)}\left({\alpha, \beta_1 \atop \gamma}; x_1, x_2\right) = \Phi_1\left({\alpha, \beta_1 \atop \gamma}; x_1, x_2\right) = \sum_{k_1,k_2=0}^{\infty} \frac{(\alpha)_{k_1+k_2}(\beta_1)_{k_1}}{(\gamma)_{k_1+k_2} k_1! k_2!} x_1^{k_1} x_2^{k_2}$$

for $|x_1| < 1$ and $|x_2| < +\infty$ (cf. [4]).

## Main objects

We can now introduce the hypergeometric type generating function $\mathcal{Z}_{a,\lambda}^{(n)}\left({s,\boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right)$ of $(n+1)$-variables $(z_0, \boldsymbol{z}) = (z_0, z_1, \ldots, z_n)$ associated with $\zeta(s, a+z_0, \lambda)$, defined by the $n$-ple power series

$$\text{(2.1)} \qquad \begin{aligned} \mathcal{Z}_{a,\lambda}^{(n)}\left({s,\boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right) &= \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{(s)_{\langle \boldsymbol{k}\rangle}(\boldsymbol{\beta})_{\boldsymbol{k}}}{(\gamma)_{\langle \boldsymbol{k}\rangle}\boldsymbol{k}!} \zeta(s+\langle \boldsymbol{k}\rangle, a+z_0, \lambda)(-\boldsymbol{z})^{\boldsymbol{k}} \\ &= \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{(s)_{k_1+\cdots+k_n}(\beta_1)_{k_1}\cdots(\beta_n)_{k_n}}{(\gamma)_{k_1+\cdots+k_n} k_1!\cdots k_n!} \\ &\quad \times \zeta(s+k_1+\cdots+k_n, a+z_0, \lambda)(-z_1)^{k_1}\cdots(-z_n)^{k_n}, \end{aligned}$$

which converges absolutely in the domain $|z_j| < |\operatorname{Im} z_0|$ $(j = 1, \ldots, n)$. The change of the order of summations in (2.1) readily implies that

$$\text{(2.2)} \qquad \mathcal{Z}_{a,\lambda}^{(n)}\left({s,\boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right) = \sum_{l=0}^{\infty} e(\lambda l)(a+l+z_0)^{-s} F_D^{(n)}\left({s,\boldsymbol{\beta} \atop \gamma}; -\frac{\boldsymbol{z}}{a+l+z_0}\right)$$

for $\sigma > 1$; the cases $\boldsymbol{\beta} = \boldsymbol{0}$ and $\boldsymbol{z} = \boldsymbol{0}$ of (2.2) both reduce to

$$\mathcal{Z}_{a,\lambda}^{(n)}\left({s,\boldsymbol{0} \atop \gamma}; \boldsymbol{z}, z_0\right) = \mathcal{Z}_{a,\lambda}^{(n)}\left({s,\boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{0}\right) = \zeta(s, a+z_0, \lambda),$$

while the cases $n = 1$ and $n = 2$ respectively to

$$\mathcal{Z}_{a,\lambda}^{(1)}\left({s,\beta \atop \gamma}; z_0, z_1\right) = \sum_{l=0}^{\infty} e(\lambda l)(a+l+z_0)^{-s} {}_2F_1\left({s,\beta \atop \gamma}; -\frac{z_1}{a+l+z_0}\right),$$

$$\begin{aligned} \mathcal{Z}_{a,\lambda}^{(2)}\left({s,\beta_1,\beta_2 \atop \gamma}; z_0, z_1, z_2\right) &= \sum_{l=0}^{\infty} e(\lambda l)(a+l+z_0)^{-s} \\ &\quad \times F_1\left({s,\beta_1,\beta_2 \atop \gamma}; -\frac{z_1}{a+l+z_0}, -\frac{z_2}{a+l+z_0}\right) \end{aligned}$$

for $\sigma > 1$. It is further possible to obtain a new class of generating functions, denoted by $\widehat{\mathcal{Z}}_{a,\lambda}^{(n)}\left({s,\boldsymbol{\beta}_{n-1} \atop \gamma}; z_0, \boldsymbol{z}\right)$, from $\mathcal{Z}_{a,\lambda}^{(n)}\left({s,\boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right)$ through the confluence operation

$$\text{(2.3)} \quad \begin{aligned} \mathcal{Z}_{a,\lambda}^{(n)}\left({s,\boldsymbol{\beta}_{n-1},\beta_n \atop \gamma}; z_0, \boldsymbol{z}_{n-1}, \frac{z_n}{\beta_n}\right) &\xrightarrow[(\beta_n \to +\infty)]{} \widehat{\mathcal{Z}}_{a,\lambda}^{(n)}\left({s,\boldsymbol{\beta}_{n-1} \atop \gamma}; z_0, \boldsymbol{z}\right) \\ &= \sum_{l=0}^{\infty} e(\lambda l)(a+l+z_0)^{-s} \widehat{F}_D^{(n)}\left({s,\boldsymbol{\beta}_{n-1} \atop \gamma}; -\frac{\boldsymbol{z}}{a+l+z_0}\right) \end{aligned}$$

for $\sigma > 1$, where the change of the order of summations in the last expression gives

$$\widehat{\mathcal{Z}}_{a,\lambda}^{(n)}\left({s,\boldsymbol{\beta}_{n-1} \atop \gamma}; z_0, \boldsymbol{z}\right) = \sum_{\boldsymbol{k}\geq 0} \frac{(s)_{\langle\boldsymbol{k}\rangle}(\boldsymbol{\beta}_{n-1})_{\boldsymbol{k}_{n-1}}}{(\gamma)_{\langle\boldsymbol{k}\rangle}\boldsymbol{k}!}\zeta(s+\langle\boldsymbol{k}\rangle, a+z_0, \lambda)(-\boldsymbol{z})^{\boldsymbol{k}} \tag{2.4}$$

in the domain $|z_j| < |\operatorname{Im} z|$ $(j = 1, \ldots, n)$.

We shall give in the remaining of this section a brief overview of the history of research related to various generating functions associated with specific values of zeta-functions‡. Several power series involving the particular values of $\zeta(s,a)$ were first studied by Srivastava [18][19][20], while Klusch [11] treated the Taylor series for $\zeta(s, a+z, \lambda)$ in the variable $z \in \mathbb{C}$, and gave its many interesting applications. Hypergeometric type generating functions of $\zeta(s)$ were first introduced and studied by Raina-Srivastava [17] and the author [6][7], independently of each other; we refer the reader to the comprehensive account [21] into this direction. Hikami-Kirillov [5] more recently investigated hypergeometric generating functions of various $L$-function values in connection with $q$-hypergeometric series and quantum invariants. Hypergeometric type generating functions associated with $\zeta_\nu(s,a,w)$ (a weighted extension of $\zeta(s,a,\lambda)$) were first introduced and studied by Bin-Saad and Al-Gonah [3] and further by Bin-Saad [2]. Li-Kanemitsu-Tsukada [14] made Maijer's $G$-function theoretic interpretation of the results in [6][7], while similar $G$-function theoretic study on the results in [8] was made by Kuzumaki [12]. We next mention several relevant asymptotic aspects into this direction. Complete asymptotic expansions of $\zeta(s, a+z, \lambda)$ for small and large $z \in \mathbb{C}$ in the sector $|\arg z| < \pi$ was established by the author [8]. Matsumoto [15] investigated complete asymptotic expansions of $\zeta_2(s, a \mid (1, w))$ (Barnes' double zeta-function) for small and large basis parameter $w \in \mathbb{C}$ in $|\arg w| < \pi$. Onodera [16] more recently studied complete asymptotic expansions of $\zeta_m(s, a+x \mid \boldsymbol{\omega})$ (Barnes' multiple zeta-function) for small and large $x \in \mathbb{R}_+$ and one of $\omega_i$'s in the basis parameters $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m) \in \mathbb{R}_+^m$, where $\mathbb{R}_+$ denotes the set of positive real numbers.

### Asymptotic expansions for small and large $(z_0, \boldsymbol{z})$

To describe our results we introduce the generalized Bernoulli polynomials $B_k(x,y)$ $(k \in \mathbb{N}_0)$ for any parameters $x, y \in \mathbb{C}$ by the power series

$$\frac{ze^{xz}}{ye^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(x,y)}{k!} z^k,$$

centered at $z = 0$; this in particular gives

$$B_0(x,y) = \begin{cases} 1 & \text{if } y = 1; \\ 0 & \text{if } y \neq 1. \end{cases}$$

Note that $B_k(x) = B_k(x,1)$ are the usual Bernoulli polynomials, and so $B_k = B_k(0)$ are the usual Bernoulli numbers. The vertical straight path from $u - i\infty$ to $u + i\infty$ (with $u \in \mathbb{R}$) is hereafter denoted by $(u)$.

We first state the asymptotic expansion of $\mathcal{Z}_{a,\lambda}^{(n)}\left({s,\boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right)$ when $(z_0, \boldsymbol{z})$ becomes small.

**Theorem 1.** *Let $\theta_0$ be any angle fixed with $\theta_0 \in [-\pi/2, \pi/2]$. Then for any integer $K \geq 0$, in the region $\sigma > 1 - K$ except at $s = 1$ the formula*

$$\mathcal{Z}_{a,\lambda}^{(n)}\left({s,\boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right) = S_{a,\lambda,K}^{+}\left({s,\boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right) + R_{a,\lambda,K}^{+}\left({s,\boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right) \tag{3.1}$$

‡The author would like to make apology for insufficiency (in many respects) of the present survey.

*holds for all* $(z_0, \boldsymbol{z})$ *in the poly-sector* $|\arg z_j - \theta_0| < \pi/2$ $(j = 0, 1, \ldots, n)$ *and for all* $(\boldsymbol{\beta}, \gamma) \in \mathbb{C}^n \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}$. *Here*

$$(3.2) \qquad S^{+}_{a,\lambda,K}\left({s, \boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right) = \sum_{k=0}^{K-1} \frac{(-1)^k (s)_k}{k!} F^{(n)}_D\left({-k, \boldsymbol{\beta} \atop \gamma}; -\frac{\boldsymbol{z}}{z_0}\right) \zeta(s+k, a, \lambda) z_0^k,$$

*and* $R^{+}_{a,\lambda,K}$ *is the remainder term expressed as*

$$(3.3) \qquad R^{+}_{a,\lambda,K}\left({s, \boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right) = \frac{1}{2\pi\sqrt{-1}} \int_{(u_K^{+})} \Gamma\left({s+w, -w \atop s}\right) F^{(n)}_D\left({-w, \boldsymbol{\beta} \atop \gamma}; -\frac{\boldsymbol{z}}{z_0}\right) \times \zeta(s+w, a, \lambda) z_0^w dw,$$

*where* $u_K^{+}$ *is a constant satisfying* $\max(1-\sigma, K-1) < u_K^{+} < K$. *Formula* (3.1) *further provides the analytic continuation of* $\mathcal{Z}^{(n)}_{a,\lambda}\left({s, \boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right)$ *over the whole s-plane except at* $s = 1$, *to the poly-sector* $|\arg z_j - \theta_0| < \pi/2$ $(j = 0, 1, \ldots, n)$, *and for all* $(\boldsymbol{\beta}, \gamma) \in \mathbb{C}^n \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}$. *Moreover if* $(z_0, \boldsymbol{z})$ *is in* $|\arg z_j - \theta_0| \leq \pi/2 - \delta$ *with any small* $\delta > 0$ $(j = 0, 1, \ldots, n)$, *and satisfies*

$$|z_j| \leq c|z_0| \qquad (j = 1, \ldots, n)$$

*for some constant* $c > 0$, *then the estimates*

$$F^{(n)}_D\left({-k, \boldsymbol{\beta} \atop \gamma}; -\frac{\boldsymbol{z}}{z_0}\right) = O(1) \qquad \text{and} \qquad R^{+}_{a,\lambda,K}\left({s, \boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right) = O(|z_0|^K)$$

*follow for all* $K > k \geq 0$ *as* $z_0 \to 0$ *through* $|\arg z_0 - \theta_0| \leq \pi/2 - \delta$, *in the same region of* $(s, \boldsymbol{\beta}, \gamma)$ *above, where the constants implied in the O-symbols may depend on* $a$, $K$, $c$, $s$, $\boldsymbol{\beta}$, $\gamma$ *and* $\delta$; *this shows that* (3.1) *with* (3.2) *and* (3.3) *gives a complete asymptotic expansion in the ascending order of* $z_0$ *as* $z_0 \to 0$ *through the sector* $|\arg z_0 - \theta_0| < \pi/2$.

It can be seen that $\lim_{K\to+\infty} R^{+}_{a,\lambda,K}\left({s, \boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right) = 0$ for $|z_j| < a$ $(j = 0, 1, \ldots, n)$; this yields the following corollary.

**Corollary 1.** *Let* $(s, \boldsymbol{\beta}, \gamma)$ *be as in Theorem 1. Then the infinite series*

$$\mathcal{Z}^{(n)}_{a,\lambda}\left({s, \boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k (s)_k}{k!} F^{(n)}_D\left({-k, \boldsymbol{\beta} \atop \gamma}; -\frac{\boldsymbol{z}}{z_0}\right) \zeta(s+k, a, \lambda) z_0^k$$

*holds for all* $(z_0, \boldsymbol{z})$ *in the poly-disk* $|z_j| < a$ $(j = 0, 1, \ldots, n)$.

**Corollary 2.** *Function* $\mathcal{Z}^{(n)}_{a,\lambda}\left({s, \boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right)$ *is continued to a one-valued meromorphic function of s over the whole s-plane, to the whole poly-sector* $|\arg z_j| < \pi$ $(j = 0, 1, \ldots, n)$, *and for all* $(\boldsymbol{\beta}, \gamma) \in \mathbb{C}^n \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}$; *its only singularity, as a function of s, is a (possible) simple ple at* $s = 1$ *with the residue* $B_0(a, e(\lambda))$.

We next state the asymptotic expansion of $\mathcal{Z}^{(n)}_{a,\lambda}\left({s, \boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right)$ when $(z_0, \boldsymbol{z})$ becomes large.

**Theorem 2.** *Let* $\theta_0$ *be any angle fixed with* $\theta_0 \in [-\pi/2, \pi/2]$. *Then for any integer* $K \geq 0$, *in the region* $\sigma > -K$ *except the point at* $s = 1$ *the formula*

$$(3.4) \qquad \mathcal{Z}^{(n)}_{a,\lambda}\left({s, \boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right) = S^{-}_{a,\lambda,K}\left({s, \boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right) + R^{-}_{a,\lambda,K}\left({s, \boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right)$$

*holds for all* $(z_0, \boldsymbol{z})$ *in the poly-sector* $|\arg z_j - \theta_0| < \pi/2$ $(j = 0, 1, \ldots, n)$ *and for all* $(\boldsymbol{\beta}, \gamma) \in \mathbb{C}^n \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}$. *Here*

$$(3.5)\qquad S^-_{a,\lambda,K}\left({s, \boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right) = \sum_{k=-1}^{K-1} \frac{(-1)^{k+1}(s)_k}{(k+1)!} F_D^{(n)}\left({s+k, \boldsymbol{\beta} \atop \gamma}; -\frac{\boldsymbol{z}}{z_0}\right) B_{k+1}(a, e(\lambda)) z_0^{-s-k},$$

*and* $R^-_{a,\lambda,K}$ *is the remainder term expressed as*

$$(3.6)\qquad R^-_{a,\lambda,K}\left({s, \boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right) = \frac{1}{2\pi\sqrt{-1}} \int_{(u_K^-)} \Gamma\left({s+w, -w \atop s}\right) F_D^{(n)}\left({-w, \boldsymbol{\beta} \atop \gamma}; -\frac{\boldsymbol{z}}{z_0}\right) \times \zeta(s+w, a, \lambda) z_0^w dw,$$

*where* $u_K^-$ *is a constant satisfying* $-\sigma - K < u_K^- < \min(-\sigma - K + 1, 0)$. *Formula* (3.4) *further provides the analytic continuation of* $\mathcal{Z}_{a,\lambda}^{(n)}\left({s, \boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right)$ *over the whole s-plane except at* $s = 1$, *to the poly-sector* $|\arg z_j - \theta_0| < \pi/2$ $(j = 0, 1, \ldots, n)$, *and for all* $(\boldsymbol{\beta}, \gamma) \in \mathbb{C}^n \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}$. *Moreover if* $(z_0, \boldsymbol{z})$ *is in* $|\arg z_j - \theta_0| \le \pi/2 - \delta$ *with any small* $\delta > 0$ $(j = 0, 1, \ldots, n)$, *and satisfies*

$$|z_j| \le c|z_0| \qquad (j = 1, \ldots, n)$$

*for some constant* $c > 0$, *then the estimates*

$$F_D^{(n)}\left({s+k, \boldsymbol{\beta} \atop \gamma}; -\frac{\boldsymbol{z}}{z_0}\right) = O(1) \qquad \text{and} \qquad R^-_{a,\lambda,K}\left({s, \boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right) = O(|z_0|^{-\sigma-K})$$

*follow for all* $K > k \ge 0$ *as* $z_0 \to \infty$ *through* $|\arg z_0 - \theta_0| \le \pi/2 - \delta$, *in the same region of* $(s, \boldsymbol{\beta}, \gamma)$ *as above, where the constants implied in the O-symbols may depend on a, K, c, s,* $\boldsymbol{\beta}$, $\gamma$ *and* $\delta$; *this shows that* (3.4) *with* (3.5) *and* (3.6) *gives a complete asymptotic expansion in the descending order of* $z_0$ *as* $z_0 \to \infty$ *through the sector* $|\arg z_0 - \theta_0| < \pi/2$.

The cases $s = -l$ ($l \in \mathbb{N}_0$) of Theorem 2 reduce to the evaluations in finite closed form of $\mathcal{Z}_{a,\lambda}^{(n)}\left({s, \boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right)$.

**Corollary 3.** *Let* $(\boldsymbol{\beta}, \gamma)$ *be as in Theorem 2, and* $(z_0, \boldsymbol{z})$ *in the poly-sector* $|\arg z_j - \theta_0| < \pi$ $(j = 0, 1, \ldots, n)$ *with any angle fixed with* $\theta_0 \in [-\pi/2, \pi/2]$. *Then for any* $l \in \mathbb{N}_0$ *we have*

$$\mathcal{Z}_{a,\lambda}^{(n)}\left({-l, \boldsymbol{\beta} \atop \gamma}; z_0, \boldsymbol{z}\right) = -\frac{1}{l+1} \sum_{k=-1}^{l} \binom{l+1}{k+1} F_D^{(n)}\left({k-l, \boldsymbol{\beta} \atop \gamma}; -\frac{\boldsymbol{z}}{z_0}\right) B_{k+1}(a, e(\lambda)) z_0^{l-k}.$$

The asymptotic expansions of $\widehat{\mathcal{Z}}_{a,\lambda}^{(n)}\left({s, \boldsymbol{\beta}_{n-1} \atop \gamma}; z_0, \boldsymbol{z}\right)$ can be derived from our main formulae (3.1) and (3.4) through the confluence operation in (2.3); this asserts the following Theorems 3 and 4.

**Theorem 3.** *Let* $\theta_0$ *be any angle fixed with* $\theta_0 \in [-\pi/2, \pi/2]$. *Then for any integer* $K \ge 0$, *in the region* $\sigma > 1 - K$ *except at* $s = 1$ *the formula*

$$(3.7)\qquad \widehat{\mathcal{Z}}_{a,\lambda}^{(n)}\left({s, \boldsymbol{\beta}_{n-1} \atop \gamma}; z_0, \boldsymbol{z}\right) = \widehat{S}^+_{a,\lambda,K}\left({s, \boldsymbol{\beta}_{n-1} \atop \gamma}; z_0, \boldsymbol{z}\right) + \widehat{R}^+_{a,\lambda,K}\left({s, \boldsymbol{\beta}_{n-1} \atop \gamma}; z_0, \boldsymbol{z}\right)$$

*holds for all* $(z_0, \boldsymbol{z})$ *in the poly-sector* $|\arg z_j - \theta_0| < \pi/2$ $(j = 0, 1, \ldots, n)$ *and for all* $(\boldsymbol{\beta}_{n-1}, \gamma) \in \mathbb{C}^{n-1} \times \{\mathbb{C} \setminus (-\mathbb{N}_0)\}$. *Here*

$$(3.8)\qquad \widehat{S}^+_{a,\lambda,K}\left({s, \boldsymbol{\beta}_{n-1} \atop \gamma}; z_0, \boldsymbol{z}\right) = \sum_{k=0}^{K-1} \frac{(-1)^k (s)_k}{k!} \widehat{F}_D^{(n)}\left({-k, \boldsymbol{\beta}_{n-1} \atop \gamma}; -\frac{\boldsymbol{z}}{z_0}\right) \zeta(s+k, a, \lambda) z_0^k,$$

*and* $\widehat{R}^{+}_{a,\lambda,K}$ *is the remainder term expressed as*

$$\widehat{R}^{+}_{a,\lambda,K}\left({s,\boldsymbol{\beta}_{n-1}\atop\gamma};z_0,\boldsymbol{z}\right)=\frac{1}{2\pi\sqrt{-1}}\int_{(u_K^{+})}\Gamma\left({s+w,-w\atop s}\right)F_D^{(n)}\left({-w,\boldsymbol{\beta}_{n-1}\atop\gamma};-\frac{\boldsymbol{z}}{z_0}\right)\times\zeta(s+w,a,\lambda)z_0^{w}dw, \tag{3.9}$$

*where* $u_K^{+}$ *is a constant satisfying* $\max(1-\sigma,K-1)<u_K^{+}<K$. *Formula* (3.7) *further provides the analytic continuation of* $\widehat{\mathcal{Z}}^{(n)}_{a,\lambda}\left({s,\boldsymbol{\beta}_{n-1}\atop\gamma};z_0,\boldsymbol{z}\right)$ *over the whole s-plane except at* $s=1$, *to the poly-sector* $|\arg z_j-\theta_0|<\pi/2$ $(j=0,1,\ldots,n)$, *and for all* $(\boldsymbol{\beta}_{n-1},\gamma)\in\mathbb{C}^{n-1}\times\{\mathbb{C}\setminus(-\mathbb{N}_0)\}$. *Moreover if* $(z_0,\boldsymbol{z})$ *is in* $|\arg z_j-\theta_0|\le\pi/2-\delta$ *with any small* $\delta>0$ $(j=0,1,\ldots,n)$, *and satisfies*

$$|z_j|\le c|z_0| \qquad (j=1,\ldots,n)$$

*for some constant* $c>0$, *then the estimates*

$$\widehat{F}_D^{(n)}\left({-k,\boldsymbol{\beta}_{n-1}\atop\gamma};-\frac{\boldsymbol{z}}{z_0}\right)=O(1) \qquad \textit{and} \qquad \widehat{R}^{+}_{a,\lambda,K}\left({s,\boldsymbol{\beta}_{n-1}\atop\gamma};z_0,\boldsymbol{z}\right)=O(|z_0|^{K})$$

*follow for all* $K>k\ge0$ *as* $z_0\to0$ *through* $|\arg z_0-\theta_0|\le\pi/2-\delta$, *in the same region of* $(s,\boldsymbol{\beta}_{n-1},\gamma)$ *above, where the constants implied in the O-symbols may depend on* $a$, $K$, $c$, $s$, $\boldsymbol{\beta}_{n-1}$, $\gamma$ *and* $\delta$; *this shows that* (3.7) *with* (3.8) *and* (3.9) *gives a complete asymptotic expansion in the ascending order of* $z_0$ *as* $z_0\to0$ *through the sector* $|\arg z_0-\theta_0|<\pi/2$.

**Corollary 4.** *Let* $(s,\boldsymbol{\beta}_{n-1},\gamma)$ *be as in Theorem 3. Then the infinite series*

$$\widehat{\mathcal{Z}}^{(n)}_{a,\lambda}\left({s,\boldsymbol{\beta}_{n-1}\atop\gamma};z_0,\boldsymbol{z}\right)=\sum_{k=0}^{\infty}\frac{(-1)^k(s)_k}{k!}\widehat{F}_D^{(n)}\left({-k,\boldsymbol{\beta}_{n-1}\atop\gamma};-\frac{\boldsymbol{z}}{z_0}\right)\zeta(s+k,a,\lambda)z_0^{k}$$

*holds for all* $(z_0,\boldsymbol{z})$ *in the poly-disk* $|z_j|<a$ $(j=0,1,\ldots,n)$.

**Theorem 4.** *Let* $\theta_0$ *be any angle fixed with* $\theta_0\in[-\pi/2,\pi/2]$. *Then for any integer* $K\ge0$, *in the region* $\sigma>-K$ *except at* $s=1$ *the formula*

$$\widehat{\mathcal{Z}}^{(n)}_{a,\lambda}\left({s,\boldsymbol{\beta}_{n-1}\atop\gamma};z_0,\boldsymbol{z}\right)=\widehat{S}^{-}_{a,\lambda,K}\left({s,\boldsymbol{\beta}_{n-1}\atop\gamma};z_0,\boldsymbol{z}\right)+\widehat{R}^{-}_{a,\lambda,K}\left({s,\boldsymbol{\beta}_{n-1}\atop\gamma};z_0,\boldsymbol{z}\right) \tag{3.10}$$

*holds for all* $(z_0,\boldsymbol{z})$ *in the poly-sector* $|\arg z_j-\theta_0|<\pi/2$ $(j=0,1,\ldots,n)$ *and for all* $(\boldsymbol{\beta}_{n-1},\gamma)\in\mathbb{C}^{n-1}\times\{\mathbb{C}\setminus(-\mathbb{N}_0)\}$. *Here*

$$\widehat{S}^{-}_{a,\lambda,K}\left({s,\boldsymbol{\beta}_{n-1}\atop\gamma};z_0,\boldsymbol{z}\right)=\sum_{k=-1}^{K-1}\frac{(-1)^k(s)_k}{k!}\widehat{F}_D^{(n)}\left({s+k,\boldsymbol{\beta}_{n-1}\atop\gamma};-\frac{\boldsymbol{z}}{z_0}\right)\times B_{k+1}(a,e(\lambda))z_0^{-s-k}, \tag{3.11}$$

*and* $\widehat{R}^{-}_{a,\lambda,K}$ *is the remainder term expressed as*

$$\widehat{R}^{-}_{a,\lambda,K}\left({s,\boldsymbol{\beta}_{n-1}\atop\gamma};z_0,\boldsymbol{z}\right)=\frac{1}{2\pi\sqrt{-1}}\int_{(u_K^{-})}\Gamma\left({s+w,-w\atop s}\right)\widehat{F}_D^{(n)}\left({-w,\boldsymbol{\beta}_{n-1}\atop\gamma};-\frac{\boldsymbol{z}}{z_0}\right)\times\zeta(s+w,a,\lambda)z_0^{w}dw, \tag{3.12}$$

*where* $u_K^{-}$ *is a constant satisfying* $-\sigma-K<u_K^{-}<\min(-\sigma-K+1,0)$. *Formula* (3.10) *further provides the analytic continuation of* $\widehat{\mathcal{Z}}^{(n)}_{a,\lambda}\left({s,\boldsymbol{\beta}_{n-1}\atop\gamma};z_0,\boldsymbol{z}\right)$ *over the whole s-plane* *except at* $s=1$, *to the poly-sector* $|\arg z_j-\theta_0|<\pi/2$ $(j=0,1,\ldots,n)$, *and for all* $(\boldsymbol{\beta}_{n-1},\gamma)\in\mathbb{C}^{n-1}\times\{\mathbb{C}\setminus(-\mathbb{N}_0)\}$. *Moreover if* $(z_0,\boldsymbol{z})$ *is in* $|\arg z_j-\theta_0|\le\pi/2-\delta$ *with any small* $\delta>0$ $(j=0,1,\ldots,n)$, *and satisfies*

$$|z_j|\le c|z_0| \qquad (j=1,\ldots,n)$$

*for some constant* $c>0$, *then the estimates*

$$\widehat{F}_D^{(n)}\left({s+k,\boldsymbol{\beta}_{n-1}\atop\gamma};-\frac{\boldsymbol{z}}{z_0}\right)=O(1) \qquad \text{and} \qquad \widehat{R}^{-}_{a,\lambda,K}\left({s,\boldsymbol{\beta}_{n-1}\atop\gamma};z_0,\boldsymbol{z}\right)=O(|z_0|^{-\sigma-K})$$

*follow for all* $K>k\ge0$ *as* $z_0\to\infty$ *through* $|\arg z_0-\theta_0|\le\pi/2-\delta$ *in the same region of* $(s,\boldsymbol{\beta}_{n-1},\gamma)$ *above, where the constants implied in the* $O$*-symbols may depend on* $a$, $K$, $c$, $s$, $\boldsymbol{\beta}_{n-1}$, $\gamma$ *and* $\delta$; *this shows that* (3.10) *with* (3.11) *and* (3.12) *gives a complete asymptotic expansion in the descending order of* $z_0$ *as* $z_0\to\infty$ *through the sector* $|\arg z_0-\theta_0|<\pi/2$.

**Corollary 5.** *Let* $(\boldsymbol{\beta},\gamma)$ *be as in Theorem 4, and* $(z_0,\boldsymbol{z})$ *in the poly-sector* $|\arg z_j-\theta_0|<\pi$ $(j=0,1,\ldots,n)$ *with any angle fixed with* $\theta_0\in[-\pi/2,\pi/2]$. *Then for any* $l\in\mathbb{N}_0$ *we have*

$$\widehat{\mathcal{Z}}_{a,\lambda}^{(n)}\left({-l,\boldsymbol{\beta}_{n-1}\atop\gamma};z_0,\boldsymbol{z}\right)=-\frac{1}{l+1}\sum_{k=-1}^{l}\binom{l+1}{k+1}\widehat{F}_D^{(n)}\left({k-l,\boldsymbol{\beta}_{n-1}\atop\gamma};-\frac{\boldsymbol{z}}{z_0}\right)B_{k+1}(a,e(\lambda))z_0^{l-k}.$$

### Asymptotics for derivatives

We define the generalized Euler-Stieltjes constants $\gamma_m(a,e(\lambda))$ $(m\in\mathbb{N}_0)$ and the modified Stirling polynomials $\sigma_{m,n}(x)$ $(m,n\in\mathbb{N}_0)$ respectively by the power series

$$\zeta(s,a,\lambda)=\frac{B_0(a,e(\lambda))}{s-1}+\sum_{m=0}^{\infty}\gamma_m(a,e(\lambda))(s-1)^m$$

centered at $s=1$, and

$$\frac{1}{m!}(1-z)^{-x}\{-\log(1-z)\}^m=\sum_{n=0}^{\infty}\frac{\sigma_{m,n}(x)}{n!}z^n$$

centered at $z=0$. Note that $\sigma_{m,n}(x)=0$ for $0\le n<m$. We further set

$$C_{k,l,m}(a,e(\lambda))=\sum_{j=0}^{m}\frac{m!}{(m-j)!}\sigma_{j,k}(l)\left(\frac{\partial}{\partial s}\right)^{m-j}\zeta(s,a,\lambda)\bigg|_{s=l+k}$$

for any $k,l,m\in\mathbb{N}_0$. Then Theorem 1 yields:

**Corollary 6.** *Let* $(\boldsymbol{\beta},\gamma,\boldsymbol{z})$ *be as in Theorem 1. For any integer* $K\ge1$ *the following asymptotic expansions hold as* $z_0\to0$ *through* $|\arg z_0-\theta_0|\le\pi/2-\delta$ *with any* $\delta>0$, *while other* $z_j$*'s move through the same sector satisfying the conditions* $|z_j|\le c|z_0|$ $(j=1,\ldots,n)$ *with some constant* $c>0$:

*i) when* $s\to1$,

$$\lim_{s\to1}\left(\frac{\partial}{\partial s}\right)^m\left\{\mathcal{Z}_{a,\lambda}^{(n)}\left({s,\boldsymbol{\beta}\atop\gamma};z_0,\boldsymbol{z}\right)-\frac{B_0(a,e(\lambda))}{s-1}\right\}$$
$$=m!\gamma_m(a,e(\lambda))+\sum_{k=1}^{K-1}\frac{(-1)^k}{k!}C_{k,1,m}(a,e(\lambda))F_D^{(n)}\left({-k,\boldsymbol{\beta}\atop\gamma};-\frac{\boldsymbol{z}}{z_0}\right)z_0^k+O(|z_0|^K);$$

*ii) when* $s = l$ $(l = 2, 3, \dots)$,

$$\left(\frac{\partial}{\partial s}\right)^m \mathcal{Z}_{a,\lambda}^{(n)}\left(\begin{matrix} s, \boldsymbol{\beta} \\ \gamma \end{matrix}; z_0, \boldsymbol{z}\right)\bigg|_{s=l} = \sum_{k=0}^{K-1} \frac{(-1)^k}{k!} C_{k,l,m}(a, e(\lambda)) F_D^{(n)}\left(\begin{matrix} -k, \boldsymbol{\beta} \\ \gamma \end{matrix}; -\frac{\boldsymbol{z}}{z_0}\right) z_0^k + O(|z_0|^K).$$

It is known that $\lim_{s\to 1}\{\zeta(s,z) - 1/(s-1)\} = \gamma_0(z) = -\psi(z) = -(\Gamma'/\Gamma)(z)$. The case $(\lambda, \boldsymbol{\beta}) = (0, \mathbf{0})$ above reduces to the classical Taylor series expansion of $\psi(a+z)$ (cf. [4]).

**Corollary 7.** *For* $|z| < a$ *we have*

$$\psi(a+z) = \psi(a) - \sum_{k=1}^{\infty} \left\{ \left( \sum_{h=1}^{k} \frac{1}{h} \right) \zeta(1+k, a) + \zeta'(1+k, a) \right\} z^k.$$

We next define the polynomials $\mathcal{P}_{l,m}, \mathcal{Q}_{k,l,m} \in \mathbb{C}[[\boldsymbol{x}]][y]$ $(k, l, m \in \mathbb{N}_0)$ by

$$\mathcal{P}_{l,m}\left(\begin{matrix} \boldsymbol{\beta} \\ \gamma \end{matrix}; \boldsymbol{x}, y\right) = \sum_{j=0}^{m} \frac{m!}{(m-j)!} \left\{ \sum_{i=0}^{j} \frac{(l+1)^{i-j-1}}{(j-i)!} \times \left(\frac{\partial}{\partial \alpha}\right)^{j-i} F_D^{(n)}\left(\begin{matrix} \alpha, \boldsymbol{\beta} \\ \gamma \end{matrix}; -\boldsymbol{x}\right)\bigg|_{\alpha=-l-1} \right\} (-y)^{m-j},$$

$$\mathcal{Q}_{k,l,m}\left(\begin{matrix} \boldsymbol{\beta} \\ \gamma \end{matrix}; \boldsymbol{x}, y\right) = \sum_{j=0}^{m} \frac{m!}{(m-j)!} \left\{ \sum_{i=0}^{j} \frac{\sigma_{i,j}(-l)}{(j-i)!} \times \left(\frac{\partial}{\partial \alpha}\right)^{j-i} F_D^{(n)}\left(\begin{matrix} \alpha, \boldsymbol{\beta} \\ \gamma \end{matrix}; -\boldsymbol{x}\right)\bigg|_{\alpha=k-l} \right\} (-y)^{m-j}.$$

**Corollary 8.** *Let* $(\boldsymbol{\beta}, \gamma, \boldsymbol{z})$ *be as in Theorem 2, and* $l, m \in \mathbb{N}_0$ *arbitrary. Then for any integer* $K \geq l+1$ *the asymptotic expansion*

$$\left(\frac{\partial}{\partial s}\right)^m \mathcal{Z}_{a,\lambda}^{(n)}\left(\begin{matrix} s, \boldsymbol{\beta} \\ \gamma \end{matrix}; z_0, \boldsymbol{z}\right)\bigg|_{s=-l} = -B_0(a, e(\lambda)) z_0^{l+1} \mathcal{P}_{l,m}\left(\begin{matrix} \boldsymbol{\beta} \\ \gamma \end{matrix}; \frac{\boldsymbol{z}}{z_0}, \log z_0\right) + \sum_{k=0}^{K-1} \frac{(-1)^{k+1}}{(k+1)!} B_{k+1}(a, e(\lambda)) z_0^{l-k} \mathcal{Q}_{k,l,m}\left(\begin{matrix} \boldsymbol{\beta} \\ \gamma \end{matrix}; \frac{\boldsymbol{z}}{z_0}, \log z_0\right) + O(|z_0|^{l-K} \log^m |z_0|)$$

*holds as* $z_0 \to \infty$ *through* $|\arg z_0 - \theta_0| \leq \pi/2 - \delta$ *with any* $\delta > 0$*, while other* $z_j$*'s move through the same sector satisfying the conditons* $|z_j| \leq c|z_0|$ $(j = 1, \dots, n)$ *with some constant* $c > 0$.

It is known that $(\partial/\partial s)\zeta(s,z)|_{s=0} = \log\{\Gamma(z)/\sqrt{2\pi}\}$ (cf. [4]). The case $(n, \boldsymbol{\beta}) = (2, \mathbf{0})$ and $\lambda \in \mathbb{Z}$ above reduces to the following variant of Stirling's formula (cf. [4]).

**Corollary 9.** *For any integer* $K \geq 0$ *the asymptotic expansion*

$$\log \Gamma(a+z) = \left(z + a - \frac{1}{2}\right) \log z + \frac{1}{2} \log(2\pi) + \sum_{k=1}^{K-1} \frac{(-1)^{k+1} B_{k+1}(a)}{k(k+1)} z^{-k} + O(|z|^{-K} \log |z|)$$

*holds as* $z \to \infty$ *through* $|\arg z| \leq \pi - \delta$ *with any small* $\delta > 0$.

### Applications of our main formulae with $n = 2$

One can observe that the case $(n, \gamma) = (2, s)$ of (2.2) and (2.4) reduce respectively to the expressions

$$(5.1)\qquad \mathcal{Z}^{(2)}_{a,\lambda}\Big(\begin{smallmatrix} s,\beta_1,\beta_2 \\ s \end{smallmatrix}; z_0, z_1, z_2\Big) = \sum_{l=0}^{\infty} e(\lambda l)(a+l+z_0)^{-s}\Big(1+\frac{z_1}{a+l+z_0}\Big)^{-\beta_1} \times \Big(1+\frac{z_2}{a+l+z_0}\Big)^{-\beta_2},$$

and

$$(5.2)\qquad \widehat{\mathcal{Z}}^{(2)}_{a,\lambda}\Big(\begin{smallmatrix} s,\beta_1 \\ s \end{smallmatrix}; z_0, z_1, z_2\Big) = \sum_{l=0}^{\infty} e(\lambda l)(a+l+z_0)^{-s}\Big(1+\frac{z_1}{a+l+z_0}\Big)^{-\beta_1} \times \exp\Big(-\frac{z_2}{a+l+z_0}\Big).$$

Theorems 1 and 2 in particular assert on (5.1) and (5.2) the following corollaries.

**Corollary 10.** *Let $\theta_0$ be as in Theorem 1. The for any integer $K \geq 0$, in the region $\sigma > 1 - K$ except at $s = 1$ Function $\mathcal{Z}^{(2)}_{a,\lambda}(\begin{smallmatrix} s,\beta_1,\beta_2 \\ s \end{smallmatrix}; z_0, z_1, z_2)$ is represented as (3.1) in the poly-sector $|\arg z_j - \theta_0| < \pi/2$ $(j = 0, 1, \dots, n)$ and for all $(\beta_1, \beta_2) \in \mathbb{C}^2$, where*

$$S^{+}_{a,\lambda,K}\Big(\begin{smallmatrix} s,\beta_1,\beta_2 \\ s \end{smallmatrix}; z_0, z_1, z_2\Big) = \sum_{k=0}^{K-1} \frac{(-1)^k (s)_k}{k!} F_1\Big(\begin{smallmatrix} -k,\beta_1,\beta_2 \\ s \end{smallmatrix}; -\frac{z_1}{z_0}, -\frac{z_2}{z_0}\Big)\zeta(s+k, a, \lambda) z_0^k,$$

*and*

$$R^{+}_{a,\lambda,K}\Big(\begin{smallmatrix} s,\beta_1,\beta_2 \\ s \end{smallmatrix}; z_0, z_1, z_2\Big) = \frac{1}{2\pi\sqrt{-1}} \int_{(u_K^+)} \Gamma\Big(\begin{smallmatrix} s+w, -w \\ s \end{smallmatrix}\Big) F_1\Big(\begin{smallmatrix} -w,\beta_1,\beta_2 \\ s \end{smallmatrix}; -\frac{z_1}{z_0}, -\frac{z_2}{z_0}\Big) \times \zeta(s+w, a, \lambda) z_0^w \, dw.$$

*These formulae give a complete asymptotic expansion of $\mathcal{Z}^{(n)}_{a,\lambda}(\begin{smallmatrix} s,\beta_1,\beta_2 \\ s \end{smallmatrix}; z_0, z_1, z_2)$ as $z_0 \to 0$ through $|\arg z_0 - \theta_0| < \pi/2$ in the sense of Theorem 1.*

**Corollary 11.** *Let $\theta_0$ be as in Theorem 1. Then for any integer $K \geq 0$, in the region $\sigma > -K$ except at $s = 1$ Function $\mathcal{Z}^{(2)}_{a,\lambda}(\begin{smallmatrix} s,\beta_1,\beta_2 \\ s \end{smallmatrix}; z_0, z_1, z_2)$ is represented as (3.4) in the poly-sector $|\arg z_j - \theta_0| < \pi/2$ $(j = 0, 1, \dots, n)$ and for all $(\beta_1, \beta_2) \in \mathbb{C}^2$, where*

$$S^{-}_{a,\lambda.K}\Big(\begin{smallmatrix} s,\beta_1,\beta_2 \\ s \end{smallmatrix}; z_0, z_1, z_2\Big) = \sum_{k=-1}^{K-1} \frac{(-1)^{k+1} (s)_k}{(k+1)!} F_1\Big(\begin{smallmatrix} s+k,\beta_1,\beta_2 \\ s \end{smallmatrix}; -\frac{z_1}{z_0}, -\frac{z_2}{z_0}\Big) \times B_{k+1}(a, e(\lambda)) z_0^{-s-k},$$

*and*

$$R^{-}_{a,\lambda,K}\Big(\begin{smallmatrix} s,\beta_1,\beta_2 \\ s \end{smallmatrix}; z_0, z_1, z_2\Big) = \frac{1}{2\pi\sqrt{-1}} \int_{(u_K^-)} \Gamma\Big(\begin{smallmatrix} s+w, -w \\ s \end{smallmatrix}\Big) F_1\Big(\begin{smallmatrix} -w,\beta_1,\beta_2 \\ s \end{smallmatrix}; -\frac{z_1}{z_0}, -\frac{z_2}{z_0}\Big) \times \zeta(s+w, a, \lambda) z_0^w \, dw.$$

*These formulae give a complete asymptotic expansion of $\mathcal{Z}^{(n)}_{a,\lambda}(\begin{smallmatrix} s,\beta_1,\beta_2 \\ s \end{smallmatrix}; z_0, z_1, z_2)$ as $z_0 \to \infty$ through $|\arg z_0 - \theta_0| < \pi/2$ in the sense of Theorem 2.*

**Corollary 12.** *Let $\theta_0$ be as in Theorem 1. Then for any integer $K \geq 0$, in the region $\sigma > 1 - K$ except at $s = 1$ Function $\widehat{\mathcal{Z}}^{(2)}_{a,\lambda}(\begin{smallmatrix} s,\beta_1 \\ s \end{smallmatrix}; z_0, z_1, z_2)$ is represented as (3.7) in the poly-sector $|\arg z_j - \theta_0| < \pi/2$ $(j = 0, 1, \ldots, n)$ and for all $\beta_1 \in \mathbb{C}$. Here*

$$\widehat{S}^{+}_{a,\lambda,K}\left(\begin{matrix} s,\beta_1 \\ s \end{matrix}; z_0, z_1, z_2\right) = \sum_{k=0}^{K-1} \frac{(-1)^k (s)_k}{k!} \Phi_1\left(\begin{matrix} -k,\beta_1 \\ s \end{matrix}; -\frac{z_1}{z_0}, -\frac{z_2}{z_0}\right) \zeta(s+k, a, \lambda) z_0^k,$$

*and*

$$\widehat{R}^{+}_{a,\lambda,K}\left(\begin{matrix} s,\beta_1 \\ s \end{matrix}; z_0, z_1, z_2\right) = \frac{1}{2\pi\sqrt{-1}} \int_{(u_K^+)} \Gamma\left(\begin{matrix} s+w, -w \\ s \end{matrix}\right) \Phi_1\left(\begin{matrix} -w,\beta_1 \\ s \end{matrix}; -\frac{z_1}{z_0}, -\frac{z_2}{z_0}\right)$$
$$\times \zeta(s+w, a, \lambda) z_0^w dw.$$

*These formulae give a complete asymptotic expansion of $\widehat{\mathcal{Z}}^{(n)}_{a,\lambda}(\begin{smallmatrix} s,\beta_1 \\ s \end{smallmatrix}; z_0, z_1, z_2)$ as $z_0 \to 0$ through $|\arg z_0 - \theta_0| < \pi/2$ in the sense of Theorem 3.*

**Corollary 13.** *Let $\theta_0$ be as in Theorem 1. Then for any integer $K \geq 0$, in the region $\sigma > -K$ except at $s = 1$ Function $\widehat{\mathcal{Z}}^{(2)}_{a,\lambda}(\begin{smallmatrix} s,\beta_1 \\ s \end{smallmatrix}; z_0, z_1, z_2)$ is represented as (3.10) in the poly-sector $|\arg z_j - \theta_0| < \pi/2$ $(j = 0, 1, \ldots, n)$ and for all $\beta_1 \in \mathbb{C}$. Here*

$$\widehat{S}^{-}_{a,\lambda,K}\left(\begin{matrix} s,\beta_1 \\ s \end{matrix}; z_0, z_1, z_2\right) = \sum_{k=-1}^{K-1} \frac{(-1)^{k+1} (s)_k}{(k+1)!} \Phi_1\left(\begin{matrix} s+k,\beta_1 \\ s \end{matrix}; -\frac{z_1}{z_0}, -\frac{z_2}{z_0}\right) B_{k+1}(a, e(\lambda)) z_0^{-s-k},$$

*and*

$$\widehat{R}^{-}_{a,\lambda,K}\left(\begin{matrix} s,\beta_1 \\ s \end{matrix}; z_0, z_1, z_2\right) = \frac{1}{2\pi\sqrt{-1}} \int_{(u_K^-)} \Gamma\left(\begin{matrix} s+w, -w \\ s \end{matrix}\right) \Phi_1\left(\begin{matrix} -w,\beta_1 \\ s \end{matrix}; -\frac{z_1}{z_0}, -\frac{z_2}{z_0}\right)$$
$$\times \zeta(s+w, a, \lambda) z_0^w dw.$$

*These formulae give a complete asymptotic expansion of $\widehat{\mathcal{Z}}^{(2)}_{a,\lambda}(\begin{smallmatrix} s,\beta_1 \\ s \end{smallmatrix}; z_0, z_1, z_2)$ as $z_0 \to \infty$ through $|\arg z_0 - \theta_0| < \pi/2$ in the sense of Theorem 4.*

## Further applications

We define for $x, y \in \mathbb{R}_+$ and for $\sigma > 1$ the functions

$$\mathcal{C}_{a,\lambda}(s, \beta; x, y) = \sum_{l=0}^{\infty} e(\lambda l)(a+l+x)^{-s} \frac{\cos\left\{\beta \operatorname{Arctan}\left(\dfrac{y}{a+l+x}\right)\right\}}{\left\{1 + \left(\dfrac{y}{a+l+x}\right)^2\right\}^{\beta/2}},$$

$$\mathcal{S}_{a,\lambda}(s, \beta; x, y) = \sum_{l=0}^{\infty} e(\lambda l)(a+l+x)^{-s} \frac{\sin\left\{\beta \operatorname{Arctan}\left(\dfrac{y}{a+l+x}\right)\right\}}{\left\{1 + \left(\dfrac{y}{a+l+x}\right)^2\right\}^{\beta/2}},$$

and their confluent forms

$$\widehat{\mathcal{C}}_{a,\lambda}(s; x, y) = \sum_{l=0}^{\infty} e(\lambda l)(a+l+x)^{-s} \cos\left(\frac{y}{a+l+x}\right),$$

$$\widehat{\mathcal{S}}_{a,\lambda}(s;x,y)=\sum_{l=0}^{\infty}e(\lambda l)(a+l+x)^{-s}\sin\Big(\frac{y}{a+l+x}\Big).$$

It is in fact possible to show that Theorems 1 and 2 are valid when $n=1$ in a wider sector

$$\max(-\pi,\arg z_0-\pi)<\arg z_1<\min(\pi,\arg z_0+\pi),$$

and this allows us to take $z_0=x$ and $z_1=e^{\pm\pi i/2}y$ with $\arg x=0$ and $\arg y=0$; the following Corollaries 14 and 15 are derived.

**Corollary 14.** *Let $(s,\beta)$ be as in Theorem 1. Then for any $s\in\mathbb{C}$ except at $s=1-k$ $(k\in\mathbb{N}_0)$, and any $x,y\in\mathbb{R}$ with $|x|,|y|<a$ the following formulae hold:*

$$\mathcal{C}_{a,\lambda}(s,\beta;x,y)=\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k(s)_k}{k!}\Big\{{}_2F_1\Big({-k,\beta \atop s};\frac{iy}{x}\Big)+{}_2F_1\Big({-k,\beta \atop s};\frac{-iy}{x}\Big)\Big\}\zeta(s+k,a,\lambda)x^k,$$

$$\widehat{\mathcal{C}}_{a,\lambda}(s;x,y)=\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k(s)_k}{k!}\Big\{{}_1F_1\Big({-k \atop s};\frac{iy}{x}\Big)+{}_1F_1\Big({-k \atop s};\frac{-iy}{x}\Big)\Big\}\zeta(s+k,a,\lambda)x^k,$$

*and similarly,*

$$\mathcal{S}_{a,\lambda}(s,\beta;x,y)=\frac{1}{2i}\sum_{k=0}^{\infty}\frac{(-1)^k(s)_k}{k!}\Big\{{}_2F_1\Big({-k,\beta \atop s};\frac{iy}{x}\Big)-{}_2F_1\Big({-k,\beta \atop s};\frac{-iy}{x}\Big)\Big\}\zeta(s+k,a,\lambda)x^k,$$

$$\widehat{\mathcal{S}}_{a,\lambda}(s;x,y)=\frac{1}{2i}\sum_{k=0}^{\infty}\frac{(-1)^k(s)_k}{k!}\Big\{{}_1F_1\Big({-k \atop s};\frac{iy}{x}\Big)-{}_1F_1\Big({-k \atop s};\frac{-iy}{x}\Big)\Big\}\zeta(s+k,a,\lambda)x^k.$$

**Corollary 15.** *Let $(s,\beta)$ be as in Theorem 2. Then for any integer $K\geq 0$ in the region $\sigma>-K$ except at $s=1-k$ $(k\in\mathbb{N}_0)$ the following asymptotic expansions hold as $x\to+\infty$, while $y$ satisfies $y\ll x$:*

$$\mathcal{C}_{a,\lambda}(s,\beta;x,y)=\frac{1}{2}\sum_{k=-1}^{K-1}\frac{(-1)^{k+1}(s)_k}{(k+1)!}\Big\{{}_2F_1\Big({s+k,\beta \atop s};\frac{iy}{x}\Big)+{}_2F_1\Big({s+k,\beta \atop s};\frac{-iy}{x}\Big)\Big\}$$
$$\times B_{k+1}(a,e(\lambda))x^{-s-k}+O(x^{-\sigma-K}),$$

$$\widehat{\mathcal{C}}_{a,\lambda}(s;x,y)=\frac{1}{2}\sum_{k=-1}^{K-1}\frac{(-1)^{k+1}(s)_k}{(k+1)!}\Big\{{}_1F_1\Big({s+k \atop s};\frac{iy}{x}\Big)+{}_1F_1\Big({s+k \atop s};\frac{-iy}{x}\Big)\Big\}$$
$$\times B_{k+1}(a,e(\lambda))x^{-s-k}+O(x^{-\sigma-K}),$$

*and similarly,*

$$\mathcal{S}_{a,\lambda}(s,\beta;x,y)=\frac{1}{2i}\sum_{k=-1}^{K-1}\frac{(-1)^{k+1}(s)_k}{(k+1)!}\Big\{{}_2F_1\Big({s+k,\beta \atop s};\frac{iy}{x}\Big)-{}_2F_1\Big({s+k,\beta \atop s};\frac{-iy}{x}\Big)\Big\}$$
$$\times B_{k+1}(a,e(\lambda))x^{-s-k}+O(x^{-\sigma-K}),$$

$$\widehat{\mathcal{S}}_{a,\lambda}(s;x,y)=\frac{1}{2i}\sum_{k=-1}^{K-1}\frac{(-1)^{k+1}(s)_k}{(k+1)!}\Big\{{}_1F_1\Big({s+k \atop s};\frac{iy}{x}\Big)-{}_1F_1\Big({s+k \atop s};\frac{-iy}{x}\Big)\Big\}$$
$$\times B_{k+1}(a,e(\lambda))x^{-s-k}+O(x^{-\sigma-K}).$$

## References


[1] P. Appell and J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques, Polynomes d'Hermite*, Gauthier-Villars, Paris, 1926.

[2] M. G. Bin Saad, *Hypergeometric series associated with the Hurwitz-Lerch zeta function*, Acta. Math. Univ. Comenianae, **LXXVIII** (2009), 269–286.

[3] M. G. Bin-Saad and A. A. Al-Gonah, *On hypergeometric-type generating relations associated with the generalized zeta functions*, Acta. Math. Univ. Comenianae **LXXV** (2006), 253–266.

[4] A. Erdélyi (ed.), W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York, 1953.

[5] K. Hikami and A. N. Kirillov, *Hypergeometric generating function of L-function, Slater's identities, and quantum invariant*, Algebra i Analiz **17** (2005), 190–208; translation in St. Petersburg Math. J **17** (2006), 143–156.

[6] M. Katsurada, *Power series with the Riemann-zeta function in the coefficients*, Proc. Japan Acad. Ser. A Math. Sci. **72** (1996), 61–63.

[7] ———, *On Mellin-Barnes type of integrals and sums associated with the Riemann zeta-function*, Publ. Inst. Math. (Beograd) (N.S.) **62(76)** (1997), 13–25.

[8] ———, *Power series and asymptotic series associated with the Lerch zeta-function*, Proc. Japan Acad. Ser. A **74** (1998), 167–170.

[9] ———, *Rapidly convergent series representations for $\zeta(2n+1)$ and their $\chi$-analogue*, Acta Arith. **90** (1999), 79–89.

[10] ———, Hypergeometric type generating functions of several variables associated with the Lerch zeta-function, (in preparation).

[11] D. Klusch, *On the Taylor expansion of the Lerch zeta-function*, J. Math. Anal. Appl. **170** (1992), 513–523.

[12] T. Kuzumaki, *Asymptotic expansions for a class of zeta-functions*, Ramanujan. J. **24** (2011), 331–343.

[13] M. Lerch, *Note dur la fonction $K(w,x,s) = sum_{n\geq 0} \exp\{2\pi inx\}(n+w)^{-s}$*, Acta Math. **11** (1887), 19–24.

[14] H.-L. Li, S. Kanemitsu and H. Tsukada, *Modular relation interpretation of the series involving the Riemann zeta values*, Proc. Japan Acad. Ser. A Math. Sci. **84** (2008), 154–158.

[15] K. Matsumoto, *Asymptotic expansions of double zeta-functions of Barnes, of Shintani, and Eisenstein series*, Nagoya Math. J. **172** (2003), 59–102.

[16] K. Onodera, *Asymptotic expansions for period deformatios of multiple Hurwitz zeta functions and their associated functions*, Internat. J. Math. **20**, (2009), 299–307.

[17] R. K. Raina and H. M. Srivastava, *Certain results associated with the generalized Riemann zeta functions*, Rev. Téc. Ing. Univ. Zulia **18** (1995), 301–304.

[18] H. M. Srivastava, *Summation of a class of series involving the Riemann zeta function*, Rev. Téc. Ing. Univ. Zulia (Edición Especial) **9** (1986), 79–82.

[19] ———, *Some infinite series associated with the Riemann zeta-function*, Yokohama Math. J. **35** (1986), 47–53.

[20] ———, *Sums of certain series of the Riemann zeta function*, J. math. Anal. Appl. **134** (1988), 129–140.

[21] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier, London, 2012.



Department of Mathematics, Hiyoshi Campus, Keio University, 4–1–1 Hiyoshi, Kouhoku-ku, Yokohama 223–8521, Japan
*E-mail address*: katsurad@z3.keio.jp