数理解析研究所講究録
第 1874 巻 2014 年 183-189

The asymptotic behavior of multiple zeta functions at non-positive integers

TOMOKAZU ONOZUKA

1 Introduction

The Euler-Zagier multiple zeta function $\zeta_d(s_1, \cdots, s_d)$ is defined by

$$\zeta_d(s_1, \cdots, s_d) := \sum_{m_1=1}^{\infty} \cdots \sum_{m_d=1}^{\infty} \frac{1}{m_1^{s_1}(m_1+m_2)^{s_2}\cdots(m_1+\cdots+m_d)^{s_d}} \tag{1.1}$$

where s_i $(i = 1, \cdots, d)$ are complex variables. Matsumoto [4] proved that the series (1.1) is absolutely convergent in

$$\{(s_1, \cdots, s_d) \in \mathbb{C}^d \mid \Re(s_d(d-k+1)) > k \quad (k = 1, \cdots, d)\}$$

where $s_d(n) = s_n + s_{n+1} + \cdots + s_d$ $(n = 1, \cdots, d)$. Akiyama, Egami and Tanigawa [1] and Zhao [7] proved the meromorphic continuation to the whole space independently.

The function $\zeta_d(s_1, \cdots, s_d)$ has singularities on

$$\begin{cases} s_d = 1, \\ s_{d-1} + s_d = 2, 1, 0, -2, -4, \cdots, \\ s_d(d-j+1) \in \mathbb{Z}_{\leq j} \ (j = 3, 4, \cdots, d), \end{cases} \tag{1.2}$$

where $\mathbb{Z}_{\leq j}$ is the set of integers less than or equal to j; $\mathbb{Z}_{\geq j}$ is defined similarly. Therefore $(-r_1, \cdots . -r_d) \in \mathbb{Z}_{\leq 0}^d$ lies on the set of singularities. Moreover, it is an indeterminacy of $\zeta_d(s_1, \cdots, s_d)$. For example, Sasaki [6] proved that

$$\lim_{s_3\to 0}\lim_{s_2\to 0}\lim_{s_1\to 0}\zeta_3(s_1, s_2, s_3) = -\frac{3}{8}, \tag{1.3}$$

$$\lim_{s_1\to 0}\lim_{s_2\to 0}\lim_{s_3\to 0}\zeta_3(s_1, s_2, s_3) = -\frac{1}{4}. \tag{1.4}$$

Since $(0, 0, 0)$ is an indeterminacy of $\zeta_3(s_1, s_2, s_3)$, (1.3) and (1.4) give different values.

Akiyama, Egami and Tanigawa [1] defined the regular values by

$$\zeta_d(-r_1, \cdots, -r_d) := \lim_{s_1\to -r_1} \cdots \lim_{s_d\to -r_d} \zeta_d(s_1, \cdots, s_d),$$

and Akiyama and Tanigawa [2] considered the reverse and central values given by

$$\zeta_d^R(-r_1,\cdots,-r_d) := \lim_{s_d\to -r_d}\cdots\lim_{s_1\to -r_1}\zeta_d(s_1,\cdots,s_d),$$
$$\zeta_d^C(-r_1,\cdots,-r_d) := \lim_{\varepsilon\to 0}\zeta_d(-r_1+\varepsilon,\cdots,-r_d+\varepsilon),$$

respectively. Further, Sasaki [6] generalized the regular and reverse values. He defined multiple zeta values for coordinatewise limits by

$$\zeta_d(\overset{i_1}{-r_1},\cdots,\overset{i_d}{-r_d}) := \lim_{\substack{s_j\to -r_j\\ i_j=d}}\cdots\lim_{\substack{s_j\to -r_j\\ i_j=1}}\zeta_d(s_1,\cdots,s_d),$$

where $\{i_1,\cdots,i_d\}=\{1,\cdots,d\}$. He obtained all multiple zeta values of depth 3 for coordinatewise limits. On the other hand, Komori [3] considered more general multiple zeta functions, and he obtained multiple zeta values at non-positive integers given by

$$\zeta_d(\overset{w}{-\boldsymbol{r}}) = \lim_{z_{w^{-1}(d)}\to -r_{w^{-1}(d)}}\cdots\lim_{z_{w^{-1}(1)}\to -r_{w^{-1}(1)}}\zeta_d(z_1,\cdots,z_d),$$
$$\zeta_d(\underset{\boldsymbol{\theta}}{-\boldsymbol{r}}) = \zeta_d(\underset{\theta_1}{-r_1},\cdots,\underset{\theta_d}{-r_d}) = \lim_{\delta\to 0}\zeta_d(-r_1+\delta\theta_1,\cdots,-r_d+\delta\theta_d),$$

where $-\boldsymbol{r}=(-r_1,\cdots,-r_d)\in\mathbb{Z}_{\le 0}^d$, $w\in\mathfrak{S}_d$ and $\boldsymbol{\theta}=(\theta_1,\cdots,\theta_d)\in\mathbb{C}^d$. To obtain these values by Komori's method, we need to compute generalized multiple Bernoulli numbers.

In the present paper, we calculate the asymptotic behavior of multiple zeta functions at non-positive integers. By using that result, we can evaluate the limit values of multiple zeta functions at non-positive integers. For example,

$$\lim_{\varepsilon\to 0}\zeta_3(\varepsilon^2,\varepsilon,\varepsilon) = -\frac{1}{3}. \tag{1.5}$$

This limit value is not contained in the above 2 kinds of values, however by the result, we can compute this value.

2 Main Theorem

In this section, we state the main theorem.

Let B_m be the mth Bernoulli number, and $B(x,y)$ be the beta function. For $(m_1,\cdots,m_d)\in\mathbb{Z}_{\ge 0}^d$, $(p_1,\cdots,p_d)\in\mathbb{Z}_{\ge 0}^d$ and $(\varepsilon_1,\cdots,\varepsilon_d)\in\mathbb{C}^d$, let $m_d(n)$, $p_d(n)$ and $\varepsilon_d(n)$ be $m_n+m_{n+1}+\cdots+m_d$, $p_n+p_{n+1}+\cdots+p_d$ and $\varepsilon_n+\varepsilon_{n+1}+\cdots+\varepsilon_d$ respectively. In addition, the Pochhammer symbol $(a)_n$ is defined by $(a)_n := \Gamma(a+n)/\Gamma(a)$.

Theorem 1. *Suppose that* $\varepsilon_j\neq 0$, $\varepsilon_d(j)\neq 0$ $(j=1,\cdots,d)$, $|\varepsilon_1|+\cdots+|\varepsilon_d|\le\frac{1}{2}$ *and* $|\varepsilon_k/\varepsilon_d(j)|\ll 1$ *as* $(\varepsilon_1,\cdots,\varepsilon_d)\to(0,\cdots,0)$ $(j=1,\cdots,d,\ k=j,\cdots,d)$. *Then for*

$m_j \in \mathbb{Z}_{\geq 0}$ $(j = 1, \cdots, d)$, we have

$$\zeta_d(-m_1+\varepsilon_1,\cdots,-m_d+\varepsilon_d) = (-1)^{m_d} m_d! \sum_{\substack{p_1+\cdots+p_d=d+M \\ p_1,\cdots,p_d\geq 0 \\ -m_d(j)-d+j+p_d(j)<2 \text{ or} \\ -m_d(j-1)-d+j+p_d(j)\geq 2\ (2\leq\forall j\leq d)}} \frac{B_{p_1}\cdots B_{p_d}}{p_1!\cdots p_d!} \times$$
$$\times \prod_{j=2}^{d} \frac{[\varepsilon_d(j)]_{-m_d(j)-d+j+p_d(j)-1}}{[\varepsilon_d(j-1)]_{-m_d(j-1)-d+j+p_d(j)-1}} + \sum_{j=1}^{d} O(\varepsilon_j)$$

as $(\varepsilon_1, \cdots, \varepsilon_d) \to (0, \cdots, 0)$, where

$$M := m_1 + \cdots + m_d,$$
$$[a]_n := \begin{cases} a(n-1)! & (n \geq 1), \\ (-1)^n(-n)!^{-1} & (n < 1). \end{cases}$$

In the theorem, ε_j $(j = 1, \cdots, d)$ should satisfy $|\varepsilon_k/\varepsilon_d(j)| \ll 1$ $(j = 1, \cdots, d,\ k = j, \cdots, d)$. Let us consider this condition. If $|\varepsilon_k/\varepsilon_d(j)| \to \infty$, then $\varepsilon_d(j)$ tends to 0 rapidly. By (1.2), $s_j + \cdots + s_d = -M$ is a singular locus. Therefore, when $|\varepsilon_k/\varepsilon_d(j)| \to \infty$, the point $(-m_1+\varepsilon_1, \cdots, -m_d+\varepsilon_d)$ approximates asymptotically to the singular locus. Hence, $|\varepsilon_k/\varepsilon_d(j)| \ll 1$ means geometrically that $(-m_1+\varepsilon_1, \cdots, -m_d+\varepsilon_d)$ does not approximate asymptotically to the singular locus.

3 Examples

By the main theorem, we can compute various multiple zeta values at non-positive integers. Let us see some examples.

In the case $d = 2$, we have

$$\zeta_2(\varepsilon_1, \varepsilon_2) = \frac{1}{3} + \frac{1}{12}\cdot\frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2} + \sum_{j=1}^{2} O(\varepsilon_j),$$
$$\zeta_2(-1+\varepsilon_1, \varepsilon_2) = \frac{1}{24} + \sum_{j=1}^{2} O(\varepsilon_j),$$
$$\zeta_2(\varepsilon_1, -1+\varepsilon_2) = \frac{1}{12} + \sum_{j=1}^{2} O(\varepsilon_j),$$
$$\zeta_2(-1+\varepsilon_1, -1+\varepsilon_2) = \frac{1}{360} + \frac{1}{720}\cdot\frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2} + \sum_{j=1}^{2} O(\varepsilon_j).$$

In the case $d = 3$, we have

$$\zeta_3(\varepsilon_1, \varepsilon_2, \varepsilon_3) = -\frac{1}{4} - \frac{1}{24} \cdot \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3} - \frac{1}{24} \cdot \frac{\varepsilon_2 + 2\varepsilon_3}{\varepsilon_1 + \varepsilon_2 + \varepsilon_3} + \sum_{j=1}^{3} O(\varepsilon_j),$$

$$\zeta_3(-1 + \varepsilon_1, \varepsilon_2, \varepsilon_3) = -\frac{17}{720} - \frac{1}{144} \cdot \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3} + \frac{1}{720} \cdot \frac{-\varepsilon_2 + 3\varepsilon_3}{\varepsilon_1 + \varepsilon_2 + \varepsilon_3} + \sum_{j=1}^{3} O(\varepsilon_j),$$

$$\zeta_3(\varepsilon_1, -1 + \varepsilon_2, \varepsilon_3) = -\frac{19}{360} + \frac{1}{360} \cdot \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2 + \varepsilon_3} + \sum_{j=1}^{3} O(\varepsilon_j),$$

$$\zeta_3(\varepsilon_1, \varepsilon_2, -1 + \varepsilon_3) = -\frac{3}{40} - \frac{1}{720} \cdot \frac{4\varepsilon_2 + 3\varepsilon_3}{\varepsilon_1 + \varepsilon_2 + \varepsilon_3} + \sum_{j=1}^{3} O(\varepsilon_j).$$

Note that the example (1.5) comes from the first example of the above, taking $\varepsilon_1 = \varepsilon^2$ and $\varepsilon_2 = \varepsilon_3 = \varepsilon$.

In the case $d = 4$, we have

$$\begin{aligned}\zeta_4(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) =& \frac{1}{5} + \frac{1}{36} \cdot \frac{\varepsilon_4}{\varepsilon_3 + \varepsilon_4} + \frac{1}{48} \cdot \frac{\varepsilon_3 + 2\varepsilon_4}{\varepsilon_2 + \varepsilon_3 + \varepsilon_4} \\ &+ \frac{1}{720} \cdot \frac{19\varepsilon_2 + 33\varepsilon_3 + 52\varepsilon_4}{\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4} + \frac{1}{144} \cdot \frac{\varepsilon_4(\varepsilon_2 + \varepsilon_3 + \varepsilon_4)}{(\varepsilon_3 + \varepsilon_4)(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)} \\ &+ \sum_{j=1}^{4} O(\varepsilon_j).\end{aligned}$$

4 Proof of the Main Theorem

In this section, we prove the main theorem. If $d = 1$, $\zeta_1(s_1)$ is the Riemann zeta function. Hence, the main theorem is clear. So we prove the theorem in the case $d > 1$.

First, we prove the meromorphic continuation of $\zeta_d(s_1, \cdots, s_d)$. $\zeta_d(s_1, \cdots, s_d)$ has an integral representation as the following,

$$\begin{aligned}&\Gamma(s_1) \cdots \Gamma(s_d) \zeta_d(s_1, \cdots, s_d) \\ &= \int_0^1 \cdots \int_0^1 \int_0^\infty \prod_{j=1}^{d} x_j^{s_d(j) - d + j - 2} \prod_{j=2}^{d} (1 - x_j)^{s_{j-1} - 1} \prod_{j=1}^{d} \frac{x_1 \cdots x_j}{e^{x_1 \cdots x_j} - 1} dx_1 \cdots dx_d. \qquad (4.1)\end{aligned}$$

Dividing the integral into two parts and integrating by parts, (4.1) can be written

$$
\begin{aligned}
\zeta_d(s_1,\cdots,s_d) =& \frac{1}{\Gamma(s_1)\cdots\Gamma(s_d)}\sum_{k=0}^{n_1}\sum_{p_1+\cdots+p_d=k}\frac{B_{p_1}\cdots B_{p_d}}{p_1!\cdots p_d!}\frac{1}{s_d(1)-d+k}\times \\
&\times\prod_{j=2}^{d}B(s_d(j)-d+j+p_d(j)-1,s_{j-1}) \\
&+\frac{1}{\Gamma(s_1)\cdots\Gamma(s_d)}\int_0^1 x^{s_d(1)-d+n_1}F_\varphi(x,n_2,\cdots,n_d)dx \\
&+\frac{1}{\Gamma(s_1)\cdots\Gamma(s_d)}\int_1^\infty \frac{x^{s_d(1)-d}}{e^x-1}F_\psi(x,n_2,\cdots,n_d)dx,
\end{aligned}
\tag{4.2}
$$

where $n_1,\cdots,n_d\in\mathbb{Z}_{\geq 0}$. (4.2) can be continued meromorphically to

$$
\left\{(s_1,\cdots,s_d)\in\mathbb{C}^d \;\middle|\; \begin{array}{l}\Re(s_d(j))>d-j-n_j\ (j=1,\cdots,d),\\ \Re(s_{j-1})>-n_j-1\ (j=2,\cdots,d)\end{array}\right\}.
$$

We use (4.2) with $s_j=-m_j+\varepsilon_j$ $(j=1,\cdots,d)$ and $n_1=\cdots=n_d=M+d$. By estimating the second term and the third terms of (4.2), we obtain

$$
\frac{1}{\Gamma(s_1)\cdots\Gamma(s_d)}\left(\int_0^1 x^{s_d(1)-d+n_1}F_\varphi(x)dx+\int_1^\infty\frac{x^{s_d(1)-d}}{e^x-1}F_\psi(x)dx\right)=\sum_{j=1}^{d}O(\varepsilon_j).
$$

Consider the first term of (4.2) by writing it as the following,

$$
\sum_{k=0}^{M+d}=\sum_{k=0}^{M+d-1}+\sum_{k=M+d},
\tag{4.3}
$$

and estimating the first term of (4.3), we obtain

$$
\sum_{k=0}^{M+d-1}=\sum_{j=1}^{d}O(\varepsilon_j).
$$

Next, we consider the second term of (4.3). First, we estimate the factors containing gamma funcions and beta functions as the following,

$$
\begin{aligned}
&\frac{1}{\Gamma(s_1)\cdots\Gamma(s_d)}\prod_{j=2}^{d}B(s_d(j)-d+j+p_d(j)-1,s_{j-1}) \\
&=\frac{1}{\Gamma(s_d)}\prod_{j=2}^{d}\frac{\Gamma(s_d(j)-d+j+p_d(j)-1)}{\Gamma(s_d(j-1)-d+j+p_d(j)-1)} \\
&=\frac{1}{(\varepsilon_d)_{-m_d}\Gamma(\varepsilon_d(1))}\prod_{j=2}^{d}\frac{(\varepsilon_d(j))_{m_d(j)-d+j+p_d(j)-1}}{(\varepsilon_d(j-1))_{m_d(j-1)-d+j+p_d(j)-1}}.
\end{aligned}
\tag{4.4}
$$

Since we have

$$\begin{aligned}\frac{1}{(\varepsilon_d)_{-m_d}\Gamma(\varepsilon_d(1))} =&((-1)^{m_d}m_d!+O(\varepsilon_d))\left(\frac{\sin(\pi\varepsilon_d(1))}{\pi}\Gamma(1-\varepsilon_d(1))\right)\\ =&(-1)^{m_d}m_d!\varepsilon_d(1)+O(\varepsilon_d(1)^2)+O(\varepsilon_d(1)\varepsilon_d)\end{aligned}$$

and

$$\begin{aligned}&\prod_{j=2}^{d}\frac{(\varepsilon_d(j))_{m_d(j)-d+j+p_d(j)-1}}{(\varepsilon_d(j-1))_{m_d(j-1)-d+j+p_d(j)-1}}\\ &\quad=\prod_{j=2}^{d}\Bigg(h(-m_d(j)-d+j+p_d(j)-1,-m_d(j-1)-d+j+p_d(j)-1)\\ &\quad\quad\times\frac{[\varepsilon_d(j)]_{-m_d(j)-d+j+p_d(j)-1}}{[\varepsilon_d(j-1)]_{-m_d(j-1)-d+j+p_d(j)-1}}\Bigg)+\\ &\quad\quad+\sum_{j=2}^{d}\left\{O\left(\frac{\varepsilon_d(j)}{\varepsilon_d(j-1)}\varepsilon_d(j)\right)+O(\varepsilon_d(j-1))+O(\varepsilon_d(j))\right\},\end{aligned}$$

we find (4.4) is

$$\begin{aligned}&\frac{1}{\Gamma(s_1)\cdots\Gamma(s_d)}\prod_{j=2}^{d}B(s_d(j)-d+j+p_d(j)-1,s_{j-1})\\ &\quad=(-1)^{m_d}m_d!\varepsilon_d(1)\times\\ &\quad\quad\times\prod_{j=2}^{d}\Bigg(h(-m_d(j)-d+j+p_d(j)-1,-m_d(j-1)-d+j+p_d(j)-1)\\ &\quad\quad\frac{[\varepsilon_d(j)]_{-m_d(j)-d+j+p_d(j)-1}}{[\varepsilon_d(j-1)]_{-m_d(j-1)-d+j+p_d(j)-1}}\Bigg)\\ &\quad\quad+\sum_{j=1}^{d}O(\varepsilon_j\varepsilon_d(1)),\end{aligned}\tag{4.5}$$

where $h(m,n)$ is defined by

$$h(m,n):=\begin{cases}0 & (m\geq 1>n),\\ 1 & (otherwise).\end{cases}$$

Using (4.5), we can estimate the second term of (4.3) as the following,

$$\begin{aligned}&(-1)^{m_d}m_d!\sum_{p_1+\cdots+p_d=d+M}\frac{B_{p_1}\cdots B_{p_d}}{p_1!\cdots p_d!}\times\\ &\quad\times\prod_{j=2}^{d}h(-m_d(j)-d+j+p_d(j)-1,-m_d(j-1)-d+j+p_d(j)-1)\times\\ &\quad\quad\times\frac{[\varepsilon_d(j)]_{-m_d(j)-d+j+p_d(j)-1}}{[\varepsilon_d(j-1)]_{-m_d(j-1)-d+j+p_d(j)-1}}+\sum_{j=1}^{d}O(\varepsilon_j).\end{aligned}\tag{4.6}$$

Finally, to remove the function h from (4.6), we restrict the summation. Then we obtain the main theorem.

References

[1] S. Akiyama, S. Egami and Y. Tanigawa, *Analytic continuation of multiple zeta-functions and their values at non-positive integers*, Acta Arith, **98** (2001), 107–116.

[2] S. Akiyama, Y. Tanigawa, *Multiple zeta values at non-positive integers*, Ramanujan J. **5** (2001), 327–351.

[3] Y. Komori, *An integral representation of multiple Hurwitz-Lerch zeta functions and generalized multiple bernoulli numbers*, Quart. J. Math. (Oxford) (2009), 1–60.

[4] K. Matsumoto, *On analytic continuation of various multiple zeta-functions*, Number Theory for the Millenium (Urbana, 2000), Vol. II, M. A. Bennett et. al. (eds.), A. K. Peters, Natick, MA, 2002, pp. 417–440.

[5] K. Matsumoto, *The analytic continuation and the asymptotic behaviour of certain multiple zeta-functions I*, J. Number Theory **101** (2003), 223–243.

[6] Y. Sasaki, *Multiple zeta values for coordinatewise limits at non-positive integers*, Acta Arith. **136** (2009), 299–317.

[7] J. Zhao, *Analytic continuation of multiple zeta functions*, Proc. Amer. Math. Soc. **128** (2000), 1275–1283.

Graduate School of Mathematics
Nagoya University
Chikusa-ku, Nagoya 464-8602, Japan
E-mail: m11022v@math.nagoya-u.ac.jp