数理解析研究所講究録
第 1878 巻 2014 年 1-6

RADIUS PROBLEMS FOR INVERSE FUNCTIONS CONCERNING WITH BI-UNIVALENT FUNCTIONS

EMEL YAVUZ DUMAN AND SHIGEYOSHI OWA

ABSTRACT. For bi-univalent functions of univalent functions in the open unit disc, there are some coefficient estimates. In the present paper, new radius problems for convex functions and starlike functions concerning with bi-univalent functions are discussed.

1. INTRODUCTION

Let $\mathcal{A}(r)$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

that are analytic in the open unit disc $\mathbb{U}(r) = \{z \in \mathbb{C} \mid |z| < r\}$.

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}(1)$ consisting of $f(z)$ which are univalent in the open unit disc $\mathbb{U}(1)$. A function $f(z) \in \mathcal{A}(1)$ is said to be starlike with respect to the origin in $\mathbb{U}(1)$ if $f(z)$ satisfies

$$\mathfrak{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{U}(1)). \tag{1.2}$$

We denote by $\mathcal{S}^*$ the class of all such starlike functions $f(z)$. Further, let $\mathcal{K}$ be the subclass of $\mathcal{A}(1)$ consisting of functions $f(z)$ which satisfy

$$\mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \mathbb{U}(1)). \tag{1.3}$$

A function $f(z)$ in the class $\mathcal{K}$ is said to be convex in $\mathbb{U}(1)$.

It is well-known that

$$f(z) = \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} k z^k \tag{1.4}$$

is the extremal function for $\mathcal{S}^*$, and that

$$f(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k \tag{1.5}$$

is the extremal function for $\mathcal{K}$ (see [2], [3]).

We also note that $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S} \subset \mathcal{A}(1)$.

Since $\mathcal{S}$ is the class of univalent functions $f(z) \in \mathcal{A}(1)$, for each function $w = f(z)$ in $\mathcal{S}$, there exists an inverse function $f^{-1}(w)$ of $f(z)$. If $f(z) \in \mathcal{S}$ and $f^{-1}(w)$ has a univalent analytic continuation to $|w| < 1$, then $f(z)$ is said to be bi-univalent in $\mathbb{U}(1)$. The concept of bi-univalent functions was given by Lewin

[5], and studied by Brannan and Taha [1], Xu, Gui and Srivastava [6], and Xu, Xiao and Srivastava [7]. Xu, Gui and Srivastava [6] showed that functions

$$\frac{z}{1-z}, \ -\log(1-z), \ \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

are bi-univalent in $\mathbb{U}(1)$, and that functions

$$z-\frac{1}{2}z^2, \ \frac{z}{1-z^2}$$

are not bi-univalent in $\mathbb{U}(1)$.

Recently, Hayami and Owa [4] have given the following theorem for bi-univalent functions.

Theorem A. *If $f(z) \in \mathcal{S}$, then it follows that $f(\mathbb{U}(1)) \not\supset \mathbb{U}(1)$ and $f(\mathbb{U}(1)) \not\subset \mathbb{U}(1)$ unless $f(z)=z$.*

But, we know that all functions $f(z) \in \mathcal{S}$ include the open disc $\mathbb{U}(1/4) = \{z \in \mathbb{C} \,|\, |z| < 1/4\}$. Therefore, we consider the subclass $\mathcal{S}(r)$ of $\mathcal{A}(r)$ consisting of $f(z)$ which are univalent in $\mathbb{U}(r)$.

Since $f(0)=0$ for $f(z) \in \mathcal{S}(r)$, there exists an open disc such that

$$f(\mathbb{U}(r)) \supset |z| < \max_{|z|<r} |f(z)|.$$

For such an open disc, we consider the inverse function $f^{-1}(w)$ of $f(z)$ such that $f^{-1}(0)=0$.

In view of the above concept, we can consider

$$\begin{aligned} w_1(z) &= f(z) && (z \in \mathbb{U}(r_1)), \\ w_2(z) &= f^{-1}(w_1) && (z \in \mathbb{U}(r_2)), \\ w_3(z) &= f^{-1}(w_2) && (z \in \mathbb{U}(r_3)), \end{aligned}$$

and

$$w_n = f^{-1}(w_{n-1}) \quad (z \in \mathbb{U}(r_n)).$$

2. Properties for Convex Functions

We first consider the inverse function $f^{-1}(w)$ of the automorphism $w=f(z)$.

Theorem 2.1. *Let us define*

$$w_1 = f(z) = \frac{z}{1-az} \quad (|z| < 1/a) \tag{2.1}$$

for some real a $(0 < a \leq 1)$. Then $w_n = f^{-1}(w_{n-1})$ satisfies

$$w_n = \frac{w_{n-1}}{1+(-1)^n a w_{n-1}} \quad (|w_{n-1}| < 1/(na)) \tag{2.2}$$

and

$$\left| w_n + \frac{(-1)^n}{(n^2-1)a} \right| < \frac{n}{(n^2-1)a} \quad (n=2,3,4,\cdots). \tag{2.3}$$

Proof. For w_1, we see that

$$|z| = \left| \frac{w_1}{1+aw_1} \right| < \frac{1}{a} \tag{2.4}$$

which gives that

$$\mathfrak{Re}\,w_1 > -\frac{1}{2a} \quad (|z| < 1/a).$$

Next, we consider

$$w_2 = f^{-1}(w_1) = \frac{w_1}{1 + aw_1} \quad (|w_1| < 1/(2a)). \tag{2.5}$$

Noting that

$$|w_1| = \left|\frac{w_2}{1 - aw_2}\right| < \frac{1}{2a}, \tag{2.6}$$

we have that

$$\left|w_2 + \frac{1}{3a}\right| < \frac{2}{3a}. \tag{2.7}$$

Therefore, the result holds true for $n = 2$.

Suppose that (2.2) and (2.3) hold true for n. Then, since

$$|w_{n-1}| = \left|\frac{w_n}{1 - (-1)^n a w_n}\right| < \frac{1}{na}, \tag{2.8}$$

we obtain that

$$w_{n+1} = \frac{w_n}{1 + (-1)^{n+1} a w_n} \tag{2.9}$$

and (2.3) shows us that w_n includes the open disc $|w_n| < 1/((n+1)a)$. Therefore, w_{n+1} satisfies that

$$|w_n| = \left|\frac{w_{n+1}}{1 - (-1)^{n+1} a w_{n+1}}\right| < \frac{1}{(n+1)a}. \tag{2.10}$$

Noting that

$$(n+1)^2 a^2 |w_{n+1}|^2 < |1 - (-1)^{n+1} a w_{n+1}|^2, \tag{2.11}$$

we show that

$$\left|w_{n+1} + \frac{(-1)^{n+1}}{((n+1)^2 - 1)a}\right| < \frac{n+1}{((n+1)^2 - 1)a}. \tag{2.12}$$

Thus, by the mathematical induction, we complete the proof of the theorem. □

Making $a = 1$ in Theorem 2.1, we have

Corollary 2.2. *The extremal function $f(z)$ given by* (1.5) *in $|z| < 1$ satisfies*

$$w_n = \frac{w_{n-1}}{1 + (-1)^n w_{n-1}} \quad (|w_{n-1}| < 1/n) \tag{2.13}$$

and

$$\left|w_n + \frac{(-1)^n}{n^2 - 1}\right| < \frac{n}{n^2 - 1} \quad (n = 2, 3, 4, \cdots). \tag{2.14}$$

3. Properties for Starlike Functions

The next our result for the inverse function $f^{-1}(w)$ of starlike functions is contained in

Theorem 3.1. *Let us define*

$$w_1 = f(z) = \frac{z}{(1-z)^2} \quad (|z| < 1). \tag{3.1}$$

Then $w_n = f^{-1}(w_{n-1})$ *satisfies*

$$w_{2n} = \frac{1 + 2w_{2n-1} - \sqrt{1 + 4w_{2n-1}}}{2w_{2n-1}} \quad (|w_{2n-1}| < 1/(4n)) \tag{3.2}$$

and

$$w_{2n+1} = \frac{w_{2n}}{(1 - w_{2n})^2} \quad (|w_{2n}| < 2n + 1 - 2\sqrt{n(n+1)}) \tag{3.3}$$

for $n = 1, 2, 3, \cdots$.

Proof. For $n = 1$,

$$w_2 = \frac{1 + 2w_1 - \sqrt{1 + 4w_1}}{2w_1} \quad (|w_1| < 1/4). \tag{3.4}$$

Since

$$|w_2| = \left| 1 + \frac{1 - \sqrt{1 + 4w_1}}{2w_1} \right| \quad (|w_1| < 1/4), \tag{3.5}$$

we obtain that

$$\min_{|w_1| = 1/4} |w_2| = 1 + 2(1 - \sqrt{2}) = 3 - 2\sqrt{2}. \tag{3.6}$$

Therefore, we have that

$$w_3 = \frac{w_2}{(1 - w_2)^2} \quad (|w_2| < 3 - 2\sqrt{2}). \tag{3.7}$$

Since

$$w_3 = \frac{1}{w_2 + \frac{1}{w_2} - 2} \quad (|w_2| < 3 - 2\sqrt{2}), \tag{3.8}$$

let us consider

$$w_2 + \frac{1}{w_2} = u + iv \tag{3.9}$$

for $|w_2| = 3 - 2\sqrt{2}$. This implies that

$$\frac{u^2}{36} + \frac{v^2}{32} = 1. \tag{3.10}$$

Thus, we obtain that

$$\min_{|w_2| = 3 - 2\sqrt{2}} |w_3| = \frac{1}{8}. \tag{3.11}$$

Therefore, (3.2) and (3.3) are hold true for $n = 1$. Next, we assume that (3.2) and (3.3) are true for $n = j$, such that

$$w_{2j} = \frac{1 + 2w_{2j-1} - \sqrt{1 + 4w_{2j-1}}}{2w_{2j-1}} \quad (|w_{2j-1}| < 1/(4j)) \tag{3.12}$$

and

$$w_{2j+1} = \frac{w_{2j}}{(1-w_{2j})^2} \quad (|w_{2j}| < 2j+1-2\sqrt{j(j+1)}). \tag{3.13}$$

It follows from (3.13) that $w_{2j} = u + iv$ satisfies

$$\frac{u^2}{4(2j+1)^2} + \frac{v^2}{16j(j+1)} = 1 \tag{3.14}$$

for $|w_{2j}| = 2j+1-2\sqrt{j(j+1)}$. This gives us that

$$\min_{|w_{2j}|=2j+1-2\sqrt{j(j+1)}} |w_{2j+1}| = \frac{1}{4(j+1)}. \tag{3.15}$$

Thus, we have that

$$w_{2(j+1)} = \frac{1+2w_{2j+1}-\sqrt{1+4w_{2j+1}}}{2w_{2j+1}} \quad (|w_{2j+1}| < 1/(4(j+1))). \tag{3.16}$$

Furthermore, since

$$|w_{2(j+1)}| = \left|1+\frac{1-\sqrt{1+4w_{2j+1}}}{2w_{2j+1}}\right| \quad (|w_{2j+1}| < 1/(4(j+1))), \tag{3.17}$$

we also have that

$$\min_{|w_{2j+1}|=\frac{1}{4(j+1)}} |w_{2(j+1)}| = 2(j+1)+1-2\sqrt{(j+1)(j+2)}. \tag{3.18}$$

Consequently, (3.2) and (3.3) are hold true for $n = j+1$. Thus, applying mathematical induction, we complete the proof of the theorem. □

Finally, we consider the following function

$$w_1 = \frac{z}{(1-az)^2} \quad (0 < a \leq 1) \tag{3.19}$$

for $|z| < 1$. Since

$$\mathfrak{Re}\left(\frac{zw_1'}{w_1}\right) = \mathfrak{Re}\left(\frac{1+az}{1-az}\right) > \frac{1-a}{1+a}, \tag{3.20}$$

w_1 is starlike with respect to the origin.

If $a = 1$, then w_1 becomes the extremal function for the class $\mathcal{S}^*$. For this function w_1 given by (3.19), how can we consider the inverse function $w_n = f^{-1}(w_{n-1})$?

References

1. D.A. Brannan and T.S. Taha, On some classes of bi-univalent functions, *Studia Univ. Babeş-Bolyai Math.* **31** (1986), 70-77.
2. P.L. Duren, *Univalent Functions*, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.
3. A.W. Goodman, *Univalent Functions*, Vol. I, Mariner Publishing Co. Inc., Tampa, FL, 1983.
4. T. Hayami and S. Owa, Coefficient bounds for bi-univalent functions, *PanAmerican Math. J.* **22** (2012), 15-26.
5. M. Lewin, On a coefficient problem for bi-univalent functions, *Proc. Amer. Math. Soc.* **18** (1967), 63-68.
6. Q.-H. Xu, Y.-C. Gui and H.M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, *Appl. Math. Lett.* **25** (2012), 990-994.

7. Q.-H. Xu, H.-G. Xiao and H.M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, *Appl. Math. Comp.* **218** (2012), 11461-11465.

İSTANBUL KÜLTÜR UNIVERSITY, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, ATAKÖY CAMPUS, 34156 BAKIRKÖY, İSTANBUL, TURKEY
E-mail address: `e.yavuz@iku.edu.tr`

KINKI UNIVERSITY, DEPARTMENT OF MATHEMATICS, HIGASHI-OSAKA, OSAKA 577-8502, JAPAN
E-mail address: `shige21@ican.zaq.ne.jp`