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On N-Fractional Calculus of the Function $((z-b)^2-c)^{\frac{1}{3}}$

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Abstract

We discuss the N-fractional calculus of $f(z)=((z-b)^2-c)^{\frac{1}{3}}$. In order to do fractional calculus of $((z-b)^2-c)^{\frac{1}{3}}$, we consider four type's factorization of the equation and calculate

1. $(f)_\gamma=\left(((z-b)^2-c)^{\frac{1}{3}}\right)_\gamma$
2. $(f)_\gamma=\left(((z-b)^2-c)^{-\frac{2}{3}}((z-b)^2-c)\right)_\gamma$
3. $(f)_\gamma=\left(((z-b)^2-c)^{-\frac{5}{3}}((z-b)^2-c)^2\right)_\gamma$
4. $(f)_\gamma=\left((((z-b)^2-c)^2-c)^{\frac{1}{3}})_1\right)_{\gamma-1}$

We have four representations of fractional calculus. And then we show that these four different forms of N-fractional calculus are consistent in special case. And some identities are reported.

1 Introduction

We adopt the following definition of the fractional calculus.

(I) Definition. (by K. Nishimoto, [1] Vol. 1)

Let $D=\{D_-,D_+\}$, $C=\{C_-,C_+\}$, C_- be a curve along the cut joining two points z and $-\infty+iIm(z)$, C_+ be a curve along the cut joining two points z and $\infty+iIm(z)$, D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ (Here D contains the points over the curve C).

Moreover, let $f=f(z)$ be a regular function in $D(z\in D)$,

$$\begin{aligned} f_\nu &= (f)_\nu = {}_C(f)_\nu \\ &= \frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^{\nu+1}} \quad (\nu\notin Z^-), \end{aligned} \tag{1}$$

$$(f)_{-m} = \lim_{\nu \to -m} (f)_\nu \quad (m \in Z^+), \tag{2}$$

where

$$-\pi \le arg(\zeta - z) \le \pi \ for\ C_-, \quad 0 \le arg(\zeta - z) \le 2\pi \ \ for\ C_+,$$

$$\zeta \ne z, \quad z \in C, \quad \nu \in R, \quad \Gamma; \ Gamma\ function,$$

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^ν be

$$N^\nu = (\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\zeta}{(\zeta - z)^{\nu+1}}) \quad (\nu \notin Z^-), \quad (Refer\ to[1]) \tag{3}$$

with

$$N^{-m} = \lim_{\nu \to -m} N^\nu \quad (m \in Z^+), \tag{4}$$

and define the binary operation $\circ$ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\alpha(N^\beta f) \quad (\alpha, \beta \in R), \tag{5}$$

then the set

$$\{N^\nu\} = \{N^\nu | \nu \in R\} \tag{6}$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \ne |f_\nu| < \infty, \nu \in R\}$, where $f = f(z)$ and $z \in C$. (vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. " F.O.G. $\{N^\nu\}$) " is an " Action product group which has continuous index ν " for the set of F . (F.O.G. ; Fractional calculus operator group)

Theorem C. Let

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} \quad (\nu \in R). \tag{7}$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \tag{8}$$

holds. [4]

(III)

In some previous papers, the following result are known as elementary properties.

Lemma. We have [1]

(i)

$$((z-c)^{\beta})_{\alpha} = e^{-i\pi\alpha}\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}(z-c)^{\beta-\alpha} \quad (|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}| < \infty)$$

(ii)

$$(log(z-c))_{\alpha} = -e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty)$$

(iii)

$$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\log(z-c), \quad (|\Gamma(\alpha)| < \infty)$$

where $z-c \neq 0$ in (i), and $z-c \neq 0, 1$ in (ii) and (iii) ,

(iv)

$$(u \cdot v)_{\alpha} := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha+1-k)} u_{\alpha-k} v_k. \quad (u = u(z), v = v(z))$$

Moreover in the previous works we refer to the next theorem [6].

Theorem D. We have

(i)

$$(((z-b)^{\beta}-c)^{\alpha})_{\gamma} = e^{-i\pi\gamma}(z-b)^{\alpha\beta-\gamma}\sum_{k=0}^{\infty}\frac{[-\alpha]_k\Gamma(\beta k-\alpha\beta+\gamma)}{k!\Gamma(\beta k-\alpha\beta)}\left(\frac{c}{(z-b)^{\beta}}\right)^k \tag{9}$$

$$(|\frac{\Gamma(\beta k-\alpha\beta+\gamma)}{\Gamma(\beta k-\alpha\beta)}| < \infty),$$

and

(ii)

$$(((z-b)^{\beta}-c)^{\alpha})_{n} = (-1)^n(z-b)^{\alpha\beta-n}\sum_{k=0}^{\infty}\frac{[-\alpha]_k[\beta k-\alpha\beta]_n}{k!}\left(\frac{c}{(z-b)^{\beta}}\right)^k \tag{10}$$

$$(n \in Z_0^+, \quad |\frac{c}{(z-b)^{\beta}}| < 1),$$

where

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda) \quad with \quad [\lambda]_0 = 1,$$

(Pochhammer's Notation).

2 N-Fractional Calculus of the Functions $f(z) = ((z-b)^2-c)^{\frac{1}{3}}$

In order to have a representation of N-fractional calculus with γ-order, we directly apply the theorem to the function at the beginning.

Theorem 1. Let

$$f = f(z) = ((z-b)^2-c)^{\frac{1}{3}} \quad \left(((z-b)^2-c)^{\frac{1}{3}} \neq 0\right) \tag{1}$$

we have

$$(f)_\gamma = e^{-i\pi\gamma}(z-b)^{-\frac{2}{3}-\gamma} \sum_{k=0}^{\infty} \frac{[-\frac{1}{3}]_k \Gamma(2k-\frac{2}{3}+\gamma)}{k!\Gamma(2k-\frac{2}{3})} \left(\frac{c}{(z-b)^2}\right)^k \tag{2}$$

Proof. According to Theorem D, we have the equation (1) directly.

Secondly, we consider the function as a product of two functions like as

$$f(z) = ((z-b)^2-c)^{-\frac{2}{3}} \cdot ((z-b)^2-c)$$

and we have the new representation for $(f)_\gamma$ as follows.

Theorem 2. We set $f = f(z)$, and S, K ,J as follows,

$$S = S(z) = \frac{c}{(z-b)^2}, \quad (|S| < 1) \tag{3}$$

$$K(k,\gamma,m) = \frac{[\frac{2}{3}]_k \Gamma(2k+\frac{4}{3}+\gamma-m)}{k!\Gamma(2k+\frac{4}{3})} S^k, \tag{4}$$

$$J(\gamma, m) = \sum_{k=0}^{\infty} K(k,\gamma,m). \tag{5}$$

We have

$$\begin{aligned}(f)_\gamma \quad &= \quad e^{-i\pi\gamma}(z-b)^{-\frac{4}{3}-\gamma+2}\{(1-S)J(\gamma,0) - 2\gamma J(\gamma,1) \\ &\quad +\gamma(\gamma-1)J(\gamma,2)\}\end{aligned} \tag{6}$$

Proof. According to Lemma (iv), we have

$$(f)_\gamma = \left(((z-b)^2-c)^{-\frac{2}{3}} \cdot ((z-b)^2-c)\right)_\gamma \tag{7}$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+1)}{k!\Gamma(\gamma+1-k)} (((z-b)^2-c)^{-\frac{2}{3}})_{\gamma-k} \cdot ((z-b)^2-c)_k \tag{8}$$

and applying Theorem D.(i) to

$$(((z-b)^2-c)^{-\frac{2}{3}})_{\gamma-k}, \tag{9}$$

we obtain

$$\begin{aligned}
(f)_\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1)}\left(((z-b)^2-c)^{-\frac{2}{3}}\right)_\gamma ((z-b)^2-c)_0 \\
&\quad + \frac{\Gamma(\gamma+1)}{\Gamma(\gamma)}\left(((z-b)^2-c)^{-\frac{2}{3}}\right)_{\gamma-1}(2(z-b)) \\
&\quad + \frac{\Gamma(\gamma+1)}{2!\Gamma(\gamma-1)}\left(((z-b)^2-c)^{-\frac{2}{3}}\right)_{\gamma-2}\cdot 2 \\
&= \left(((z-b)^2-c)^{-\frac{2}{3}}\right)_\gamma ((z-b)^2-c) + 2\gamma\left(((z-b)^2-c)^{-\frac{2}{3}}\right)_{\gamma-1}\cdot(z-b) \\
&\quad + 2\gamma(\gamma-1)\left(((z-b)^2-c)^{-\frac{2}{3}}\right)_{\gamma-2} \\
&= e^{-i\pi\gamma}(z-b)^{-\frac{4}{3}-\gamma}((z-b)^2-c)\sum_{k=0}^{\infty}\frac{[\frac{2}{3}]_k\Gamma(2k+\frac{4}{3}+\gamma)}{k!\Gamma(2k+\frac{4}{3})}\left(\frac{c}{(z-b)^2}\right)^k \\
&\quad + 2\gamma(z-b)e^{-i\pi(\gamma-1)}(z-b)^{-\frac{4}{3}-\gamma+2}\sum_{k=0}^{\infty}\frac{[\frac{2}{3}]_k\Gamma(2k+\frac{4}{3}+\gamma-1)}{k!\Gamma(2k+\frac{4}{3})}\left(\frac{c}{(z-b)^2}\right)^k \\
&\quad + 2\gamma(\gamma-1)e^{-i\pi(\gamma-2)}(z-b)^{-\frac{4}{3}-\gamma+2}\sum_{k=0}^{\infty}\frac{[\frac{2}{3}]_k\Gamma(2k+\frac{4}{3}+\gamma-2)}{k!\Gamma(2k+\frac{4}{3})}\left(\frac{c}{(z-b)^2}\right)^k
\end{aligned} \tag{10}$$

Then we have the representation

$$(f(z))_\gamma = e^{-i\pi\gamma}(z-b)^{-\frac{4}{3}-\gamma+2}\{(1-S)J(\gamma,0)-2\gamma J(\gamma,1)+2\gamma(\gamma-1)J(\gamma,2)\}. \tag{11}$$

This is the same one as the equation (6).

Next, we consider the function as another product form like as

$$f(z) = ((z-b)^2-c)^{-\frac{5}{3}} \cdot ((z-b)^2-c)^2$$

and we have the new representation for $(f)_\gamma$ as follows.

Theorem 3. We set $f = f(z)$, and S, H ,G as follows,

$$S = S(z) = \frac{c}{(z-b)^2}, \quad (|S| < 1) \tag{12}$$

$$H(k,\gamma,m) = \frac{[\frac{5}{3}]_k \Gamma(2k + \frac{10}{3} + \gamma - m)}{k!\Gamma(2k+\frac{10}{3})} S^k, \tag{13}$$

$$G(\gamma, m) = \sum_{k=0}^{\infty} H(k,\gamma,m). \tag{14}$$

We have

$$\begin{aligned}(f)_\gamma &= e^{-i\pi\gamma}(z-b)^{-\frac{10}{3}-\gamma+4}\{(1-S)^2G(\gamma,0) - 4\gamma(1-S)G(\gamma,1) + 6\gamma(\gamma-1)(1-\frac{1}{3}S)G(\gamma,2) \\ &\quad -4\gamma(\gamma-1)(\gamma-2)G(\gamma,3) + \gamma(\gamma-1)(\gamma-2)(\gamma-3)G(\gamma,4)\}\end{aligned} \tag{15}$$

Proof. According to Lemma (iv), we have

$$(f)_\gamma = \left(((z-b)^2-c)^{-\frac{5}{3}} \cdot (((z-b)^2-c)^2)\right)_\gamma \tag{16}$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+1)}{k!\Gamma(\gamma+1-k)}(((z-b)^2-c)^{-\frac{5}{3}})_{\gamma-k} \cdot (((z-b)^2-c)^2)_k \tag{17}$$

and applying Theorem D.(i) to

$$(((z-b)^2-c)^{-\frac{5}{3}})_{\gamma-k}, \tag{18}$$

we obtain

$$\begin{aligned}(f)_\gamma = &\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1)}\left(((z-b)^2-c)^{-\frac{5}{3}}\right)_\gamma (((z-b)^2-c)^2)_0 \\ &+\frac{\Gamma(\gamma+1)}{\Gamma(\gamma)}\left(((z-b)^2-c)^{-\frac{5}{3}}\right)_{\gamma-1}(4((z-b)^2-c)(z-b)) \\ &+\frac{\Gamma(\gamma+1)}{2!\Gamma(\gamma-1)}\left(((z-b)^2-c)^{-\frac{5}{3}}\right)_{\gamma-2}\cdot(12(z-b)^2-4c) \\ &+\frac{\Gamma(\gamma+1)}{3!\Gamma(\gamma-2)}\left(((z-b)^2-c)^{-\frac{5}{3}}\right)_{\gamma-3}\cdot(24(z-b)) \\ &+\frac{\Gamma(\gamma+1)}{4!\Gamma(\gamma-3)}\left(((z-b)^2-c)^{-\frac{5}{3}}\right)_{\gamma-4}\cdot 24\end{aligned}$$

$$= \left(((z-b)^2-c)^{-\frac{5}{3}}\right)_\gamma (((z-b)^2-c)^2) + 4\gamma \left(((z-b)^2-c)^{-\frac{5}{3}}\right)_{\gamma-1} \cdot ((z-b)^3 - c(z-b))$$
$$+\gamma(\gamma-1)\left(((z-b)^2-c)^{-\frac{5}{3}}\right)_{\gamma-2} (6(z-b)^2-2c)$$
$$+4\gamma(\gamma-1)(\gamma-2)\left(((z-b)^2-c)^{-\frac{5}{3}}\right)_{\gamma-3} (z-b)$$
$$+\gamma(\gamma-1)(\gamma-2)(\gamma-3)\left(((z-b)^2-c)^{-\frac{5}{3}}\right)_{\gamma-4}$$
$$= e^{-i\pi\gamma}(z-b)^{-\frac{10}{3}-\gamma+4}((z-b)^2-c)\sum_{k=0}^{\infty}\frac{[\frac{5}{3}]_k\Gamma(2k+\frac{10}{3}+\gamma)}{k!\Gamma(2k+\frac{10}{3})}\left(\frac{c}{(z-b)^2}\right)^k (1-\frac{c}{(z-b)^2})^2$$
$$+4\gamma(z-b)e^{-i\pi(\gamma-1)}(z-b)^{-\frac{10}{3}-\gamma+4}\sum_{k=0}^{\infty}\frac{[\frac{5}{3}]_k\Gamma(2k+\frac{10}{3}+\gamma-1)}{k!\Gamma(2k+\frac{10}{3})}\left(\frac{c}{(z-b)^2}\right)^k (1-\frac{c}{(z-b)^2})$$
$$+6\gamma(\gamma-1)e^{-i\pi(\gamma-2)}(z-b)^{-\frac{10}{3}-\gamma+4}\sum_{k=0}^{\infty}\frac{[\frac{5}{3}]_k\Gamma(2k+\frac{10}{3}+\gamma-2)}{k!\Gamma(2k+\frac{10}{3})}\left(\frac{c}{(z-b)^2}\right)^k (1-\frac{1}{3}\frac{c}{(z-b)^2})$$
$$+4\gamma(\gamma-1)(\gamma-2)e^{-i\pi(\gamma-3)}(z-b)^{-\frac{10}{3}-\gamma+4}\sum_{k=0}^{\infty}\frac{[\frac{5}{3}]_k\Gamma(2k+\frac{10}{3}+\gamma-3)}{k!\Gamma(2k+\frac{10}{3})}\left(\frac{c}{(z-b)^2}\right)^k$$
$$+\gamma(\gamma-1)(\gamma-2)(\gamma-3)e^{-i\pi(\gamma-4)}(z-b)^{-\frac{10}{3}-\gamma+4}\sum_{k=0}^{\infty}\frac{[\frac{5}{3}]_k\Gamma(2k+\frac{10}{3}+\gamma-4)}{k!\Gamma(2k+\frac{10}{3})}\left(\frac{c}{(z-b)^2}\right)^k \tag{19}$$

Then we have the representation

$$(f(z))_\gamma = e^{-i\pi\gamma}(z-b)^{-\frac{10}{3}-\gamma+4}\{(1-S)^2G(\gamma,0) - 4\gamma(1-S)G(\gamma,1) + 6\gamma(\gamma-1)(1-\frac{1}{3}S)G(\gamma,2)$$
$$-4\gamma(\gamma-1)(\gamma-2)G(\gamma,3) + \gamma(\gamma-1)(\gamma-2)(\gamma-3)G(\gamma,4)\}. \tag{20}$$

This is the same one as the equation (15).

Next, we choose another process of the fractional calculus which is devided into two stages as like as

$$(f(z))_\gamma = ((f(z))_1)_{\gamma-1}\cdot \tag{21}$$

We have an another result.

Theorem 4. We set $f = f(z)$, and S, R ,W as follows,

$$S = S(z) = \frac{c}{(z-b)^2}, \quad (|S|<1) \tag{22}$$

$$R(k,\gamma,m) = \frac{[\frac{2}{3}]_k\Gamma(2k+\frac{4}{3}+\gamma-m)}{k!\Gamma(2k+\frac{4}{3})}S^k, \tag{23}$$

$$W(\gamma;m)=\sum_{k=0}^{\infty}R(k,\gamma,m). \tag{24}$$

Then we have

$$(f)_\gamma=\frac{2}{3}e^{-i\pi\gamma}(z-b)^{-\frac{4}{3}-\gamma+2}\{-W(\gamma;1)-(\gamma-1)W(\gamma,2)\}. \tag{25}$$

Proof. We have

$$\begin{aligned}\left(((z-b)^2-c)^{\frac{1}{3}}\right)_1 &= \frac{1}{3}((z-b)^2-c)^{-\frac{2}{3}}\cdot 2(z-b)\\ &= \frac{2}{3}((z-b)^2-c)^{-\frac{2}{3}}(z-b) \end{aligned} \tag{26}$$

Then

$$\begin{aligned}&\left((((z-b)^2-c)^{\frac{1}{3}})_1\right)_{\gamma-1}=\frac{2}{3}\left(((z-b)^2-c)^{-\frac{2}{3}}(z-b)\right)_{\gamma-1}\\ &=\frac{2}{3}\sum_{k=0}^{\infty}\frac{\Gamma(\gamma)}{k!\Gamma(\gamma-k)}\left(((z-b)^2-c)^{-\frac{2}{3}}\right)_{\gamma-1-k}(z-b)_k\\ &=\frac{2}{3}\{\frac{\Gamma(\gamma)}{\Gamma(\gamma)}(((z-b)^2-c)^{-\frac{2}{3}})_{\gamma-1}(z-b)+\frac{\Gamma(\gamma)}{\Gamma(\gamma-1)}(((z-b)^2-c)^{-\frac{2}{3}})_{\gamma-1-1}\}\\ &=\frac{2}{3}\{e^{-i\pi(\gamma-1)}(z-b)^{-\frac{4}{3}-\gamma+2}\sum_{k=0}^{\infty}\frac{[\frac{2}{3}]_k\Gamma(2k+\frac{4}{3}+\gamma-1)}{k!\Gamma(2k+\frac{4}{3})}\left(\frac{c}{(z-b)^2}\right)^k\\ &+(\gamma-1)e^{-i\pi(\gamma-2)}(z-b)^{-\frac{4}{3}-\gamma+2}\sum_{k=0}^{\infty}\frac{[\frac{2}{3}]_k\Gamma(2k+\frac{4}{3}+\gamma-2)}{k!\Gamma(2k+\frac{4}{3})}\left(\frac{c}{(z-b)^2}\right)^k \end{aligned} \tag{27}$$

And we put

$$R(k,\gamma,m)=\frac{[\frac{2}{3}]_k\Gamma(2k+\frac{4}{3}+\gamma-m)}{k!\Gamma(2k+\frac{4}{3})}\left(\frac{c}{(z-b)^2}\right)^k,$$

$$W(\gamma;m)=\sum_{k=0}^{\infty}R(k,\gamma,m).$$

So we have

$$(f(z))_\gamma=\frac{2}{3}e^{-i\pi\gamma}(z-b)^{-\frac{4}{3}-\gamma+2}\{-W(\gamma,1)+(\gamma-1)W(\gamma,2)\},\quad (\gamma\notin Z^-). \tag{28}$$

We have the equation (25) from above equation directly.

3 Identities

We have four kinds of representation on N-fractional calculus of the function $f(z) = ((z-b)^2 - c)^{-\frac{1}{3}}$ like as Theorem 1, 2 ,3 and 4. Accordingly we have the following identities with using J and G and W and L given in §2.

Theorem 5. We have

(i)

$$\sum_{k=0}^{\infty} \frac{[\frac{1}{3}]_k \Gamma(2k - \frac{2}{3} + \gamma)}{k!\Gamma(2k - \frac{2}{3})} S^k = (1-S)J(\gamma,0) - 2\gamma J(\gamma,1) + 2\gamma(\gamma-1)J(\gamma,2), \quad (\gamma \notin Z^-) \tag{1}$$

and

(ii)

$$\sum_{k=0}^{\infty} \frac{[\frac{1}{3}]_k \Gamma(2k - \frac{2}{3} + \gamma)}{k!\Gamma(2k - \frac{2}{3})} S^k = (1-S)^2 G(\gamma,0) - 4\gamma(1-S)G(\gamma,1)$$
$$+6\gamma(\gamma-1)(1-\frac{1}{3}S)G(\gamma,2) - 4\gamma(\gamma-1)(\gamma-2)G(\gamma,3)$$
$$+\gamma(\gamma-1)(\gamma-2)(\gamma-3)G(\gamma,4), \quad (\gamma \notin Z^-) \tag{2}$$

(iii)

$$\sum_{k=0}^{\infty} \frac{[\frac{1}{3}]_k \Gamma(2k - \frac{2}{3} + \gamma)}{k!\Gamma(2k - \frac{2}{3})} S^k = \frac{2}{3}\{-L(\gamma,1) + (\gamma-1)L(\gamma,2)\}. \quad (\gamma \notin Z^-) \tag{3}$$

Proof. From Theorems 2 and 3 and 4 we can obtain above equations directly.

4 A Special Case

In order to make sure of the formulations of Theorem 1, 2 , 3 and 4, we consider the case of the integer $\gamma = 1$.
From Theorem 1, in case of $\gamma = 1$ the equation becomes

$$(f)_1 = e^{-i\pi}(z-b)^{-\frac{1}{3}} \sum_{k=0}^{\infty} \frac{[-\frac{1}{3}]_k \Gamma(2k - \frac{2}{3} + 1)}{k!\Gamma(2k - \frac{2}{3})} S^k$$

$$= e^{-i\pi}(z-b)^{-\frac{1}{3}}\{2\sum_{k=0}^{\infty}\frac{[-\frac{1}{3}]_k k}{k!}S^k - \frac{2}{3}\sum_{k=0}^{\infty}\frac{[-\frac{1}{3}]_k k}{k!}S^k\}$$
$$= e^{-i\pi}(z-b)^{-\frac{1}{3}}\{2S(-\frac{1}{3})\sum_{k=0}^{\infty}\frac{[\frac{2}{3}]_k}{k!}S^k - \frac{2}{3}\sum_{k=0}^{\infty}\frac{[-\frac{1}{3}]_k k}{k!}S^k\}$$
$$= e^{-i\pi}(z-b)^{-\frac{1}{3}}\{2S(-\frac{1}{3})(1-S)^{-\frac{2}{3}} - \frac{2}{3}(1-S)^{\frac{1}{3}}\}$$
$$= (-1)(z-b)^{-\frac{1}{3}}(-\frac{2}{3})(1-S)^{-\frac{2}{3}}$$
$$= \frac{2}{3}(z-b)^{-\frac{1}{3}}(1-S)^{-\frac{2}{3}} \tag{1}$$

When $\gamma = 1$, from Theorem 2 , we have

$$(f)_1 = e^{-i\pi}(z-b)^{-\frac{1}{3}}\{(1-S)J(1,0) - 2J(1,1)\}$$
$$= (-1)(z-b)^{-\frac{1}{3}}\{(1-S)\sum_{k=0}^{\infty}\frac{[\frac{2}{3}]_k\Gamma(2k+\frac{4}{3}+1)}{k!\Gamma(2k+\frac{4}{3})}S^k$$
$$-2\sum_{k=0}^{\infty}\frac{[\frac{2}{3}]_k\Gamma(2k+\frac{4}{3})}{k!\Gamma(2k+\frac{4}{3})}S^k\} \tag{2}$$

And we notice following relations,

$$\sum_{k=0}^{\infty}\frac{[\lambda]_k}{k!}z^k = (1-z)^{-\lambda} \tag{3}$$

$$\sum_{k=0}^{\infty}\frac{[\lambda]_k k}{k!}T^k = \sum_{k=0}^{\infty}\frac{[\lambda]_k}{(k-1)!}T^k = \sum_{k=0}^{\infty}\frac{[\lambda]_{k+1}}{k!}T^{k+1}$$
$$= \lambda T\sum_{k=0}^{\infty}\frac{[\lambda+1]_k}{k!}T^k = \lambda T(1-T)^{-1-\lambda} \tag{4}$$

$$[\lambda]_{k+1} = \frac{\Gamma(\lambda+1+k)}{\Gamma(\lambda)} = \lambda[\lambda+1]_k \tag{5}$$

Then, we have the following relations with applying to the above euations.

$$(f)_1 = -(z-b)^{-\frac{1}{3}}\{2S(1-S)\sum_{k=0}^{\infty}\frac{[\frac{2}{3}]_{k+1}}{k!}S^k + \frac{4}{3}(1-S)\sum_{k=0}^{\infty}\frac{[\frac{2}{3}]_k}{k!}S^k\} - 2\sum_{k=0}^{\infty}\frac{[\frac{2}{3}]_k}{k!}S^k\}$$
$$= -(z-b)^{-\frac{1}{3}}\{2(1-S)S(\frac{2}{3})\sum_{k=0}^{\infty}\frac{[\frac{5}{3}]_k}{k!}S^k + \frac{4}{3}(1-S)(1-S)^{-\frac{2}{3}} - 2(1-S)^{-\frac{2}{3}}\}$$

$$= -(z-b)^{-\frac{1}{3}}\{\frac{4}{3}(1-S)S(1-S)^{-\frac{5}{3}} + \frac{4}{3}(1-S)(1-S)^{-\frac{2}{3}} - 2(1-S)^{-\frac{2}{3}}\}$$

$$= -(z-b)^{-\frac{1}{3}}(1-S)^{-\frac{2}{3}}(\frac{4}{3}S + \frac{4}{3} - \frac{4}{3}S - 2)$$

$$= \frac{2}{3}(z-b)^{\frac{1}{3}}(1-S)^{-\frac{2}{3}}. \qquad (6)$$

And from Theorem 3, we have

$$(f)_1 = e^{-i\pi}(z-b)^{-\frac{1}{3}}\{(1-S)^2G(1,0) - 4(1-S)G(1,1)\}$$

$$= e^{-i\pi}(z-b)^{-\frac{1}{3}}(1-S)\{(1-S)\sum_{k=0}^{\infty}\frac{[\frac{5}{3}]_k\Gamma(2k+\frac{10}{3}+1)}{k!\Gamma(2k+\frac{10}{3})}S^k$$

$$-4\sum_{k=0}^{\infty}\frac{[\frac{5}{3}]_k\Gamma(2k+\frac{10}{3})}{k!\Gamma(2k+\frac{10}{3})}S^k\}$$

$$= -(z-b)^{-\frac{1}{3}}(1-S)\{2(1-S)S\sum_{k=0}^{\infty}\frac{[\frac{5}{3}]_{k+1}}{k!}S^k$$

$$+\frac{10}{3}(1-S)\sum_{k=0}^{\infty}\frac{[\frac{5}{3}]_k}{k!}S^k - 4\sum_{k=0}^{\infty}\frac{[\frac{5}{3}]_k}{k!}S^k\}$$

$$= -(z-b)^{-\frac{1}{3}}(1-S)\{\frac{10}{3}S(1-S)^{\frac{5}{3}} + \frac{10}{3}(1-S)^{-\frac{2}{3}} - 4(1-S)^{-\frac{5}{3}}\}$$

$$= -(z-b)^{-\frac{1}{3}}(1-S)^{-\frac{2}{3}}(\frac{10}{3}S - 4 + \frac{10}{3} - \frac{10}{3}S)$$

$$= \frac{2}{3}(z-b)^{-\frac{1}{3}}(1-S)^{-\frac{2}{3}}. \qquad (7)$$

Next, frpm Theorem 4 we have

$$(f)_1 = \frac{2}{3}(-1)(z-b)^{-\frac{4}{3}+1}\{-L(1,1)\}$$

$$= \frac{2}{3}(z-b)^{-\frac{1}{3}}L(1,1)$$

$$= \frac{2}{3}(z-b)^{-\frac{1}{3}}\sum_{k=0}^{\infty}\frac{[\frac{2}{3}]_k\Gamma(2k+\frac{4}{3})}{k!\Gamma(2k+\frac{4}{3})}\left(\frac{c}{(z-b)^2}\right)^k$$

$$= \frac{2}{3}(z-b)^{-\frac{1}{3}}\sum_{k=0}^{\infty}\frac{[\frac{2}{3}]_k}{k!}\left(\frac{c}{(z-b)^2}\right)^k$$

$$= \frac{2}{3}(z-b)^{-\frac{1}{3}}\left(1-\frac{c}{(z-b)^2}\right)^{-\frac{2}{3}}$$

$$= \frac{2}{3}(z-b)\left((z-b)^2-c\right)^{-\frac{2}{3}} \qquad (8)$$

Therefore we have the same results from four different forms of N-fractional calculus for the function $((z-b)^2-c)^{\frac{1}{3}}$.

Now these results are consistent with the one of the classical calculus of

$$\frac{d}{dz}((z-b)^2-c)^{\frac{1}{3}}. \tag{9}$$

Here we confirm again the result for Theorem 1.

When $\gamma=1$, from Theorem 1.(2), we have

$$\begin{aligned}
\left(((z-b)^2-c)^{\frac{1}{3}}\right)_1 &= -(z-b)^{-\frac{1}{3}}\sum_{k=0}^{\infty}\frac{[-\frac{1}{3}]_k\Gamma(2k-\frac{2}{3}+\gamma)}{k!\Gamma(2k-\frac{2}{3})}S^k \\
&= -(z-b)^{-\frac{1}{3}}\{2\sum_{k=0}^{\infty}\frac{[-\frac{1}{3}]_k k}{k!}S^k-\frac{2}{3}\sum_{k=0}^{\infty}\frac{[-\frac{1}{3}]_k}{k!}S^k\} \\
&= -(z-b)^{-\frac{1}{3}}\{2S(-\frac{1}{3})\sum_{k=0}^{\infty}\frac{[\frac{2}{3}]_k}{k!}S^k-\frac{2}{3}\sum_{k=0}^{\infty}\frac{[-\frac{1}{3}]_k}{k!}S^k\} \\
&= -(z-b)^{-\frac{1}{3}}\{-\frac{2}{3}S(1-S)^{-\frac{2}{3}}-\frac{2}{3}(1-S)^{\frac{1}{3}}\} \\
&= -(z-b)^{-\frac{1}{3}}(1-S)^{-\frac{2}{3}}\{-\frac{2}{3}S-\frac{2}{3}(1-S)\} \\
&= -(z-b)^{-\frac{1}{3}}(-\frac{2}{3})(1-S)^{-\frac{2}{3}} \\
&= \frac{2}{3}(z-b)^{-\frac{1}{3}}(1-S)^{-\frac{2}{3}}
\end{aligned} \tag{10}$$

We have

$$\begin{aligned}
\frac{2}{3}(z-b)^{-\frac{1}{3}}(1-S)^{-\frac{2}{3}} &= \frac{2}{3}(z-b)^{-\frac{1}{3}}(\frac{(z-b)^2-c)}{(z-b)^2})^{-\frac{2}{3}} \\
&= \frac{2}{3}(z-b)((z-b)^2-c)^{-\frac{2}{3}}.
\end{aligned} \tag{11}$$

This result also coincides with the one cbtained by the classical calculus.

So we conclude that according to the definition of fractional differintegration , we have three forms for γ-th differintegrate of the function $((z-b)^2-c)^{\frac{1}{3}}$ by Theorems 1 , 2 , 3 and 4.

We made sure that they have the same results as the classical result when the differential order is in the case of $\gamma=1$.

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