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The Solutions to The Homogeneous Bessel Equations by Means of The N- Fractional Calculus (The Calculus in The 21 th Century) (Again)

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Abstract

In a previous article of the author,. the solutions to the homogeneous Bessel equations are discussed by means of the our N-fractional calculus, omitting the additional arbitrary constants of the integrations.

In this article, the solutions that contain the arbitrary constants of the integrations are discussed by means of our N-Fractional calculus again.

Some ones of them are shown as follows, for example.

$$\varphi_{[1](K.M)} = z^{\nu}e^{iz}\{e^{K}(e^{-i2z}\cdot z^{-(\nu+1/2)})_{\nu-1/2} + M\}$$

(fractional differintegrated form)

$$= e^{K}(-i2)^{\nu-1/2}z^{-1/2}e^{-iz}\ {}_2F_0(1/2-\nu, 1/2+\nu\ ;\ \frac{i}{2z}) + M z^{\nu}e^{iz} \qquad (|i/2z|<1)$$

$$= A\cdot H_{\nu}^{(2)}(z) + Mz^{\nu}e^{iz} \qquad (A = \sqrt{\pi}\cdot 2^{\nu-1}e^{-i\pi\nu}\cdot e^{K}) \qquad (1)$$

and

$$\varphi_{[6](K.M)} = z^{-\nu}e^{iz}\{e^{K}(z^{\nu-1/2}\cdot e^{-i2z})_{-(\nu+1/2)} + M\}$$

(fractional differintegrated form)

$$= e^{K}e^{i\pi(\nu+1/2)}\Gamma(-2\nu)z^{\nu}e^{-iz}\ {}_1F_1(1/2+\nu; 1+2\nu\ ;\ i2z) + M z^{-\nu}e^{iz} \qquad (|i2z|<1)$$

$$= A\cdot H_{\nu}^{(2)}(z) + Mz^{-\nu}e^{iz} \qquad (A = \sqrt{\pi}\cdot 2^{\nu-1}e^{-i\pi\nu}\cdot e^{K}) \qquad (2)$$

$$(|\Gamma(-2\nu-k)/\Gamma(-\nu+1/2)|<\infty)$$

$$= B^{*}\cdot J_{\nu}^{(2)}(z) + Mz^{-\nu}e^{iz} \qquad (B^{*} = 2^{\nu}\Gamma(-2\nu)\Gamma(1+\nu)e^{i\pi(\nu+1/2)}\cdot e^{K})$$

where K and M are the additional arbitrary constants of the integration,

${}_pF_q(\cdots\cdots)$ is the generalized Gauss Hypergeometric function,

$H_{\nu}^{(2)}(z)$ is the Hankel function and

$J_{\nu}^{(2)}(z) = e^{-iz}\dfrac{(z/2)^{\nu}}{\Gamma(1+\nu)}\ {}_1F_1(1/2+\nu\ ; 1+2\nu\ ; i2z) = J_{\nu}(z)$ is the first kind Bessel function.

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1).

Let $D=\{D_-, D_+\}$, $C=\{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty+i\,\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty+i\,\mathrm{Im}(z)$,

D_- be a domain surrounded by C_-; D_+ be a domain surrounded by C_+.

(Here D contains the points over the curve C.)

Moreover, let $f=f(z)$ be a regular function in D ($z\in D$),

$$f_\nu=(f)_\nu={}_C(f)_\nu=\frac{\Gamma(\nu+1)}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z)^{\nu+1}}d\zeta \qquad (\nu\notin Z^-), \tag{1}$$

$$(f)_{-m}=\lim_{\nu\to -m}(f)_\nu \qquad (m\in Z^+), \tag{2}$$

where $\quad -\pi\le\arg(\zeta-z)\le\pi$ for C_-, $\quad 0\le\arg(\zeta-z)\le 2\pi$ for C_+,

$\zeta\ne z$, $z\in C$, $\nu\in R$, Γ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu>0$, and integrals of order $-\nu$ for $\nu<0$), with respect to z, of the function f, if $|(f)_\nu|<\infty$.

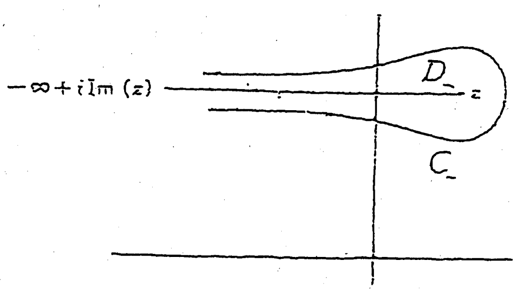

Fig. 1.

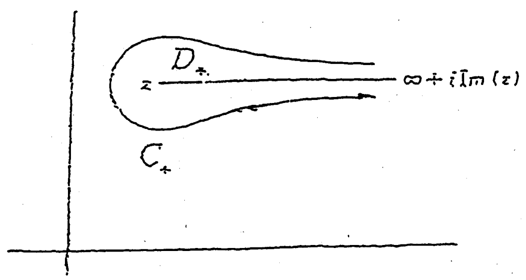

Fig. 2.

Notice that (1) is reduced to Goursat's integral for $\nu=n$ ($\in Z^+$) and is reduced to the famous Cauchy's integral for $\nu=0$. That is, (1) is an extention of Cauchy's integral and of Goursat's one, conversely Cauchy's and Goursat's ones are special cases of (1).

Moreover, notice that (1) is the representation which unifies the derivatives and integrations.

(II) On the fractional calculus operator N^{ν} [3]

Theorem A. *Let fractional calculus operator (Nishimoto's Operator) N^{ν} be*

$$N^{\nu} = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\zeta}{(\zeta - z)^{\nu+1}} \right) \qquad (\nu \notin Z^{-}), \quad [\textit{Refer to}\,(1)] \tag{3}$$

with

$$N^{-m} = \lim_{\nu \to -m} N^{\nu} \qquad (m \in Z^{+}), \tag{4}$$

and define the binary operation $\circ$ *as*

$$N^{\beta} \circ N^{\alpha} f = N^{\beta} N^{\alpha} f = N^{\beta}(N^{\alpha} f) \qquad (\alpha, \beta \in R), \tag{5}$$

then the set

$$\{N^{\nu}\} = \{N^{\nu} \mid \nu \in R\} \tag{6}$$

is an Abelian product group (having continuous index ν *) which has the inverse transform operator* $(N^{\nu})^{-1} = N^{-\nu}$ *to the fractional calculus operator* N^{ν} *, for the function* f *such that* $f \in F = \{f\,;\ 0 \neq |f_{\nu}| < \infty,\ \nu \in R\}$, *where* $f = f(z)$ *and* $z \in C$.
(*vis.* $-\infty < \nu < \infty$).

(For our convenience, we call $N^{\beta} \circ N^{\alpha}$ as product of N^{β} and N^{α}.)

Theorem B. " F.O.G. $\{N^{\nu}\}$ " *is an " Action product group which has continuous index* ν " *for the set of* F . (*F.O.G.* ; Fractional calculus operator group) [3]

Theorem C. *Let*

$$S := \{\pm N^{\nu}\} \cup \{0\} = \{N^{\nu}\} \cup \{-N^{\nu}\} \cup \{0\} \quad (\nu \in R) . \tag{7}$$

Then the set S is a commutative ring for the function $f \in F$, *when the identity*

$$N^{\alpha} + N^{\beta} = N^{\gamma} \qquad (N^{\alpha}, N^{\beta}, N^{\gamma} \in S) \,; \tag{8}$$

holds. [5]

(III). **Lemma.** We have [1]

(i) $$((z-c)^{b})_{\alpha} = e^{-i\pi\alpha} \frac{\Gamma(\alpha - b)}{\Gamma(-b)} (z-c)^{b-\alpha} \qquad \left(\left| \frac{\Gamma(\alpha - b)}{\Gamma(-b)} \right| < \infty \right),$$

(ii) $$(\log(z-c))_{\alpha} = -e^{-i\pi\alpha} \Gamma(\alpha) (z-c)^{-\alpha} \qquad (|\Gamma(\alpha)| < \infty),$$

(iii) $$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c) \qquad (|\Gamma(\alpha)| < \infty),$$

where $z - c \neq 0$ for (i) and $z - c \neq 0, 1$ for (ii), (iii) ,

(iv) $$(u \cdot v)_{\alpha} := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)} u_{\alpha-k} v_{k} \qquad \left(\begin{array}{l} u = u(z), \\ v = v(z) \end{array} \right).$$

§ 1. Preliminary

(I) The theorem below is reported by the author already (cf. J.F C, Vol. 27, May (2005), 83 - 88.). [31]

Theorem D. *Let*

$$P = P(\alpha,\beta,\gamma) := \frac{\sin\pi\alpha \cdot \sin\pi(\gamma-\alpha-\beta)}{\sin\pi(\alpha+\beta)\cdot\sin\pi(\gamma-\alpha)} \quad (\,|P(\alpha,\beta,\gamma)| = M < \infty) \qquad (1)$$

and

$$Q = Q(\alpha,\beta,\gamma) := P(\beta,\alpha,\gamma); \qquad (\,|P(\beta,\alpha,\gamma)| = M < \infty) \qquad (2)$$

When $\alpha,\beta,\gamma \notin Z_0^+$, *we have* ;

(i) $$((z-c)^\alpha \cdot (z-c)^\beta)_\gamma = e^{-i\pi\gamma} P(\alpha,\beta,\gamma) \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}(z-c)^{\alpha+\beta-\gamma}, \qquad (3)$$

$$(\,\mathrm{Re}(\alpha+\beta+1) > 0, \quad (1+\alpha-\gamma) \notin Z_0^- \,),$$

(ii) $$((z-c)^\beta \cdot (z-c)^\alpha)_\gamma = e^{-i\pi\gamma} Q(\alpha,\beta,\gamma) \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}(z-c)^{\alpha+\beta-\gamma}, \qquad (4)$$

$$(\,\mathrm{Re}(\alpha+\beta+1) > 0, \quad (1+\beta-\gamma) \notin Z_0^- \,)$$

(iii) $$((z-c)^{\alpha+\beta})_\gamma = e^{-i\pi\gamma} \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}(z-c)^{\alpha+\beta-\gamma}, \qquad (5)$$

where

$$z-c \neq 0, \quad \left|\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\right| < \infty .$$

Then the inequalities below are established from this theorem.

Corollary 1. *We have the inequalities*

(i) $$((z-c)^\alpha \cdot (z-c)^\beta)_\gamma \neq ((z-c)^\beta \cdot (z-c)^\alpha)_\gamma, \qquad (6)$$

and

(ii) $$((z-c)^\alpha \cdot (z-c)^\beta)_\gamma \neq ((z-c)^{\alpha+\beta})_\gamma, \qquad (7)$$

where

$$\alpha,\beta,\gamma \notin Z_0^+, \quad \alpha \neq \beta, \quad z-c \neq 0.$$

Corollary 2.

(i) *When* $\alpha, \beta, \gamma \notin Z_0^+$, *and*

$$P(\alpha, \beta, \gamma) = Q(\beta, \alpha, \gamma) = 1, \qquad (8)$$

we have

$$((z-c)^{\alpha} \cdot (z-c)^{\beta})_{\gamma} = ((z-c)^{\beta} \cdot (z-c)^{\alpha})_{\gamma} = ((z-c)^{\alpha+\beta})_{\gamma}, \qquad (9)$$

$$(\mathrm{Re}(\alpha+\beta+1) > 0, \ (1+\alpha-\gamma) \notin Z_0^-, \ (1+\beta-\gamma) \notin Z_0^-).$$

(ii) *When* $\gamma = m \in Z_0^+$, *we have* ;

$$((z-c)^{\alpha} \cdot (z-c)^{\beta})_m = ((z-c)^{\beta} \cdot (z-c)^{\alpha})_m = ((z-c)^{\alpha+\beta})_m. \qquad (10)$$

(II) The Teorem below is reported by the author already (cf. J. Frac. Calc. Vol. 29, May (2006),pp.35 - 44.) . [7]

Theorem E. *We have*

(i)
$$\left(((z-b)^{\beta} - c)^{\alpha}\right)_{\gamma} = e^{-i\pi\gamma}(z-b)^{\alpha\beta-\gamma} \times \sum_{k=0}^{\infty} \frac{[-\alpha]_k \Gamma(\beta k - \alpha\beta + \gamma)}{k! \Gamma(\beta k - \alpha\beta)} \left(\frac{c}{(z-b)^{\beta}}\right)^k \qquad (11)$$

$$\left(\left| \frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)} \right| < \infty \right)$$

and

(ii)
$$\left(((z-b)^{\beta} - c)^{\alpha}\right)_n = (-1)^n (z-b)^{\alpha\beta-n} \times \sum_{k=0}^{\infty} \frac{[-\alpha]_k [\beta k - \alpha\beta]_n}{k!} \left(\frac{c}{(z-b)^{\beta}}\right)^k \quad (n \in Z_0^+) \qquad (12)$$

where

$$\left| \frac{c}{(z-b)^{\beta}} \right| < 1,$$

and

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda) \ \textit{with} \ [\lambda]_0 = 1 ;$$

(Notation of Pochhammer).

§2. The Solutions to The Homogeneous Bessel Equations by Means of The N-Fractional Calculus (Calculus in The 21 th Century)

Theorem 1 - 1. *Let be $\varphi = \varphi(z) \in F$, then the homogeneous Bessel equation*

$$L[\varphi;z;\nu] = \varphi_2 \cdot z^2 + \varphi_1 \cdot z + \varphi \cdot (z^2 - \nu^2) = 0 \quad (z \neq 0) \tag{1}$$

$$(\varphi_\alpha = d^\alpha \varphi / dz^\alpha \text{ for } \alpha > 0,\ \varphi_0 = \varphi = \varphi(z)).$$

has the solutions of the forms (fractional differintegrated forms)

Group I.

(i) $$\varphi = z^\nu e^{iz}\{e^K(e^{-i2z} \cdot z^{-(\nu+1/2)})_{\nu-1/2} + M\} \equiv \varphi_{[1](K,M)} \quad \text{(denote)} \tag{2}$$

(ii) $$\varphi = z^\nu e^{iz}\{e^K(z^{-(\nu+1/2)} \cdot e^{-i2z})_{\nu-1/2} + M\} \equiv \varphi_{[2](K,M)} \tag{3}$$

(iii) $$\varphi = z^\nu e^{-iz}\{e^K(e^{i2z} \cdot z^{-(\nu+1/2)})_{\nu-1/2} + M\} \equiv \varphi_{[3](K,M)} \tag{4}$$

(iv) $$\varphi = z^\nu e^{-iz}\{e^K(z^{-(\nu+1/2)} \cdot e^{i2z})_{\nu-1/2} + M\} \equiv \varphi_{[4](K,M)} \tag{5}$$

Group II.

(i) $$\varphi = z^{-\nu} e^{iz}\{e^K(e^{-i2z} \cdot z^{\nu-1/2})_{-(\nu+1/2)} + M\} \equiv \varphi_{[5](K,M)} \tag{6}$$

(ii) $$\varphi = z^{-\nu} e^{iz}\{e^K(z^{\nu-1/2} \cdot e^{-i2z})_{-(\nu+1/2)} + M\} \equiv \varphi_{[6](K,M)} \tag{7}$$

(iii) $$\varphi = z^{-\nu} e^{-iz}\{e^K(e^{i2z} \cdot z^{\nu-1/2})_{-(\nu+1/2)} + M\} \equiv \varphi_{[7](K,M)} \tag{8}$$

(iv) $$\varphi = z^{-\nu} e^{-iz}\{e^K(z^{\nu-1/2} \cdot e^{i2z})_{-(\nu+1/2)} + M\} \equiv \varphi_{[8](K,M)} \tag{9}$$

where K and M are the additional arbitrary constants of the integrations.

Proof.

Set

$$\varphi = z^\lambda \phi, \qquad \phi = \phi(z). \tag{10}$$

We have then

$$\varphi_1 = \lambda z^{\lambda-1}\phi + z^\lambda \phi_1 \tag{11}$$

and

$$\varphi_2 = \lambda(\lambda-1)z^{\lambda-2}\phi + 2\lambda z^{\lambda-1}\phi_1 + z^\lambda \phi_2 \tag{12}$$

Therefore, we obtain

$$\phi_2 \cdot z + \phi_1 \cdot (2\lambda + 1) + \phi \cdot (z + \frac{\lambda^2 - \nu^2}{z}) = 0 \tag{13}$$

from (1), applying (10), (11) and (12).

Choose λ such that $\lambda^2 - \nu^2 = 0$, we have then

$$\lambda = \nu, \quad -\nu. \tag{14}$$

(I) Case $\lambda = \nu$;

In this case we have

$$\varphi = z^{\nu}\phi \tag{15}$$

from (10) and hence

$$\phi_2 \cdot z + \phi_1 \cdot (2\nu + 1) + \phi \cdot z = 0 \tag{16}$$

from (13).

Next set

$$\phi = e^{\alpha z}u \qquad (u = u(z)), \tag{17}$$

we have then

$$\phi_1 = \alpha e^{\alpha z}u + e^{\alpha z}u_1 \tag{18}$$

and

$$\phi_2 = \alpha^2 e^{\alpha z}u + 2\alpha e^{\alpha z}u_1 + e^{\alpha z}u_2 . \tag{19}$$

Therefore, we obtain

$$u_2 \cdot z + u_1 \cdot (2\alpha z + 2\nu + 1) + u \cdot \{(\alpha^2 + 1)z + \alpha(2\nu + 1)\} = 0 \tag{20}$$

from (16), applying (17), (18) and (19).

Choose α such that

$$\alpha^2 + 1 = 0, \tag{21}$$

we have then

$$\alpha = i, \quad -i . \tag{22}$$

(i) Case $\alpha = i$;

In this case we have

$$\phi = e^{iz}u \tag{23}$$

from (17) and hence

$$u_2 \cdot z + u_1 \cdot (2iz + 2\nu + 1) + u \cdot i(2\nu + 1) = 0 \tag{24}$$

from (20).

Operate N-fractional calculus operator (NFCO) N^{γ} to the both sides of equation (24), we have then

$$(u_2 \cdot z)_\gamma + (u_1 \cdot (2iz + 2\nu + 1))_\gamma + u_\gamma \cdot i(2\nu + 1) = 0, \quad (\gamma \notin Z^-). \tag{25}$$

Now we have

$$(u_2 \cdot z)_\gamma = \sum_{k=0}^{1} \frac{\Gamma(\gamma + 1)}{k!\Gamma(\gamma + 1 - k)} (u_2)_{\gamma - k}(z)_k = u_{2+\gamma}z + \gamma\, u_{1+\gamma} \tag{26}$$

and

$$(u_1 \cdot (2iz + 2\nu + 1))_\gamma = u_{1+\gamma} \cdot (2iz + 2\nu + 1) + \gamma u_\gamma\, i2 . \tag{27}$$

Hence we obtain

$$u_{2+\gamma}\cdot z+u_{1+\gamma}\cdot(2iz+2\nu+1+\gamma)+u_\gamma\cdot i(2\gamma+2\nu+1)=0 \tag{28}$$

from (25), applying (26) and (27).

Choose γ such that

$$2\gamma+2\nu+1=0$$

that is,

$$\gamma=-(\nu+1/2)\ . \tag{29}$$

we have then

$$u_{3/2-\nu}\cdot z+u_{1/2-\nu}\cdot(2iz+\nu+1/2)=0 \tag{30}$$

from (28), using (29).

Set

$$u_{1/2-\nu}=w \tag{31}$$

we have then

$$w_1+w\cdot(2i+\frac{\nu+1/2}{z})=0 \tag{32}$$

from (30). The solution to this variable separable form equation is given by

$$w=e^K\cdot e^{-i2z}z^{-(\nu+1/2)} \tag{33}$$

where K is the additional arbitrary constant of the integration. (See the **Note 1.**)

Therefore, we obtain

$$u=w_{\nu-1/2}=e^K(e^{-i2z}\cdot z^{-(\nu+1/2)})_{\nu-1/2}+M\equiv u_{[1]} \tag{34}$$

where M is an additional arbitrary constant of the integration again such that

$$M_{1/2-\nu}=0\ . \tag{35}$$

Next we obtain

$$u=w_{\nu-1/2}=e^K(z^{-(\nu+1/2)}\cdot e^{-i2z})_{\nu-1/2}+M\equiv u_{[2]}\ , \tag{36}$$

changing the order

e^{-i2z} and $z^{-(\nu+1/2)}$ in the parenthesis $(\ \cdot\ )_{\nu-1/2}$ in (34).

Notice that when $(\nu-1/2)\in Z_0^+$,

(34) and (36) overlap each other.

We have then

$$\phi = e^{iz}u_{[1]} = e^{iz}\{e^{K}(e^{-i2z}\cdot z^{-(\nu+1/2)})_{\nu-1/2} + M\} \equiv \phi_{[1]} \tag{37}$$

and

$$\phi = e^{iz}u_{[2]} = e^{iz}\{e^{K}(z^{-(\nu+1/2)}\cdot e^{-i2z})_{\nu-1/2} + M\} \equiv \phi_{[2]} \tag{38}$$

from (23), applying (34) and (36), respectively.

Therefore, we obtain

$$\varphi = z^{\nu}\phi_{[1]} = z^{\nu}e^{iz}\{e^{K}(e^{-i2z}\cdot z^{-(\nu+1/2)})_{\nu-1/2} + M\} \equiv \varphi_{[1]} \tag{2}$$

and

$$\varphi = z^{\nu}\phi_{[2]} = z^{\nu}e^{iz}\{e^{K}(z^{-(\nu+1/2)}\cdot e^{-i2z})_{\nu-1/2} + M\} \equiv \varphi_{[2]} \tag{3}$$

from (15), applying (37) and (38), respectively.

(ii) Case $\alpha = -i$;

Set $-i$ instead of i in (2) and (3), we have then

$$\varphi = z^{\nu}e^{-iz}\{e^{K}(e^{i2z}\cdot z^{-(\nu+1/2)})_{\nu-1/2} + M\} \equiv \varphi_{[3]} \tag{4}$$

and

$$\varphi = z^{\nu}e^{-iz}\{e^{K}(z^{-(\nu+1/2)}\cdot e^{i2z})_{\nu-1/2} + M\} \equiv \varphi_{[4]} \tag{5}$$

frespectively.

(II) Case $\lambda = -\nu$;

Set $-\nu$ instead of ν in $\varphi_{[1]} \sim \varphi_{[4]}$, we obtain

$$\varphi = z^{-\nu}e^{iz}\{e^{K}(e^{-i2z}\cdot z^{\nu-1/2})_{-(\nu+1/2)} + M\} \equiv \varphi_{[5](K,M)} \tag{6}$$

$$\varphi = z^{-\nu}e^{iz}\{e^{K}(z^{\nu-1/2)}\cdot e^{-i2z})_{-(\nu+1/2)} + M\} \equiv \varphi_{[6](K,M)} \tag{7}$$

$$\varphi = z^{-\nu}e^{-iz}\{e^{K}(e^{i2z}\cdot z^{\nu-1/2})_{-(\nu+1/2)} + M\} \equiv \varphi_{[7](K,M)} \tag{8}$$

and

$$\varphi = z^{-\nu}e^{-iz}\{e^{K}(z^{\nu-1/2}\cdot e^{i2z})_{-(\nu+1/2)} + M\} \equiv \varphi_{[8](K,M)} \tag{9}$$

respectively.

Note 1. We have

$$\frac{w_1}{w} = -\left(i2 + \frac{\nu+1/2}{z}\right) \tag{39}$$

from (32). The solution to this variable separable form equation is given by

$$\log w = -\{i2z + (\nu+1/2)\log z\} + K\log e \tag{40}$$

$$= -\{i2z\ \log e + (\nu+1/2)\log z\} + \log e^{k} \tag{41}$$

Hence

$$w = e^{K}\cdot e^{-i2z}z^{-(\nu+1/2)} \quad . \tag{33}$$

§3. The Familiar Forms of The Solutions in Section 2

Theorem 1 - 2. We have the familiar form solutions from the ones in Theorem 1 - 1, as follows ;

Group I.

(i) $\varphi_{[1](K,M)} = e^K(-i2)^{\nu-1/2}z^{-1/2}e^{-iz}\ {}_2F_0(1/2-\nu, 1/2+\nu\ ;\ i/2z) + Mz^{\nu}e^{iz}$ (1)

$= A\cdot H_{\nu}^{(2)}(z) + Mz^{\nu}e^{iz}$ $(A = e^K\sqrt{\pi}\ 2^{\nu-1}e^{-i\pi\nu})$, $(|i/2z|<1)$ (1)'

(ii) $\varphi_{[2](K,M)} = e^K e^{-i\pi(\nu-1/2)}\Gamma(2\nu)z^{-\nu}e^{-iz}\ {}_1F_1(1/2-\nu\ ; 1-2\nu\ ;\ i2z) + Mz^{\nu}e^{iz}$ (2)

$= B\cdot J_{-\nu}^{(2)}(z) + Mz^{\nu}e^{iz}$ $(B = e^K\ 2^{-\nu}\Gamma(2\nu)\Gamma(1-\nu)e^{-i\pi(\nu-1/2)})$ (2)'

$(|\Gamma(2\nu-k)/\Gamma(\nu+1/2)|<\infty)$, $(|i2z|<1)$

(iii) $\varphi_{[3](K,M)} = e^K(i2)^{\nu-1/2}z^{-1/2}e^{iz}\ {}_2F_0(1/2-\nu, 1/2+\nu\ ;\ 1/i2z) + Mz^{\nu}e^{-iz}$ (3)

$= C\cdot H_{\nu}^{(1)}(z) + Mz^{\nu}e^{-iz}$ $(C = e^K\sqrt{\pi}\ 2^{\nu-1}e^{i\pi\nu})$, $(|1/i2z|<1)$ (3)'

(iv) $\varphi_{[4](K,M)} = e^K e^{i\pi(\nu-1/2)}\Gamma(2\nu)z^{-\nu}e^{iz}\ {}_1F_1(1/2-\nu\ ; 1-2\nu\ ;\ -i2z) + Mz^{\nu}e^{-iz}$ (4)

$= D\cdot J_{-\nu}^{(1)}(z) + Mz^{\nu}e^{-iz}$ $(D = e^K\ 2^{-\nu}\Gamma(2\nu)\Gamma(1-\nu)e^{i\pi(\nu-1/2)})$ (4)'

$(|\Gamma(2\nu-k)/\Gamma(\nu+1/2)|<\infty)$, $(|i2z|<1)$,

Group II.

(i) $\varphi_{[5](K,M)} = e^K(-i2)^{-(\nu+1/2)}z^{-1/2}e^{-iz}\ {}_2F_0(1/2-\nu, 1/2+\nu\ ;\ i/2z) + Mz^{-\nu}e^{iz}$ (5)

$= A^*\cdot H_{-\nu}^{(2)}(z) + Mz^{-\nu}e^{iz}$ $(A^* = e^K\sqrt{\pi}\ 2^{-(\nu+1)}e^{i\pi\nu})$, $(|i/2z|<1)$ (5)'

(ii) $\varphi_{[6](K,M)} = e^K e^{i\pi(\nu+1/2)}\Gamma(-2\nu)z^{\nu}e^{-iz}\ {}_1F_1(1/2+\nu\ ; 1+2\nu\ ;\ i2z) + Mz^{-\nu}e^{iz}$ (6)

$= B^*\cdot J_{\nu}^{(2)}(z) + Mz^{-\nu}e^{iz}$ $(B^* = e^K\ 2^{\nu}\Gamma(-2\nu)\Gamma(1+\nu)e^{i\pi(\nu+1/2)})$ (6)'

$(|\Gamma(-2\nu-k)/\Gamma(-\nu+1/2)|<\infty)$, $(|i2z|<1)$,

(iii) $\varphi_{[7](K,M)} = e^K(i2)^{-(\nu+1/2)}z^{-1/2}e^{iz}\ {}_2F_0(1/2-\nu, 1/2+\nu\ ;\ 1/i2z) + Mz^{-\nu}e^{-iz}$ (7)

$= C^*\cdot H_{-\nu}^{(1)}(z) + Mz^{-\nu}e^{-iz}$ $(C^* = e^K\sqrt{\pi}\ 2^{-(\nu+1)}e^{-i\pi\nu})$, $(|1/i2z|<1)$ (7)'

(iv) $\varphi_{[8](K,M)} = e^K e^{-i\pi(\nu+1/2)}\Gamma(-2\nu)z^{\nu}e^{iz}\ {}_1F_1(1/2+\nu\ ; 1+2\nu\ ;\ -i2z) + Mz^{-\nu}e^{-iz}$ (8)

$= D^*\cdot J_{\nu}^{(1)}(z) + Mz^{-\nu}e^{-iz}$ $(D^* = e^K\ 2^{\nu}\Gamma(-2\nu)\Gamma(1+\nu)e^{-i\pi(\nu+1/2)})$ (8)'

$(|\Gamma(-2\nu-k)/\Gamma(-\nu+1/2)|<\infty)$, $(|i2z|<1)$, *Where* $J_\nu^{(1)}(z)$ *and* $J_\nu^{(2)}(z)$ *are the first kind Bessel functions and* $H_\nu^{(1)}(z)$ *and* $H_\nu^{(2)}(z)$ *are the Hankel functions.* (Refer to the next section)

Proof of Group 1. We have

$$e^K z^\nu e^{iz}(e^{-i2z}\cdot z^{-(\nu+1/2)})_{\nu-1/2)} = e^K z^\nu e^{iz}\sum_{k=0}^{\infty}\frac{\Gamma(1/2+\nu)}{k!\,\Gamma(1/2+\nu-k)}(e^{-i2z})_{\nu-1/2-k}(z^{-(\nu+1/2)})_k \quad (9)$$

(by Lemma (iv))

$$= e^K z^\nu e^{iz}\sum_{k=0}^{\infty}\frac{\Gamma(1/2+\nu)}{k!\,\Gamma(1/2+\nu-k)}(-i2)^{\nu-1/2-k}e^{-i2z}e^{-i\pi k}\frac{\Gamma(\nu+1/2+k)}{\Gamma(\nu+1/2)}z^{-(\nu+1/2+k)} \quad (10)$$

$$= e^K z^{-1/2}e^{-iz}(-i2)^{\nu-1/2}\sum_{k=0}^{\infty}\frac{[1/2-\nu]_k[1/2+\nu]_k}{k!}(-i2z)^{-k} \quad (11)$$

$$= e^K(-i2)^{\nu-1/2}z^{-1/2}e^{-iz}\,{}_2F_0(1/2-\nu,1/2+\nu\,;i/2z) \quad (|i/2z|<1) \quad (12)$$

$$= A\cdot H_\nu^{(2)} \qquad (|i/2z|<1) \quad (13)$$

since

$$\Gamma(\lambda-k) = (-1)^{-k}\frac{\Gamma(\lambda)\Gamma(1-\lambda)}{\Gamma(k+1-\lambda)} = (-1)^{-k}\frac{\Gamma(\lambda)}{[1-\lambda]_k} \quad (k\in Z_0^+), \quad (14)$$

$$(e^{\lambda z})_\gamma = \lambda^\gamma e^{\lambda z}\ , \quad (15)$$

$$(z^\lambda)_\gamma = e^{-i\pi\gamma}\frac{\Gamma(\gamma-\lambda)}{\Gamma(-\lambda)}z^{\lambda-\gamma} \qquad (|\Gamma(\gamma-\lambda)/\Gamma(-\lambda)|<\infty) \quad (16)$$

and

$$[\lambda]_k = \frac{\Gamma(\lambda-k)}{\Gamma(\lambda)} \quad (17)$$

Therefore, we have

$$\varphi_{[1](K,M)} = A\cdot H_\nu^{(2)}(z) + M z^\nu e^{iz}\ . \quad (1)'$$

Next we have

$$e^K z^\nu e^{iz}(z^{-(\nu+1/2)}\cdot e^{-i2z})_{\nu-1/2)}$$

$$= e^K z^\nu e^{iz}\sum_{k=0}^{\infty}\frac{\Gamma(1/2+\nu)}{k!\,\Gamma(1/2+\nu-k)}(z^{-(\nu+1/2)})_{\nu-1/2-k}(e^{-i2z})_k \quad (18)$$

$$= e^{K} z^{\nu} e^{iz} \sum_{k=0}^{\infty} \frac{(-1)^k [1/2-\nu]_k}{k!} e^{-i\pi(\nu-1/2-k)} \frac{\Gamma(2\nu-k)}{\Gamma(\nu+1/2)} z^{-2\nu+k} (-i2)^k e^{-i2z} \qquad (19)$$

$$(|\Gamma(2\nu-k)/\Gamma(\nu+1/2)| < \infty)$$

$$= e^{K} e^{-i\pi(\nu-1/2)} \Gamma(2\nu) z^{-\nu} e^{-iz} \sum_{k=0}^{\infty} \frac{[1/2-\nu]_k}{k![1-2\nu]_k} (i2z)^k \qquad (20)$$

$$= e^{K} e^{-i\pi(\nu-1/2)} \Gamma(2\nu) z^{-\nu} e^{-iz} {}_1F_1(1/2-\nu; 1-2\nu; i2z) \quad (|i2z| < 1) \qquad (12)$$

$$= B \cdot J_{-\nu}^{(2)}(z) \quad \cdot \quad (|i/2z| < 1) \,. \qquad (13)$$

Therefore, we have

$$\varphi_{[2](K,M)} = B \cdot J_{-\nu}^{(2)}(z) + M z^{\nu} e^{iz} \quad . \qquad (2)'$$

Next we have

$$\varphi_{[3](K,M)} = C \cdot H_{\nu}^{(1)}(z) + M z^{\nu} e^{iz} \,, \qquad (3)'$$

and

$$\varphi_{[4](K,M)} = D \cdot J_{-\nu}^{(1)}(z) + M z^{\nu} e^{iz} \,, \qquad (4)'$$

setting

$-i$ instead of i in $\varphi_{[1](K,M)}$ and in $\varphi_{[2](K,M)}$

respectively.

Proof of Group II.

Set $-\nu$ instead of ν in $\varphi_{[1](K,M)} \sim \varphi_{[4](K,M)}$,

we have then $\varphi_{[5](K,M)} \sim \varphi_{[8](K,M)}$, respectively.

§4.The Hankel Function and The First Kind Bessel Function

[I] We have the representations as follows.

$$H_{\nu}^{(1)}(z) \sim (2/\pi)^{1/2} e^{-i\pi(\nu/2+1/4)} z^{-1/2} e^{iz} {}_2F_0(1/2+\nu, 1/2-\nu; 1/2iz) \qquad (1)$$

$$(-\pi < \arg z < 2\pi) \quad (|1/2iz| < 1)$$

and

$$H_{\nu}^{(2)}(z) \sim (2/\pi)^{1/2} e^{i\pi(\nu/2+1/4)} z^{-1/2} e^{-iz} {}_2F_0(1/2-\nu, 1/2+\nu; -1/2iz) \qquad (2)$$

$$(-2\pi < \arg z < \pi) \quad (|-1/2iz| < 1) \,.$$

(cf. A Treaties on the Theory of Bessel function; by G.N. Watson,(1962), p.198; Cambridge).

Where $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are the Hankel function.

However, here we set

$$(2/\pi)^{1/2} e^{-i\pi(\nu/2+1/4)} z^{-1/2} e^{iz} {}_2F_0(1/2+\nu, 1/2-\nu; 1/2iz) \equiv H_\nu^{(1)}(z) \qquad (3)$$

$$(|1/2iz| < 1) \qquad \text{(denote)}$$

and

$$(2/\pi)^{1/2} e^{i\pi(\nu/2+1/4)} z^{-1/2} e^{-iz} {}_2F_0(1/2 - \nu, 1/2+\nu; -1/2iz) \equiv H_\nu^{(2)}(z) \qquad (4)$$

$$(|-1/2iz| < 1)$$

we have then

$$z^{-1/2} e^{iz} {}_2F_0(1/2+\nu, 1/2-\nu; 1/2iz) \equiv \sqrt{\pi}\, 2^{-1/2} e^{i\pi(\nu/2+1/4)} H_\nu^{(1)}(z) \qquad (5)$$

and

$$z^{-1/2} e^{-iz} {}_2F_0(1/2+\nu, 1/2-\nu; -1/2iz) \equiv \sqrt{\pi}\, 2^{-1/2} e^{-i\pi(\nu/2+1/4)} H_\nu^{(2)}(z) \qquad (6)$$

from (3) and (4), respectively.

[II] Next we have

$$J_\nu(z) = e^{iz} \frac{(z/2)^\nu}{\Gamma(1+\nu)} {}_1F_1(1/2+\nu; 1+2\nu; -2iz) \equiv J_\nu^{(1)}(z) \qquad (|-2iz|<1) \qquad (7)$$

and

$$J_\nu(z) = e^{-iz} \frac{(z/2)^\nu}{\Gamma(1+\nu)} {}_1F_1(1/2+\nu; 1+2\nu; 2iz) \equiv J_\nu^{(2)}(z) \qquad (|2iz|<1) \qquad (8)$$

(cf. Volume of Watson; p.191). Where $J_\nu(z)$ is the famouse first kind Bessel function.

Here $J_\nu^{(1)}(z)$ and $J_\nu^{(2)}(z)$ are denoted by the author, for our convenience, refering to the Hankel function.

We have then

$$z^\nu e^{iz} {}_1F_1(1/2+\nu; 1+2\nu; -2iz) = 2^\nu \Gamma(1+\nu) J_\nu^{(1)}(z) \qquad (|-2iz|<1) \qquad (9)$$

and

$$z^\nu e^{-iz} {}_1F_1(1/2+\nu; 1+2\nu; 2iz) = 2^\nu \Gamma(1+\nu) J_\nu^{(2)}(z) \qquad (|2iz|<1) \qquad (10)$$

from (7) and (8), respectively.

Therefore, we have the presentations that are shown in section 3. using (5), (6), (9) and (10), respectively.

§5. Commentary

[I] Set $K = M = 0$, we have then the below respectively.

Theorem 1. *Let be* $\varphi = \varphi(z) \in F$ *,then the homogeneous Bessel equation*

$$L[\varphi;z;\nu] = \varphi_2 \cdot z^2 + \varphi_1 \cdot z + \varphi \cdot (z^2 - \nu^2) = 0 \quad (z \neq 0) \tag{1}$$

$$(\varphi_\alpha = d^\alpha \varphi / dz^\alpha \text{ for } \alpha > 0\,,\ \varphi_0 = \varphi = \varphi(z)\,)$$

has particular solutions of the forms (fractional differintegrated forms)

Group I.

(i) $$\varphi = z^\nu e^{iz}(e^{-i2z} \cdot z^{-(\nu+1/2)})_{\nu-1/2} \equiv \varphi_{[1]} \quad \text{(denote)} \tag{2}$$

(ii) $$\varphi = z^\nu e^{iz}(z^{-(\nu+1/2)} \cdot e^{-i2z})_{\nu-1/2} \equiv \varphi_{[2]} \tag{3}$$

(iii) $$\varphi = z^\nu e^{-iz}(e^{i2z} \cdot z^{-(\nu+1/2)})_{\nu-1/2} \equiv \varphi_{[3]} \tag{4}$$

(iv) $$\varphi = z^\nu e^{-iz}(z^{-(\nu+1/2)} \cdot e^{i2z})_{\nu-1/2} \equiv \varphi_{[4]} \tag{5}$$

Group II.

(i) $$\varphi = z^{-\nu} e^{iz}(e^{-i2z} \cdot z^{\nu-1/2})_{-(\nu+1/2)} \equiv \varphi_{[5]} \tag{6}$$

(ii) $$\varphi = z^{-\nu} e^{iz}(z^{\nu-1/2} \cdot e^{-i2z})_{-(\nu+1/2)} \equiv \varphi_{[6]} \tag{7}$$

(iii) $$\varphi = z^{-\nu} e^{-iz}(e^{i2z} \cdot z^{\nu-1/2})_{-(\nu+1/2)} \equiv \varphi_{[7]} \tag{8}$$

(iv) $$\varphi = z^{-\nu} e^{-iz}(z^{\nu-1/2} \cdot e^{i2z})_{-(\nu+1/2)} \equiv \varphi_{[8]}\ . \tag{9}$$

from **Theorem 1 - 1**, and

Corollary 1. *We have*

Group I.

(i) $$\varphi_{[1]} = (-i2)^{\nu-1/2} z^{-1/2} e^{-iz}\, {}_2F_0(1/2-\nu, 1/2+\nu; i/2z) \quad (|i/2z| < 1) \tag{10}$$

$$= A \cdot H_\nu^{(2)}(z) \quad (A = \sqrt{\pi}\ 2^{\nu-1} e^{-i\pi\nu}) \tag{10'}$$

(ii) $$\varphi_{[2]} = e^{-i\pi(\nu-1/2)} \Gamma(2\nu) z^{-\nu} e^{-iz}\, {}_1F_1(1/2-\nu\,; 1-2\nu; 2iz) \quad (|2iz| < 1) \tag{11}$$

$$(|\Gamma(2\nu - k)/\Gamma(\nu+1/2)| < \infty)$$

$$= B \cdot J_{-\nu}^{(2)}(z) \quad (B = 2^{-\nu}\Gamma(2\nu)\Gamma(1-\nu) e^{-i\pi(\nu-1/2)}) \tag{11'}$$

(iii) $\varphi_{[3]} = (i2)^{\nu-1/2} z^{-1/2} e^{iz} {}_2F_0(1/2-\nu, 1/2+\nu; 1/2iz) \qquad (|1/2iz|<1)$ (12)

$= C \cdot H_\nu^{(1)}(z) \qquad (C = \sqrt{\pi}\ 2^{\nu-1} e^{i\pi\nu})$ (12)'

(iv) $\varphi_{[4]} = e^{i\pi(\nu-1/2)} \Gamma(2\nu) z^{-\nu} e^{iz} {}_1F_1(1/2-\nu; 1-2\nu; -2iz) \qquad (|-2iz|<1)$ (13)

$(|\Gamma(2\nu-k)/\Gamma(\nu+1/2)|<\infty)$

$= D \cdot J_{-\nu}^{(1)}(z) \qquad (D = 2^{-\nu}\Gamma(2\nu)\Gamma(1-\nu)e^{i\pi(\nu-1/2)})$. (13)'

Group II.

(i) $\varphi_{[5]} = (-i2)^{-(\nu+1/2)} z^{-1/2} e^{-iz} {}_2F_0(1/2-\nu, 1/2+\nu; i/2z) \qquad (|i/2z|<1)$ (14)

$= A^* \cdot H_{-\nu}^{(2)}(z) \qquad (A^* = \sqrt{\pi}\ 2^{-\nu-1} e^{i\pi\nu})$ (14)'

(ii) $\varphi_{[6]} = e^{i\pi(\nu+1/2)} \Gamma(-2\nu) z^{\nu} e^{-iz} {}_1F_1(1/2+\nu; 1+2\nu; 2iz) \qquad (|2iz|<1)$ (15)

$(|\Gamma(-2\nu-k)/\Gamma(-\nu+1/2)|<\infty)$

$= B^* \cdot J_\nu^{(2)}(z) \qquad (B^* = 2^{\nu}\Gamma(-2\nu)\Gamma(1+\nu)e^{i\pi(\nu+1/2)})$ (15)'

(iii) $\varphi_{[7]} = (i2)^{-(\nu+1/2)} z^{-1/2} e^{iz} {}_2F_0(1/2-\nu, 1/2+\nu; 1/2iz) \qquad (|1/2iz|<1)$ (16)

$= C^* \cdot H_{-\nu}^{(1)}(z) \qquad (C^* = \sqrt{\pi}\ 2^{-\nu-1} e^{-i\pi\nu})$ (16)'

(iv) $\varphi_{[8]} = e^{-i\pi(\nu+1/2)} \Gamma(-2\nu) z^{\nu} e^{iz} {}_1F_1(1/2+\nu; 1+2\nu; -2iz) \qquad (|-2iz|<1)$ (17)

$(|\Gamma(-2\nu-k)/\Gamma(-\nu+1/2)|<\infty)$

$= D^* \cdot J_\nu^{(1)}(z) \qquad (D^* = 2^{\nu}\Gamma(-2\nu)\Gamma(1+\nu)e^{-i\pi(\nu+1/2)})$. (17)'

from **Theorem 1 - 2.**

[II] In the volume of Prof. K.B. Oldham and J. Spanier, the below is shown. That is,

> As an example of the way differintegration can be used to tackle classical differential equations, we here consider Bessel's equation, which arises in connection with the vibrations of a circular drumhead, as well as in other important physical applications. The modified Bessel equation, which differs only in the sign of the third term, and which arises in a number of diffusion problems, is equally amenable to the approach we here take.
>
> The equation

(10.3.1) $$x^2\frac{d^2w}{dx^2}+x\frac{dw}{dx}+\left[x-\frac{\nu^2}{4}\right]w=0$$

is a form of Bessel's equation. As is the rule for second-order differential equations, its general solution is a combination of two linearly independent functions w_1 and w_2 of x, each of which depends on the parameter ν. The usual method of solving (10.3.1) is via an infinite series approach, but we shall demonstrate how differintegration procedures lead to a ready solution in terms of elementary functions.

We start by making either of the substitutions

$$w = x^{\pm\frac{1}{2}\nu}u,$$

where ν denotes the nonnegative square root of ν^2, so that equation (10.3.1) is transformed to

(10.3.2) $$x\frac{d^2u}{dx^2}+[1\pm\nu]\frac{du}{dx}+u=0.$$

(From p.186 ; The Fractional Calculus (1974) ; by K.B. Oldham and J. Spanier. Academic Press, Inc. London, LTD.)

And the solutions to the equation (10.3.1) above are shown as follows.

$$w_1(\nu,x)=\sqrt{\pi}\,J_{-\nu}(2\sqrt{\pi}) \qquad \text{and} \qquad w_2(\nu,x)=\sqrt{\pi}\,J_{\nu}(2\sqrt{\pi})\ .$$

Note. The equation (10. 3. 1) above is misprinted. The correct form is

$$x^2\frac{d^2w}{dx^2}+x\frac{dw}{dx}+[x^2-\frac{\nu^2}{4}]w=0\ ,$$

[III] **Compare the our method and results with that** of Frobenius and that of Prof. K.B. Oldham and J. Spanier , and that of others.

Our definition of fractional calculus and its application to the so called Special differential equations are the most excellent ones in the field of fractional calculus.

Notice that, in our NFCO-method the homogeneous and nonhomogeneous linear second order ordinary differential equations are reduced to "variable separable form one and to linear first order one" respectively.

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