数理解析研究所講究録
第 1878 巻 2014 年 67-73

Notes on a certain class of analytic functions

Junichi Nishiwaki and Shigeyoshi Owa

Abstract

Let $\mathcal{A}$ be the class of analytic functions $f(z)$ in the open unit disk $\mathbb{U}$. Furthermore, the subclass $\mathcal{B}$ of $\mathcal{A}$ concerned with the class of uniformly convex functions or the class $\mathcal{S}_p$ is defined. By virtue of some properties of uniformly convex functions and the class $\mathcal{S}_p$, an extreme function of the class $\mathcal{B}$ and its power series are considered.

1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. A function $f(z)\in\mathcal{A}$ is said to be in the class of uniformly convex (or starlike) functions denoted by $\mathcal{UCV}$ (or $\mathcal{UST}$) if $f(z)$ is convex (or starlike) in $\mathbb{U}$ and maps every circle or circular arc in $\mathbb{U}$ with center at ζ in $\mathbb{U}$ onto the convex arc (or the starlike arc with respect to $f(\zeta)$). These classes are introduced by Goodman [1] (see also [2]). For the class $\mathcal{UCV}$, it is defined as the one variable characterization by Rønning [4] and [5], that is, a function $f(z)\in\mathcal{A}$ is said to be in the class $\mathcal{UCV}$ if it satisfies

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\left|\frac{zf''(z)}{f'(z)}\right|\qquad(z\in\mathbb{U}).$$

It is independently studied by Ma and Minda [3]. But the one variable characterization for the class $\mathcal{UST}$ is still open. Further, a function $f(z)\in\mathcal{A}$ is said to be the corresponding class denoted by $\mathcal{S}_p$ if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\left|\frac{zf'(z)}{f(z)}-1\right|\qquad(z\in\mathbb{U}).$$

This class $\mathcal{S}_p$ was introduced by Rønning [4]. We easily know that the relation $f(z)\in\mathcal{UCV}$ if and only if $zf'(z)\in\mathcal{S}_p$. In view of these classes, we introduce the subclass $\mathcal{B}$ of $\mathcal{A}$ consisting

2010 Mathematics Subject Classification: Primary 30C45

Keywords and Phrases: Analytic function, unifomly convex function, extreme function, power series.

of all functions $f(z)$ which satisfy

$$\operatorname{Re}\left(\frac{z}{f(z)}\right) > \left|\frac{z}{f(z)} - 1\right| \qquad (z \in \mathbb{U}).$$

We try to derive some properties of functions $f(z)$ belonging to the class $\mathcal{B}$.

Remark 1.1. For $f(z) \in \mathcal{B}$, we write $w(z) = \dfrac{f(z)}{z} = u + iv$, then w lies in the domain which is the part of the complex plane which contains $w = 1$ and is bounded by a kind of teardrop-shape domain such that

$$u^4 - 2u^3 + 2u^2v^2 - 2uv^2 + v^4 + v^2 < 0.$$

Example 1.1. *Let us consider the function $f(z) \in \mathcal{A}$ as given by*

$$f(z) = z + \frac{1}{\sqrt{2}}z^2.$$

Then we easily see that the function $f(z)$ is not univalent. And $\dfrac{f(z)}{z}$ maps $\mathbb{U}$ onto the circular domain which is 1 *as the center and $\dfrac{1}{\sqrt{2}}$ as the radius, that is, $f(z) \in \mathcal{B}$.*

2 An extreme function for the class $\mathcal{B}$

In this section, we would like to exhibit an extreme function of the class $\mathcal{B}$ and its power series. For our results, we need to recall here some properties of the class $\mathcal{S}_p$.

Lemma 2.1. (Rønning [4]). *The extremal function $f(z)$ for the class $\mathcal{S}_p$ is given by*

$$\frac{zf'(z)}{f(z)} = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2.$$

By using the expansion of logarithmic part of $\dfrac{zf'(z)}{f(z)}$ in Lemma 2.1, we get

Lemma 2.2. (Ma and Minda [3]). *The power series of $\dfrac{zf'(z)}{f(z)}$ is following*

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\\ &= 1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2k-1}\right)z^n.\end{aligned}$$

The digamma function $\psi(z+1)$ is defined by

$$\psi(z+1) = \frac{\Gamma'(z+1)}{\Gamma(z+1)} = \psi(z) + \frac{1}{z},$$

where $\Gamma(z)$ is the gamma function defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^t dt.$$

When z is natural number, we obtain

$$\psi(n+1) = \sum_{k=1}^{n} \frac{1}{k} - \gamma \qquad (n \in \mathbb{N}),$$

where γ is Euler's constant and $-\gamma = \psi(1)$.

From Remark 1.1 and Lemma 2.1, we have the first result for the class $\mathcal{B}$.

Theorem 2.1. *The extreme function $f(z)$ for the class $\mathcal{B}$ is given by*

$$f(z) = \frac{z}{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}.$$

Proof. Let us consider the function $\frac{f(z)}{z}$ as given by

$$\frac{f(z)}{z} = \frac{1}{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}.$$

It sufficies to show that $\frac{f(z)}{z}$ maps $\mathbb{U}$ onto the interior of the domain such that

$$u^4 - 2u^3 + 2u^2v^2 - 2uv^2 + v^4 + v^2 < 0,$$

implying that $\frac{f(z)}{z}$ maps the unit circle onto the boundary of the domain. Taking $z = e^{i\theta}$, we obtain that

$$\frac{1}{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2} = \frac{1}{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+e^{i\frac{\theta}{2}}}{1-e^{i\frac{\theta}{2}}}\right)\right)^2}$$

$$= \frac{1}{1 + \frac{2}{\pi^2}\left(\log i - \log\left(\tan\frac{\theta}{4}\right)\right)^2}$$

$$= \frac{1}{\frac{1}{2} + \frac{2}{\pi^2}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^2 - i\frac{2}{\pi}\log\left(\tan\frac{\theta}{4}\right)}$$

$$= \frac{\frac{1}{2} + \frac{2}{\pi^2}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^2}{\frac{1}{4} + \frac{6}{\pi^2}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^2 + \frac{4}{\pi^4}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^4}$$

$$+ i\frac{\frac{2}{\pi}\log\left(\tan\frac{\theta}{4}\right)}{\frac{1}{4} + \frac{6}{\pi^2}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^2 + \frac{4}{\pi^4}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^4}.$$

Writing $\frac{f(z)}{z} = u + iv$, we see that

$$\log\left(\tan\frac{\theta}{4}\right) = \frac{\pi(u \pm \sqrt{u^2 - v^2})}{2v}.$$

Thus we have

$$v = \frac{\frac{2}{\pi}\log\left(\tan\frac{\theta}{4}\right)}{\frac{1}{4} + \frac{6}{\pi^2}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^2 + \frac{4}{\pi^4}\left(\log\left(\tan\frac{\theta}{4}\right)\right)^4}$$

$$= \frac{\frac{2}{\pi}\frac{\pi(u \pm \sqrt{u^2 - v^2})}{2v}}{\frac{1}{4} + \frac{6}{\pi^2}\left(\frac{\pi(u \pm \sqrt{u^2 - v^2})}{2v}\right)^2 + \frac{4}{\pi^4}\left(\frac{\pi(u \pm \sqrt{u^2 - v^2})}{2v}\right)^4}.$$

Therefore, we arrive that

$$u^4 - 2u^3 + 2u^2v^2 - 2uv^2 + v^4 + v^2 = 0.$$

This completes the proof of the theorem. □

Considering the power series of the function $f(z)$ in Theorem 2.1, we derive

Theorem 2.2. *The power series of the extreme function for the class $\mathcal{B}$ is given by*

$$f(z) = \frac{z}{1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}$$

$$= z + \sum_{n=2}^{\infty}\sum_{p=1}^{n-1}(-1)^p\left(\frac{8}{\pi^2}\right)^p \sum_{\sum_{j=1}^{p} m_j = n-1}\left(\prod_{j=1}^{p}\frac{1}{m_j}\sum_{k=1}^{m_j}\frac{1}{2k-1}\right)z^n \qquad (m_j \in \mathbb{N}).$$

Proof. Let us suppose that

$$\frac{f(z)}{z}=\frac{1}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}$$

as the proof of Theorem 2.1. Then from Lemma 2.2, we have

$$\begin{aligned}
\frac{f(z)}{z}&=\frac{1}{1+\frac{8}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2k-1}\right)z^n}\\
&=1-\frac{8}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2k-1}\right)z^n+\left(\frac{8}{\pi^2}\right)^2\left\{\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2k-1}\right)z^n\right\}^2\\
&\quad-\left(\frac{8}{\pi^2}\right)^3\left\{\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2k-1}\right)z^n\right\}^3+\cdots\\
&\quad+(-1)^n\left(\frac{8}{\pi^2}\right)^n\left\{\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2k-1}\right)z^n\right\}^n+\cdots\\
&=1-\frac{8}{\pi^2}\left(\frac{1}{1}\sum_{k=1}^{1}\frac{1}{2k-1}\right)z\\
&\quad+\left\{-\frac{8}{\pi^2}\left(\frac{1}{2}\sum_{k=1}^{2}\frac{1}{2k-1}\right)+\left(\frac{8}{\pi^2}\right)^2\left(\frac{1}{1}\sum_{k=1}^{1}\frac{1}{2k-1}\right)\left(\frac{1}{1}\sum_{k=1}^{1}\frac{1}{2k-1}\right)\right\}z^2\\
&\quad+\left[-\frac{8}{\pi^2}\left(\frac{1}{3}\sum_{k=1}^{3}\frac{1}{2k-1}\right)+\left(\frac{8}{\pi^2}\right)^2\left\{\left(\frac{1}{1}\sum_{k=1}^{1}\frac{1}{2k-1}\right)\left(\frac{1}{2}\sum_{k=1}^{1}\frac{1}{2k-1}\right)\right.\right.\\
&\qquad\left.\left.+\left(\frac{1}{2}\sum_{k=1}^{1}\frac{1}{2k-1}\right)\left(\frac{1}{1}\sum_{k=1}^{1}\frac{1}{2k-1}\right)\right\}-\left(\frac{8}{\pi^2}\right)^3\left(\frac{1}{1}\sum_{k=1}^{1}\frac{1}{2k-1}\right)^3\right]z^3\\
&\quad+\cdots\\
&\quad+\left\{-\frac{8}{\pi^2}\sum_{\sum_{j=1}^{1}m_j=n}\left(\prod_{j=1}^{1}\frac{1}{m_j}\sum_{k=1}^{m_j}\frac{1}{2k-1}\right)+\left(\frac{8}{\pi^2}\right)^2\sum_{\sum_{j=1}^{2}m_j=n}\left(\prod_{j=1}^{2}\frac{1}{m_j}\sum_{k=1}^{m_j}\frac{1}{2k-1}\right)\right.\\
&\quad+\left(\frac{8}{\pi^2}\right)^3\sum_{\sum_{j=1}^{3}m_j=n}\left(\prod_{j=1}^{3}\frac{1}{m_j}\sum_{k=1}^{m_j}\frac{1}{2k-1}\right)+\cdots+\left(\frac{8}{\pi^2}\right)^p\sum_{\sum_{j=1}^{p}m_j=n}\left(\prod_{j=1}^{p}\frac{1}{m_j}\sum_{k=1}^{m_j}\frac{1}{2k-1}\right)\\
&\quad\left.+\cdots+\left(\frac{8}{\pi^2}\right)^n\sum_{\sum_{j=1}^{n}m_j=n}\left(\prod_{j=1}^{n}\frac{1}{m_j}\sum_{k=1}^{m_j}\frac{1}{2k-1}\right)\right\}z^n+\cdots\qquad(m_j\in\mathbb{N})
\end{aligned}$$

$$= 1 - \frac{8}{\pi^2}\left(\frac{1}{1}\sum_{k=1}^{1}\frac{1}{2k-1}\right)z + \sum_{p=1}^{2}(-1)^p\left(\frac{8}{\pi^2}\right)^p \sum_{\sum_{j=1}^{p} m_j=2}\left(\prod_{j=1}^{p}\frac{1}{m_j}\sum_{k=1}^{m_j}\frac{1}{2k-1}\right)z^2$$

$$+\sum_{p=1}^{3}(-1)^p\left(\frac{8}{\pi^2}\right)^p \sum_{\sum_{j=1}^{p} m_j=3}\left(\prod_{j=1}^{p}\frac{1}{m_j}\sum_{k=1}^{m_j}\frac{1}{2k-1}\right)z^3+\cdots$$

$$+\sum_{p=1}^{n}(-1)^p\left(\frac{8}{\pi^2}\right)^p \sum_{\sum_{j=1}^{p} m_j=n}\left(\prod_{j=1}^{p}\frac{1}{m_j}\sum_{k=1}^{m_j}\frac{1}{2k-1}\right)z^n+\cdots$$

$$= 1+\sum_{n=1}^{\infty}\sum_{p=1}^{n}(-1)^p\left(\frac{8}{\pi^2}\right)^p \sum_{\sum_{j=1}^{p} m_j=n}\left(\prod_{j=1}^{p}\frac{1}{m_j}\sum_{k=1}^{m_j}\frac{1}{2k-1}\right)z^n.$$

This completes the proof of the theorem. □

By using digamma function in Theorem 2.2, we have

Corollary 2.1. *The power series of the extreme function for the class $\mathcal{B}$ is rewritten as following*

$$f(z) = z+\sum_{n=2}^{\infty}\sum_{p=1}^{n-1}(-1)^p\left(\frac{8}{\pi^2}\right)^p \times$$

$$\sum_{\sum_{j=1}^{p} m_j=n-1}\left\{\prod_{j=1}^{p}\frac{1}{m_j}\left(\psi(m_l+1)-\frac{1}{2}\psi([m_l/2]+1)-\frac{1}{2}\psi(1)\right)\right\}z^n \qquad (m_j\in\mathbb{N}),$$

where [ ] is the Gauss symbol

References

[1] A. W. Goodman, *On uniformly convex functions*, Annal. Polon. Math. **56**(1991), 87 – 92.

[2] A. W. Goodman, *On uniformly starlike functions*, J. Math. Anal. Appl. **155**(1991), 364 – 370.

[3] W. Ma and D. Minda, *Uniformly convex functions*, Annal. Polon. Math. **57**(1992), 165 – 175.

[4] F. Rønning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc. **118**(1993), 189 – 196.

[5] F. Rønning, *On uniform starlikeness and related properties of univalent functions*, Complex Variables **24**(1994), 233 – 239.

Junichi Nishiwaki
Department of Mathematics and Physics
Setsunan University
Neyagawa, Osaka 572-8508 Japan
email:*jerjun2002@yahoo.co.jp*

Shigeyoshi Owa
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502 Japan
email:*shige21@ican.zaq.ne.jp*