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ON FUNCTIONS STARLIKE IN ONE DIRECTION

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ABSTRACT. In this work we investigate the notion of starlikeness in one direction.

Let $\mathcal{H}$ denote the class of all analytic functions in the unit disc $\mathbb{D} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. Recall that a set $E \subset \mathbb{C}$ is said to be starlike with respect to a point $w_0 \in E$ if and only if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E, while a set E is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment joining any two points of E lies entirely in E.

Theorem 1. *Let $p(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic in $\mathbb{D}$. Assume also that $p(z) \neq 0$ in $\mathbb{D}$. Then there exists a number θ_0, $0 \le \theta_0 \le \pi$ for which*

$$\arg p\left(e^{i(\pi+\theta_0)}\right) = \arg p\left(e^{i\theta_0}\right). \tag{1}$$

Proof. From the hypothesis, we can define the function

$$\varphi(\theta) = \arg p\left(e^{i(\pi+\theta)}\right) - \arg p\left(e^{i\theta}\right), \tag{2}$$

where $0 \le \theta \le \pi$. Then we have the following three cases as

$$(i)\ \ \varphi(0) = 0, \quad (ii)\ \ \varphi(0) > 0, \quad (iii)\ \ \varphi(0) < 0.$$

For the case (i), it proves (1) and for the case (ii), we have

$$\begin{aligned}\varphi(\pi) &= \arg p\left(e^{2i\pi}\right) - \arg p\left(e^{i\pi}\right)\\ &= -\left(\arg p\left(e^{i\pi}\right) - \arg p\left(e^{i0}\right)\right)\\ &< 0.\end{aligned}$$

Applying the intermediate value theorem, there exists a number θ_0, $0 < \theta_0 < \pi$ for which

$$\varphi(\theta_0) = 0$$

or

$$\arg p\left(e^{i(\pi+\theta_0)}\right) = \arg p\left(e^{i\theta_0}\right).$$

For the case (iii), applying the same method as the above, we can complete the proof of (1). □

Definition 1.
Let us call the diametal line of $p(z)$ with respect to $\theta = \theta_0$ if the straight line passes through 3 points $p\left(e^{i(\pi+\theta_0)}\right)$, $w = 0$ and $p\left(e^{i\theta_0}\right)$.

Definition 2.
In the Definition 1, if the diametal line of $p(z)$ with respect to $\theta = \theta_0$ has only 2 common points with the image curve of $|z| = 1$ under the mapping $w = p(z)$, then we call that $p(z)$ is starlike in the direction of diametal line with respect to $\theta = \theta_0$.

These ideas of being starlike in one direction was introduced by M. S. Robertson in [1].

Lemma 2. *Let $p(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic $\mathbb{D}$. Assume also that $p(z) \neq 0$ in $\mathbb{D}$. Then there exists at least one the diametal line of $p(z)$.*

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Proof. From the above argument in the proof of Theorem 1 and Definitions 1 and 2, we can easily prove Lemma 2. □

Lemma 3. *(Carathéodory's lemma) Let* $p(z) = \sum_{n=0}^{\infty} c_n z^n$ *be analytic and* $\mathfrak{Re}\, p(z) \geq 0$ *in* $\mathbb{D}$*. Then we have*

$$\mathfrak{Re}\{2c_0\} \geq |c_n| \quad n \in \mathbb{N} = \{1, 2, 3, \ldots\}. \tag{3}$$

Proof. Let us put

$$I = \frac{1}{2\pi i} \int_{|z|=1} p(e^{i\alpha} z)(2 - z^n - 1/z^n) \frac{dz}{z}, \quad (n\alpha = -\arg c_n). \tag{4}$$

Then we have

$$I = 2c_0 - |c_n|$$

and

$$\begin{aligned} \mathfrak{Re}\{I\} &= \mathfrak{Re}\left\{\frac{2}{\pi} \int_0^{2\pi} p(e^{i(\alpha+\theta)}) \sin^2(n\theta/2) d\theta\right\} \\ &= \mathfrak{Re}\{2c_0 - |c_n|\} \\ &\geq 0 \end{aligned}$$

or

$$\mathfrak{Re}\{2c_0\} \geq |c_n| \quad n \in \mathbb{N}.$$

□

Theorem 4. *Let* $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ *be analytic in* $\mathbb{D}$*. Assume also that* $p(z) \neq 0$ *in* $\mathbb{D}$*,* $0 < c_1$ *and suppose that* $p(z)$ *be starlike in the direction of the diametal line of* $p(z)$ *with respect to* $\theta = 0$ *and diametal line coincides real axis. Then we have*

$$|c_n| \leq nc_1 \quad n \in \mathbb{N} = \{1, 2, 3, \ldots\}. \tag{5}$$

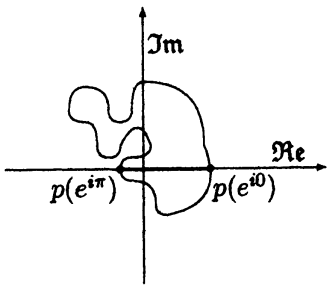

Fig. 1. A picture of Theorem 4.

Proof. From the hypothesis of Theorem 4, we can have the following

$$\begin{aligned} &\mathfrak{Re}\left\{\frac{1}{c_1}\left((p(z) - 1)\frac{1 - z^2}{z}\right)\right\} \\ = \ &\mathfrak{Re}\left\{1 + \frac{c_2}{c_1} z + (c_3 - c_1)\frac{z^2}{c_1} + (c_4 - c_2)\frac{z^3}{c_1} + \cdots + (c_n - c_{n-2})\frac{z^{n-1}}{c_1} + \cdots\right\} \\ = \ &\mathfrak{Re}\left\{(p(e^{i\theta} - 1))\,(-2i \sin\theta)\right\} \\ \geq \ &0, \end{aligned}$$

where $z = e^{i\theta}$ and $0 \leq \theta \leq \pi$. Then, from Caratheodory's lemma 3, that $\mathfrak{Re}\{2c_0\} \geq |c_n|$, we have

$$\left|\frac{c_2}{c_1}\right| \leq 2 \tag{6}$$

and

$$\left|\frac{c_n - c_{n-2}}{c_1}\right| \leq 2, \quad n = 3, 4, 5, \dots. \tag{7}$$

From (6) and (7) we obtain (5).

□

Theorem 5. *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in $\mathbb{D}$. Assume also that $p(z) \neq 0$ in $\mathbb{D}$, $c_1 \neq 0$ and suppose that $p(z)$ be starlike in the direction of the diametal line of $p(z)$ with respect to $\theta = \beta$. Then we have*

$$nc_1 \geq |c_n| \quad n \in \mathbb{N} = \{1, 2, 3, \dots\}. \tag{8}$$

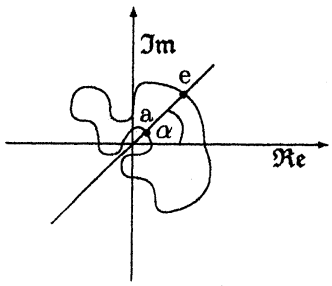

Fig. 2. $a = p(e^{i(\pi+\beta)})$, $e = p(e^{i\beta})$.

Proof. Let us put $\alpha = \arg p(e^{i\beta})$, $\gamma = \arg c_1$ and

$$\begin{aligned} I(z) &= \frac{e^{-i\gamma}}{|c_1|}\left((p(e^{i\beta}z) - 1)\frac{1 - z^2}{z}\right) \\ &= \frac{c_1 e^{i(\beta-\gamma)}}{|c_1|} + \frac{c_2 e^{i(2\beta-\gamma)}}{|c_1|} z + \frac{e^{i(\beta-\gamma)}}{|c_1|}\left(c_3 e^{2i\beta} - c_1\right) z^2 \\ &\quad + \frac{e^{i(2\beta-\gamma)}}{|c_1|}\left(c_4 e^{2i\beta} - c_2\right) z^3 + \dots \\ &\quad + \frac{e^{i((n-2)\beta-\gamma)}}{|c_1|}\left(c_n e^{2i\beta} - c_{n-2}\right) z^{n-1} + \dots. \end{aligned}$$

From the hypothesis of Theorem 5, we have

$$\mathfrak{Re}\{I(z)\} \geq 0 \quad z \in \mathbb{D}.$$

Applying Caratheodory's lemma 3, we have

$$\begin{aligned}
\Re\mathfrak{e}\left\{2e^{i(\beta\alpha+\gamma)}\right\} &\geq \left|\frac{c_2}{c_1}\right|, \\
\Re\mathfrak{e}\left\{2e^{i(\beta\alpha+\gamma)}\right\} &\geq \left|\frac{c_3e^{2i\beta}-c_1}{c_1}\right| \\
\Re\mathfrak{e}\left\{2e^{i(\beta\alpha+\gamma)}\right\} &\geq \left|\frac{c_4e^{2i\beta}-c_2}{c_1}\right| \\
&\vdots \\
\Re\mathfrak{e}\left\{2e^{i(\beta\alpha+\gamma)}\right\} &\geq \left|\frac{c_ne^{2i\beta}-c_{n-2}}{c_1}\right| \\
&\vdots
\end{aligned}$$

Applying the mathematical induction, we can easily obtain Theorem 5. □

For other results on the several problems connected to the starlikeness we refer also to the recent paper [4].

References

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