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# ON THE RELATION BETWEEN COMPLETENESS AND H-CLOSEDNESS

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## 1. INTRODUCTION AND THE MAIN RESULT

In this resume, we state the relation between completeness and H-closedness for topological partially ordered spaces (or shortly pospaces). Though H-closedness is a generalization of compactness, H-closedness does not correspond with compactness for even chains and antichains (equipped with some pospace topologies). Indeed, since the pospaces which are antichains coincide with the Hausdorff topological spaces, we have that H-closed non-compact topological spaces are also such pospaces which are antichains (e.g. Example 1). There is also another extremal example which is a countable linearly ordered H-closed non-compact pospace (e.g. Example 4.6 in [GPR]).

In [GPR], they have shown that a linearly ordered topological semilattice is H-closed if and only if it is H-closed as a topological pospace. They also have given the following characterization of H-closedness for a linearly ordered pospace to be H-closed (Corollary 3.5 in [GPR]): A linearly ordered pospace X is H-closed if and only if the following conditions hold:
(i) $X$ is a complete set with respect to the partial order on $X$;
(ii) $x=\bigvee A$ for $A=\downarrow A\setminus\{x\}$ implies $x\in \mathrm{cl}A$, whenever $A\neq\emptyset\subseteq X$; and
(iii) $x=\bigwedge B$ for $B=\uparrow B\setminus\{x\}$ implies $x\in \mathrm{cl}B$, whenever $B\neq\emptyset\subseteq X$.

This result can rewrite the following statement: A linearly ordered pospace $X$ is H-closed if and only if $X$ is a complete lattice with $\bigvee L\in \mathrm{cl}{\downarrow}L$ and $\bigwedge L\in \mathrm{cl}{\uparrow}L$ for any nonempty chain $L\subseteq X$. Naturally the following question arises: Is there a similar characterization of H-closedness for topological semilattices or pospaces? It's easy to see that a discrete countable antichain is not H-closed as a pospace but directed complete and down-directed complete. This means that there is no similar characterization of H-closedness for pospaces. However, we have given the necessary and sufficient condition for pospaces without infinite antichains to be H-closed[Y].

By a partial order on a set $X$ we mean a reflexive, transitive and anti-symmetric binary relation $\leq$ on $X$. A set endowed with a partial order is called a partially ordered set (or poset). For an element $x$ of a poset $X$, $\uparrow x:=\{y\in X\mid x\leq y\}$ (resp. $\downarrow x:=\{y\in X\mid y\leq x\}$) is called the upset (resp. the downset) of $x$. For a subset $Y\subseteq X$, $\uparrow Y:=\bigcup_{y\in Y}\uparrow y$ (resp. $\downarrow Y:=\bigcup_{y\in Y}\downarrow y$) is called the upset (resp. the downset) of $Y$. For a subset $A$ of a poset, $A$ is said to be a chain if $A$ is linearly ordered, and is said to be an antichain if any distinct elements are incomparable. A maximal chain (resp. antichain) is a chain (resp. antichain) which is properly

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contained in no other chain (resp. antichain). The Axiom of Choice implies the existence of maximal chains in any poset. A subset $D$ of a poset $X$ is (up-)directed (resp. down-directed) if every finite subset of $D$ has an upper (resp. lower) bound in $D$. A poset $X$ is said to be down-directed complete (resp. (up-)directed complete) if each down-directed (resp. up-directed) set $S$ of $X$ has $\bigwedge S$ (resp. $\bigvee S$). It is well-known that a poset $X$ is directed complete if and only if each chain $L$ of $X$ has $\bigvee L$.

For a subset of a topological space $A$, denote the closure of the set $A$ by cl$A$. Recall that a Hausdorff space $X$ with a partial order is called a topological partially ordered space (or pospace) if the partial order is a closed subset of $X \times X$. A partial order $\leq$ is said to be continuous or closed if $x \not\leq y$ in $X$ implies that there are open neighborhoods $U$ and $V$ of $x$ and $y$ respectively such that $\uparrow U \cap V = \emptyset$ (equivalently $U \cap \downarrow V = \emptyset$). A partial order $\leq$ on a Hausdorff space $X$ is continuous if and only if $(X, \leq)$ is a pospace [W]. In any pospace, $\downarrow x$ and $\uparrow x$ are both closed for any element $x$ of it.

A pospace $X$ is said to be an H-closed pospace if $X$ is a closed subspace of every pospace in which it is contained. Obviously that the notion of H-closedness is a generalization of compactness. Now we state the main result in [Y].

**Theorem 1.** *Let $X$ be a pospace without infinite antichains. Then $X$ is an H-closed pospace if and only if $X$ is directed complete and down-directed complete such that $\bigvee L \in \mathrm{cl}{\downarrow}L$ and $\bigwedge L \in \mathrm{cl}{\uparrow}L$ for any nonempty chain $L \subseteq X$.*

## 2. Examples

First, we describe a well-known example which is an H-closed non-compact topological space (i.e. an H-closed non-compact pospace which is an antichain).

**Example 1.** *Let $X = [0,1]$. Equip $X$ with the topology that has the union generated by a subbasis $\tau \cup \{X \setminus N\}$, where $N = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ and $\tau$ is the Euclidean topology on $[0,1]$. Then $N$ is closed and discrete in $X$, so $X$ is not compact. Recall a well-known fact that a Hausdorff space is H-closed if and only if for every open cover of it there is a finite subfamily whose union is dense. This fact implies that $X$ is H-closed.*

For posets $X, Y$, denote by $X + Y$ the disjoint union with the extended order as follows: for any $x \in X$ and $y \in Y$, $x < y$. The following example is a countable pospace without infinite antichains and without the finite upper bound of the cardinals of the maximal antichain containing a certain point.

**Example 2.** *Let $[n]$ be an antichain consisting of $n$ elements, $A = [1] + [2] + [3] + \cdots$ the countable union of $[n]$, and $X = \{0\} \sqcup A$ an order disjoint union. Then any antichain is of form a subset of either $[n]$ or $\{0\} \sqcup [n]$. Hence $X$ has no infinite antichain and there is no finite upper bound of the cardinals of the maximal antichain containing* 0.

The following examples are non-H-closed pospaces.

**Example 3.** *Let $Y = ([-1,1] \setminus \{0\})$ be a pospace with the usual order and the interval topology, $A = [-1,0)$, and $B = (0,1]$. Define $X := Y \sqcup \{0_-, 0_+\}$ a disjoint union with the extended order as follows: ${\downarrow}0_- = \{0_-\} \sqcup A$, ${\uparrow}0_- = \{0_-\}$, ${\downarrow}0_+ = \{0_+\}$, and ${\uparrow}0_+ = \{0_+\} \sqcup B$. Since $X$ is not directed complete and has* *no infinite antichains, Theorem 1 implies that $X$ is not H-closed with any pospace topology.*

*Similarly, the poset obtained from $[-1, 0)$ by adding incomparable maximal points $\{a', a''\}$ is not H-closed with respect to any pospace topology.*

## 3. Final remark

Note that all H-closed pospaces which the author knows are directed complete. Naturally the following question arises:

**Question.** *Is there an H-closed pospace which is not directed complete?*

## References

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