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On Poisson Anderson model with polynomially decaying single site potential
（多項式減衰するポテンシャルを配した Poisson Anderson model について）

Ryoki Fukushima
Research Institute of Mathematical Sciences
Kyoto University

In this talk, we discussed spectral properties of a Schrödinger operator with random potential

$$H_{\omega}=-\kappa\Delta+V_{\omega}.$$

The random potential is defined by

$$V_{\omega}(x)=\sum_{i}v(x-\omega_{i})$$

with v a nonnegative and integrable function and $(\omega=\sum_{i}\delta_{\omega_{i}}, \mathbb{P})$ a Poisson point process with unit intensity. In particular, we are interested in the case $v(x)=|x|^{-\alpha}\wedge 1$ $(\alpha>d)$, which models an electron receiving long range interaction from randomly scattered impurities. (The truncation $\wedge 1$ is made for technical reason and we shall neglect it except for the final subsection for simplicity.)

One of the quantity of interest in the theory of random operators is the integrated density of states defined by

$$N(\lambda)=\lim_{R\to\infty}\frac{1}{(2R)^{d}}\mathbb{E}[\#\{k\in\mathbb{N};\lambda_{\omega,k}((-R, R)^{d})\leq\lambda\}], \tag{1}$$

where $\lambda_{\omega,k}((-R, R)^{d})$ is the k-th smallest eigenvalue of H_{ω} in $(-R, R)^{d}$ with the Dirichlet boundary condition. For many random Schrödinger operators, $N(\lambda)$ decays exponentially fast as λ approaches the bottom of the spectrum, which stands in sharp contrast to the classical Weyl type asymptotics. This reflects the fact that the low lying spectra come from spatially rare “pockets” where the random potential takes atypically small value, and plays an important role in the proof of the so-called Anderson localization.

The study of the integrated density of states for the Poisson Anderson model dates back to Donsker and Varadhan [3] and Nakao [8]. They studied the case $v(x)=o(|x|^{-d-2})$ and proved that

$$N(\lambda)=\exp\left\{-c_{1}(d,\kappa)\lambda^{-\frac{d}{2}}(1+o(1))\right\} \tag{2}$$

as $\lambda\downarrow 0$, which verifies the exponential decay predicted by Lifshiz [7]. It is worth mentioning that the above asymptotics is independent of the tail of v. Thus as long as $v(x)=o(|x|^{-d-2})$, the interactions are of short-range nature.

On the other hand, if v has a heavier tail, then it exhibits a long-range nature. Indeed, Pastur [9] proved that when $v(x)\sim|x|^{-\alpha}$ $(d<\alpha<d+2)$,

$$N(\lambda)=\exp\left\{-c_{2}(d,\alpha)\lambda^{-\frac{d}{\alpha-d}}(1+o(1))\right\} \tag{3}$$

as $\lambda \downarrow 0$. It is interesting that the asymptotics is determined by d and α and is independent of κ. Pastur called it a *classical behavior* of the integrated density of states. A main result of this talk is the second order asymptotics of the integrated density of states, which in particular shows that the quantum effect appears in the second order term.

Theorem 1. ([4]) *Suppose* $v(x) = |x|^{-\alpha} \wedge 1$ *with* $d < \alpha < d+2$. *Then*

$$N(\lambda) = \exp\left\{-c_2(d,\alpha)\lambda^{-\frac{d}{\alpha-d}} - (c_3(d,\alpha,\kappa) + o(1))\lambda^{-\frac{\alpha+d-2}{2(\alpha-d)}}\right\} \tag{4}$$

as $\lambda \downarrow 0$.

In the following sections, we review outlines of arguments both in light-tailed and heavy-tailed cases. Let us first recall the following well known Feynman-Kac representation of the Laplace transform of the integrated density of states (see, e.g., [1], Theorem VI.1.1):

$$\int_0^\infty e^{-tl}\mathrm{d}N(l) = (4\pi\kappa t)^{-\frac{d}{2}}\mathbb{E} \otimes E_{0,0}^t\left[\exp\left\{-\int_0^t V_\omega(X_s)\mathrm{d}s\right\}\right],$$

where $E_{0,0}^t$ denotes the expectation with respect to the $\kappa\Delta$-Brownian bridge from 0 to 0 in the time interval $[0,t]$. In view of Tauberian theory, the first order asymptotics of $N(\lambda)$ as $\lambda \downarrow 0$ follows once we know the asymptotics of the right-hand side as $t \to \infty$. In fact, it turns out that the right-hand side have stretched exponential asymptotics both in light and heavy tailed case and thus the prefactor $(4\pi\kappa t)^{-\frac{d}{2}}$ is unimportant. Moreover, one can show that replacing the Brownian bridge by the Brownian motion has only negligible effect on the asymptotics.

1 Light-tailed case

When $\alpha > d+2$, which is referred to as the light tailed case, Donsker and Varadhan [3] determined the asymptotics

$$\begin{aligned}\mathbb{E} \otimes E_0&\left[\exp\left\{-\int_0^t V_\omega(X_s)\mathrm{d}s\right\}\right]\\ &= \exp\left\{-\inf_{U:\,\mathrm{open}}\{\lambda_1(U) + |U|\}\,t^{\frac{d}{d+2}}(1+o(1))\right\}\end{aligned} \tag{5}$$

as t goes to ∞, where $|U|$ and $\lambda_1(U)$ stand for the volume of U and the smallest Dirichlet eigenvalue of $-\Delta/2$ in U, respectively. It follows from Faber-Krahn's inequality that the unique minimizer of the above variational problem is the ball with a certain radius R_0, up to translation.

Let us start with the proof of the lower bound, which illustrates how the variational problem comes into play. We assume $V_\omega(x) = \sum_i \infty \cdot 1_{B(\omega_i,1)}$ for simplicity. Then we have the following simple lower bound:

$$\begin{aligned}\mathbb{E} \otimes E_0&\left[\exp\left\{-\int_0^t V_\omega(X_s)\mathrm{d}s\right\}\right]\\ &\geq \mathbb{P}(\#\{\omega_i \text{ in 1-neighborhood of } U\} = 0)P_0(X_s \in U \text{ for all } s \in [0,t])\end{aligned} \tag{6}$$

for any open set U. Since

$$\mathbb{P}(\#\{\omega_i \text{ in 1-neighborhood of } U\} = 0) = \exp\{-|U|\} \tag{7}$$

by definition and

$$P_0(X_s \in U \text{ for all } s \in [0,t]) = \exp\{-t\lambda_1(U)(1+o(1))\} \tag{8}$$

by the Kac formula, we have

$$\mathbb{E} \otimes E_0 \left[\exp\left\{-\int_0^t V_\omega(X_s)\mathrm{d}s\right\}\right] \geq \exp\{-|U| - t\lambda_1(U)(1+o(1))\}. \tag{9}$$

Thus scaling $U = t^{1/(d+2)}U'$ and optimizing over U' give us the correct lower bound. Note that the lower bound comes from single event which is a maximizer of probability among certain strategies.

The proof of upper bound requires more sophisticated tool called the large deviation principle for empirical measure. We still assume $V_\omega(x) = \sum_i \infty \cdot 1_{B(\omega_i,1)}$ and only explain outline of the argument. The empirical measure of process $\{X_s\}_{0\leq s\leq t}$ is formally defined by $L_t = \int_0^t \delta_{X_s}\mathrm{d}s$. The starting point of the argument is

$$\begin{aligned}
&\mathbb{E} \otimes E_0 \left[\exp\left\{-\int_0^t V_\omega(X_s)\mathrm{d}s\right\}\right] \\
&\quad = \mathbb{P} \otimes P_0 \left(\#\{\omega_i \text{ in the 1-neighborhood of supp}L_t\} = 0\right) \\
&\quad = E_0 \left[\exp\left\{-|\mathrm{supp}(L_t * 1_{B(0,1)})|\right\}\right] \\
&\quad = E_0 \left[\exp\left\{-t^{d/(d+2)}|\mathrm{supp}(L_{t^{d/(d+2)}} * 1_{B(0,t^{-1/(d+2)})})|\right\}\right],
\end{aligned}$$

where the last step is due to the Brownian scaling. Now we apply the following large deviation principle and its consequence proved in [2].

Theorem 2. *Let $N > 0$ and X be the $\kappa\Delta$-Brownian motion on the torus $\mathbb{R}^d/N\mathbb{Z}^d$. Then the mollified empirical measure $L_s * 1_{B(0,s^{-1/d})}$ satisfies a large deviation principle in the space $\mathcal{P}_s$ of probability measures with density, equipped with L^1-topology, with scale s and rate function $I(\nu) = \kappa\|\nabla\sqrt{\mathrm{d}\nu/\mathrm{d}x}\|_2^2$. Consequently, for any functional F on $\mathcal{P}_s$ which is upper semi-continuous in L^1-topology,*

$$E_0 \left[\exp\{-sF(L_s * 1_{B(0,s^{-1/d})})\}\right] \leq \exp\left\{-s \inf_{\nu\in\mathcal{P}_s}\{F(\nu) + I(\nu)\}(1+o(1))\right\}$$

as $s \to \infty$.

This result is restricted to the Brownian motion on a torus but it is no problem here since projecting on a torus only decrease volume. Also, after projecting on a torus, $|\mathrm{supp}(\nu)|$ is an upper semi-continuous in L^1 topology. Therefore we may apply this result to obtain

$$\begin{aligned}
&E_0 \left[\exp\left\{-t^{d/(d+2)}|\mathrm{supp}(L_{t^{d/(d+2)}} * 1_{B(0,t^{-1/(d+2)})})|\right\}\right] \\
&\quad \leq \exp\left\{-t^{d/(d+2)} \inf_{\|\phi\|_2=1} \left\{\kappa\|\nabla\phi\|_2^2 + |\mathrm{supp}\phi|\right\}(1+o(1))\right\}
\end{aligned}$$

as $t \to \infty$ which is easily seen to coincide with the desired bound. The above argument extend to general light-tailed v with little extra effort. Indeed, for the lower bound it suffices to consider the same event as above, that is, there is no ω_i in $B(0, R_0 t^{1/(d+2)})$ and $\{X_s\}_{0\le s\le t}$ stays there. The potential V_ω takes positive value inside the ball due to the tail but one can check that it is negligible. For the upper bound, it suffices to consider compactly supported v of finite height. Then the $|\mathrm{supp}(L_t * 1_{B(0,1)})|$ above is replaced by more complicated functional of L_t but it turns out to behave very similar manner to the volume of support.

Finally, applying an exponential Tauberian theorem, e.g., the one in [6], one finds the so-called Lifshitz tail

$$N(\lambda) = \exp\left\{-c_1(d,\kappa)\lambda^{-\frac{d}{2}}(1+o(1))\right\} \tag{10}$$

as $\lambda \downarrow 0$. Note that the probability $\mathbb{P}(\#\{\omega_i \text{ in } B(0, r\lambda^{-1/2})\} = 0)$ has the same asymptotics as the right-hand side for suitable $r > 0$ and inspecting the above argument, one finds that the lower bound is indeed proved by considering such a event. So at a heuristic level, we see that in the light tailed case, "the Lifshitz tail reflects the cost to lower the first eigenvalue by making large vacant region".

2 Heavy-tailed case

2.1 Earlier studies

Let us first explain what causes the difference between light and heavy tailed cases. In [9], a two-sided bound

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{-t\left\{\kappa\|\nabla\phi\|_2^2 + \int V_\omega(x)\phi(x)^2\mathrm{d}x\right\}\right\}\right]\\
&\quad\le \mathbb{E}\otimes E_0\left[\exp\left\{-\int_0^t V_\omega(X_s)\mathrm{d}s\right\}\right]\\
&\quad\le \mathbb{E}[\exp\{-tV_\omega(0)\}]
\end{aligned}$$

is proved for any nonnegative and smooth ϕ with unit L^2-norm. The first inequality relies on the so-called Peierls' inequality but the reader may also find a similarity to the large deviation principle in the last section. The second inequality is a consequence of Jensen's inequality. Now, the both sides are simple functionals of Poisson point process and can be computed as follows:

$$\mathbb{E}[\exp\{-tV_\omega(0)\}] = \exp\{-a_1 t^{d/\alpha}\},$$

where $a_1 = |B(0,1)|\Gamma((\alpha-d)/\alpha)$ and

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{-t\left\{\kappa\|\nabla\phi\|_2^2 + \int V_\omega(x)\phi(x)^2\mathrm{d}x\right\}\right\}\right]\\
&\quad= \exp\left\{-t\kappa\|\nabla\phi\|_2^2 - \int (1-e^{-\int v(x-y)\phi(y)^2\mathrm{d}y})\mathrm{d}x\right\}.
\end{aligned}$$

As long as $\alpha > d+2$, this is larger than the Donsker-Varadhan bound. This is consistent with the explanation in the last section that in the optimal strategy in the light tailed case, the contribution from V_ω is negligible compared with $\|\nabla\phi\|_2^2$.

However, if $\alpha < d+2$ then the above upper bound is smaller than the Donsker-Varadhan bound and in fact, one can show that the upper bound above is sharp. (For instance, it suffices to choose ϕ in the lower bound as the first eigenfunction of the Dirichlet Laplacian in $B(0, t^{1/\alpha-\epsilon})$ for a small $\epsilon > 0$.) Pastur proved that

$$\mathbb{E} \otimes E_0 \left[\exp\left\{-\int_0^t V_\omega(X_s)\mathrm{d}s\right\}\right] = \exp\left\{-a_1 t^{d/\alpha}(1+o(1))\right\} \tag{11}$$

as $t \to \infty$ for $\alpha < d+2$ in this way and derived

$$N(\lambda) = \exp\left\{-c_2(d,\alpha)\lambda^{-d/(\alpha-d)}(1+o(1))\right\}$$

as $\lambda \downarrow 0$ by a Tauberian theorem. As is expected from the fact that this is derived from $\mathbb{E}[\exp\{-tV_\omega(0)\}]$, the probability $\mathbb{P}(V_\omega(0) \le \lambda)$ has the same asymptotics as the right-hand side. Thus at a heuristic level, "the (leading term of) Lifshitz tail reflects the cost to lower the first eigenvalue by making V_ω small at one point".

2.2 Second order asymptotics of the Wiener functional

Recently, the author [4] studied the second order asymptotics of (11) to get better understanding of the Brownian motion in the random potential V_ω. (See also [5] for more recent progress.) One of the main theorem in [4] is the following.

Theorem 3. *For $d < \alpha < d+2$,*

$$\mathbb{E} \otimes E_0 \left[\exp\left\{-\int_0^t V_\omega(B_s)\mathrm{d}s\right\}\right] = \exp\left\{-a_1 t^{\frac{d}{\alpha}} - (a_2 + o(1))t^{\frac{\alpha+d-2}{2\alpha}}\right\} \tag{12}$$

as $t \to \infty$, where

$$a_2 = \left(\frac{\kappa\alpha|\partial B(0,1)|}{2}\Gamma\left(\frac{2\alpha-d+2}{\alpha}\right)\right)^{\frac{1}{2}}.$$

Moreover, the constant a_2 admits a variational expression

$$a_2 = \inf_{\|\phi\|_{L^2}=1}\left\{\int \kappa|\nabla\phi|(x)^2 + C(d,\alpha)|x|^2\phi(x)^2\mathrm{d}x\right\} \tag{13}$$

with some explicit constant $C(d,\alpha)$.

The intuition behind this result is that the best strategy in the heavy tailed case should be to make $V_\omega(0) \sim a_1\frac{d}{\alpha}t^{-(\alpha-d)/\alpha}$, whose probability is

$$\mathbb{P}\left(V_\omega(0) \sim a_1\frac{d}{\alpha}t^{-\frac{\alpha-d}{\alpha}}\right) = \exp\left\{-a_1\frac{\alpha-d}{\alpha}t^{\frac{d}{\alpha}}(1+o(1))\right\},$$

and force $\{X_s\}_{0\le s\le t}$ to stay in $B(0, t^{(\alpha-d+2)/(4\alpha)})$. This means that the process stays around the bottom of the "valley" of the potential. Since the potential locally looks like a quadratic function around the bottom due to the strong correlation, we are naturally lead to the above variational expression of a_2. See also [5] for more recent progress, where it is proved in a sense that the above strategy is indeed the best one.

The proof of Theorem 3 is essentially based on the large deviation theory and we refrain from presenting the detail. Instead, we shall focus on how to find the second term of the Lifshitz tail.

Remark 1. In [4], the proof of Theorem 1 is partly embedded in the proof of almost sure asymptotics of

$$E_0\left[\exp\left\{-\int_0^t V_\omega(B_s)\mathrm{d}s\right\}\right],$$

which is another main result. If one is interested only in the second term of Lifshitz tail, then the argument given in the next subsection is a bit simpler and easier to read.

2.3 Second term of the Lifshitz tail

The upper bound in Theorem 1 follows from Theorem 3 by the same way as the Tauberian theory. As a result, one finds that the constants in Theorem 1 are

$$c_2(d,\alpha) = \frac{\alpha-d}{\alpha}\left(\frac{d}{\alpha}\right)^{\frac{d}{\alpha-d}} a_1^{\frac{\alpha}{\alpha-d}},$$

$$c_3(d,\alpha,\kappa) = a_2\left(\frac{da_1}{\alpha}\right)^{\frac{\alpha+d-2}{2(\alpha-d)}}.$$

However, it seems difficult to derive the second order lower bound from Theorem 3 by Tauberian argument. We show how to derive the lower bound in this subsection. For some technical reason, we restrict ourselves to the case $\alpha \ge 2$. Also, we do need the truncation $v(x) = |x|^\alpha \wedge 1$ in this subsection and write $\bar{v}(x) = |x|^{-\alpha}$. The truncation makes a slight difference

$$\mathbb{E}[\exp\{-tV_\omega(0)\}] = \exp\{-a_1 t^{d/\alpha} + o(1)\}$$

but since this has little effect, we neglect the above $o(1)$ in the sequel.

Our starting point to obtain the lower bound is a well-known bound

$$\begin{aligned} N(\lambda) &= \sup_{N\ge 1}\frac{1}{(2N)^d}\mathbb{E}[\#\{k\ge 1 : \lambda_k^\omega((-N,N)^d)\le\lambda\}] \\ &\ge \sup_{N\ge 1}\frac{1}{(2N)^d}\mathbb{P}(\lambda_1^\omega((-N,N)^d)\le\lambda). \end{aligned}$$

This reduces the problem to finite volume and we choose $N = 2M\lambda^{-\frac{\alpha-d+2}{4(\alpha-d)}}$ with sufficiently large $M > 0$ so that the factor $(2N)^{-d}$ is negligible.

Let us briefly explain the outline of the argument before going into detail. We are going to bound $\mathbb{P}(\lambda_1^\omega((-N,N)^d) \le \lambda)$ from below by constructing a specific event. As explained above, we expect that $\mathbb{P}(\lambda_1^\omega((-N,N)^d) \le \lambda)$ is asymptotically close to $\mathbb{P}(V_\omega(0) \le \lambda)$ and hence we consider an event like $\{V_\omega(0) \le \lambda\}$. But conditioned on $\{V_\omega(0) \le \lambda\}$, the potential V_ω locally looks like parabola (see the discussion after Theorem 3) and thus $\lambda_1^\omega((-N,N)^d)$ becomes slightly larger than $V_\omega(0)$. This means that we have to make $V_\omega(0)$ slightly smaller than λ and this gives rise to the second term. We need to show how much $\lambda_1^\omega((-N,N)^d)$ is larger than λ conditioned on $\{V_\omega(0) \le \lambda\}$. Since conditional probability is not very easy to deal with, we will use a transformed measure instead.

Now let us introduce a transformed measure defined by

$$\frac{\mathrm{d}\tilde{\mathbb{P}}_\rho}{\mathrm{d}\mathbb{P}}(\omega) = e^{a_1\rho^{d/\alpha} - \rho V_\omega(0)}.$$

This is a substitute for the conditional measure since $V_\omega(0) \sim \frac{da_1}{\alpha}\rho^{d/\alpha}$ under $\tilde{\mathbb{P}}_\rho$ when ρ is large. By taking $\rho = (\frac{da_1}{\alpha\lambda})^{\alpha/(\alpha-d)}$ and $\lambda \downarrow 0$, we have $V_\omega(0) \sim \lambda$ under $\tilde{\mathbb{P}}_\rho$.

We collect several properties of the measure $\tilde{\mathbb{P}}_\rho$ which we shall use later.

Lemma 1. (i) *$(\omega, \tilde{\mathbb{P}}_\rho)$ is a Poisson point process with intensity $e^{-\rho v(y)}\mathrm{d}y$.*
(ii) *$\tilde{\mathbb{E}}_\rho[V_\omega(x)] = \frac{da_1}{\alpha}\rho^{-\frac{\alpha-d}{\alpha}} + C(d,\alpha)\rho^{-\frac{\alpha-d+2}{\alpha}}|x|^2 + o(\rho^{-\frac{\alpha-d+2}{2\alpha}})$ as $\rho \to \infty$, uniformly in $x \in B_M(\rho) := B(0, M\rho^{\frac{\alpha-d+2}{4\alpha}})$.*
(iii) *$\rho^{\frac{2\alpha-d}{2\alpha}}\left(V_\omega(0) - \frac{da_1}{\alpha}\rho^{-\frac{\alpha-d}{\alpha}}\right)$ under $\tilde{\mathbb{P}}_\rho$ converges in law to a non-degenerate Gaussian random variable.*

The proof of this lemma is essentially of computational nature and we omit the detail. The following elementary lemma is useful to prove the above lemma and also in the sequel. We say that a function $f(\rho)$ is of order $o(\rho^{-\infty})$ if it decays faster than any polynomial of ρ^{-1}.

Lemma 2. (i) *For any $M > 0$,*

$$\sup_{\|u\|_\infty \le 1}\left|\int_{B_{2M}(\rho)} u(y)e^{-\rho v(y)}\mathrm{d}y\right| = o(\rho^{-\infty})$$

as $\rho \to \infty$.
(ii) *For any $M > 0$ and $\gamma > 0$,*

$$\int_{B_{2M}(\rho)} |y|^{-\gamma}e^{-\rho v(y)}\mathrm{d}y = o(\rho^{-\infty})$$

as $\rho \to \infty$.
(iii) *For any $M > 0$ and $\gamma > d$,*

$$\int_{\mathbb{R}^d\setminus B_{2M}(\rho)} |y|^{-\gamma}e^{-\rho v(y)}\mathrm{d}y = O\left(\rho^{\frac{d-\gamma}{\alpha}}\right)$$

as $\rho \to \infty$.

The following is the key lemma to prove Theorem 3.

Lemma 3. *Suppose $\alpha \geq 2$. Then for any $\epsilon > 0$ and $M > 0$,*

$$\mathbb{P}\left(\sup_{x\in B_M(\rho)} |V_\omega(x) - Q_\rho(x)| \leq \epsilon\rho^{-\frac{\alpha-d+2}{2\alpha}}\right) \geq \exp\left\{-a_1\frac{\alpha-d}{\alpha}\rho^{d/\alpha}\right\} \tag{14}$$

when ρ is sufficiently large.

Remark 2. The event on the left-hand side includes $\{V_\omega(0) \lesssim \frac{da_1}{\alpha}\rho^{-\frac{\alpha-d}{\alpha}}\}$ whose probability is asymptotic to the right-hand side. Thus this lemma says that conditioned on this event, V_ω locally looks like a parabola.

Proof In view of Lemma 1-(ii), we have an inclusion

$$\begin{aligned}
&\left\{\sup_{x\in B_M(\rho)} |V_\omega(x) - Q_\rho(x)| \leq \epsilon\rho^{-\frac{\alpha-d+2}{2\alpha}}\right\}\\
&\supset \left\{V_\omega(0) - \frac{da_1}{\alpha}\rho^{-\frac{\alpha-d}{\alpha}} \in \left(0, \frac{\epsilon}{2}\rho^{-\frac{\alpha-d+2}{2\alpha}}\right)\right\} \setminus\\
&\qquad \left\{\sup_{x\in B_M(\rho)} |V_\omega(x) - V_\omega(0) - \tilde{\mathbb{E}}_\rho[V_\omega(x) - V_\omega(0)]| \geq \frac{\epsilon}{4}\rho^{-\frac{\alpha-d+2}{2\alpha}}\right\}\\
&=: E_1 \setminus E_2
\end{aligned}$$

for sufficiently large ρ. From this it follows that

$$\begin{aligned}
&\text{the left hand side of (14)}\\
&\geq e^{-a_1\rho^{d/\alpha}}\tilde{\mathbb{E}}_\rho\left[e^{\rho V_\omega(0)} : E_1 \setminus E_2\right]\\
&\geq \exp\left\{-a_1\rho^{d/\alpha} + \rho\left(\frac{da_1}{\alpha}\rho^{-\frac{\alpha-d}{\alpha}}\right)\right\}\tilde{\mathbb{P}}_\rho(E_1 \setminus E_2)\\
&\geq \exp\left\{-a_1\frac{\alpha-d}{\alpha}\rho^{d/\alpha}\right\}(\tilde{\mathbb{P}}_\rho(E_1) - \tilde{\mathbb{P}}_\rho(E_2)).
\end{aligned}$$

It remains to show that $\tilde{\mathbb{P}}_\rho(E_1) - \tilde{\mathbb{P}}_\rho(E_2)$ is bounded from below. The first term is rather easy since

$$\tilde{\mathbb{P}}_\rho(E_1) = \tilde{\mathbb{P}}_\rho\left(\rho^{\frac{2\alpha-d}{2\alpha}}\left(V_\omega(0) - \frac{da_1}{\alpha}\rho^{-\frac{\alpha-d}{\alpha}}\right) \in \left(0, \frac{\epsilon}{2}\rho^{\frac{\alpha-2}{2\alpha}}\right)\right), \tag{15}$$

which is bounded from below by a positive constant for $\alpha \geq 2$ because of Lemma 1-(iii). To estimate $\tilde{\mathbb{P}}_\rho(E_2)$, we use an well-known expectation formula for the Poisson point process to see

$$\begin{aligned}
&V_\omega(x) - V_\omega(0) - \tilde{\mathbb{E}}_\rho[V_\omega(x) - V_\omega(0)]\\
&\quad = \int (v(x-y) - v(-y))\left(\omega(\mathrm{d}y) - e^{-\rho v(y)}\mathrm{d}y\right).
\end{aligned} \tag{16}$$

For abbreviation, we write $\bar{\omega}_\rho(dy)$ for $\omega(dy) - \nu e^{-\rho v(y)}dy$ in this proof. This is a slight abuse of notation since $\bar{\omega}_\rho(dy)$ has infinite total variation. But we will only consider functions which are $e^{-\rho v(y)}dy$-integrable and therefore all the integrals appearing below make sense.

We divide the integral in (16) into $y \in B_{2M}(\rho)$ and $y \notin B_{2M}(\rho)$ and show that each part has order $o(\rho^{-(\alpha-d+2)/2\alpha})$ with probability close to 1. Fix an arbitrary small $\epsilon > 0$. Let us begin with

$$\begin{aligned}
&\sup_{x\in B_M(\rho)} \left| \int_{B_{2M}(\rho)} (v(x-y) - v(-y))\bar{\omega}_\rho(dy) \right| \\
&\le \sup_{x\in B_M(\rho)} \left\{ \int_{B_{2M}(\rho)} |v(x-y) - v(-y)|\omega(dy) \right. \\
&\qquad \left. + \int_{B_{2M}(\rho)} |v(x-y) - v(-y)| e^{-\rho v(y)} dy \right\} \\
&\le \int_{B_{2M}(\rho)} \bar{\omega}_\rho(dy) + 2\int_{B_{2M}(\rho)} e^{-\rho v(y)} dy.
\end{aligned}$$

The $\tilde{\mathbb{P}}_\rho$-mean of the first term is zero. Moreover, its variance and the second term are both of $o(\rho^{-\infty})$ by Lemma 2-(i). Hence we obtain

$$\tilde{\mathbb{P}}_\rho\left(\sup_{x\in B_M(\rho)} \left| \int_{B_{2M}(\rho)} (v(x-y) - v(-y))\bar{\omega}_\rho(dy) \right| > \epsilon\rho^{-\frac{\alpha-d+2}{2\alpha}} \right) = o(\rho^{-\infty})$$

as $\rho \to \infty$ using Chebyshev's inequality.

Now we turn to the remaining part. Since $v(x-y) = \bar{v}(x-y) (= |x-y|^{-\alpha})$ for $x \in B_M(\rho)$ and $y \notin B_{2M}(\rho)$, we can use Taylor's theorem to see

$$\begin{aligned}
&\sup_{x\in B_M(\rho)} \left| \int_{\mathbb{R}^d\setminus B_{2M}(\rho)} (v(x-y) - v(-y))\bar{\omega}_\rho(dy) \right| \\
&= \sup_{x\in B_M(\rho)} \left| \int_{\mathbb{R}^d\setminus B_{2M}(\rho)} \langle x, \nabla\bar{v}(-y)\rangle \bar{\omega}_\rho(dy) \right| \\
&\quad + \sup_{x\in B_M(\rho)} \left| \int_{\mathbb{R}^d\setminus B_{2M}(\rho)} \frac{1}{2}\langle x, \mathrm{Hess}_{\bar{v}}(-y)x\rangle \bar{\omega}_\rho(dy) \right| \\
&\quad + \sup_{x\in B_M(\rho)} \left| \int_{\mathbb{R}^d\setminus B_{2M}(\rho)} \int_0^1 \frac{(1-\theta)^2}{2} \frac{d^3}{d\theta^3}\bar{v}(\theta x - y) d\theta \bar{\omega}_\rho(dy) \right|.
\end{aligned} \tag{17}$$

The first term on the right-hand side is bounded as

$$\sup_{x\in B_M(\rho)} \left| \int_{\mathbb{R}^d\setminus B_{2M}(\rho)} \langle x, \nabla\bar{v}(-y)\rangle \bar{\omega}_\rho(dy) \right| \le M\rho^{\frac{\alpha-d+2}{4\alpha}} \left| \int_{\mathbb{R}^d\setminus B_{2M}(\rho)} \nabla\bar{v}(-y)\bar{\omega}_\rho(dy) \right|.$$

The integral on the right hand side has zero $\tilde{\mathbb{P}}_\rho$-mean and its variance is

$$\begin{aligned}
\widetilde{\mathrm{Var}}_\rho\left(\int_{\mathbb{R}^d\setminus B_{2M}(\rho)} \nabla\bar{v}(-y)\omega(dy) \right) &= \int_{\mathbb{R}^d\setminus B_{2M}(\rho)} |\nabla\bar{v}(-y)|^2 e^{-\rho v(y)} dy \\
&= O\left(\rho^{\frac{d-2\alpha-2}{\alpha}}\right)
\end{aligned}$$

due to Lemma 2-(iii). Hence Chebyshev's inequality yields

$$\tilde{\mathbb{P}}_\rho\left(\sup_{x\in B_M(\rho)}\left|\int_{\mathbb{R}^d\setminus B_{2M}(\rho)}\langle x,\nabla\bar{v}(-y)\rangle\bar{\omega}_\rho(\mathrm{d}y)\right|>\epsilon\rho^{-\frac{\alpha-d+2}{2\alpha}}\right)=O\left(\rho^{-\frac{\alpha+d-2}{2\alpha}}\right)\quad(18)$$

as $\rho\to\infty$. For the second term on the right hand side of (17), we can employ the same argument as above to obtain

$$\tilde{\mathbb{P}}_\rho\left(\sup_{x\in B_M(\rho)}\left|\int_{\mathbb{R}^d\setminus B_{2M}(\rho)}\langle x,\mathrm{Hess}_{\bar{v}}(-y)x\rangle\bar{\omega}_\rho(\mathrm{d}y)\right|>\epsilon\rho^{-\frac{\alpha-d+2}{2\alpha}}\right)=O\left(\rho^{-d/\alpha}\right)$$

Finally, we bound the third term on the right hand side of (17) as

$$\begin{aligned}\sup_{x\in B_M(\rho)}&\left|\int_{\mathbb{R}^d\setminus B_{2M}(\rho)}\int_0^1\frac{(1-\theta)^2}{2}\frac{\mathrm{d}^3}{\mathrm{d}\theta^3}\bar{v}(\theta x-y)\mathrm{d}\theta\bar{\omega}_\rho(\mathrm{d}y)\right|\\&\le\int_{\mathbb{R}^d\setminus B_{2M}(\rho)}\sup_{x\in B_M(\rho),\theta\in[0,1]}\left|\frac{\mathrm{d}^3}{\mathrm{d}\theta^3}\bar{v}(\theta x-y)\right|\bar{\omega}_\rho(\mathrm{d}y)\\&\quad+2\nu\int_{\mathbb{R}^d\setminus B_{2M}(\rho)}\sup_{x\in B_M(\rho),\theta\in[0,1]}\left|\frac{\mathrm{d}^3}{\mathrm{d}\theta^3}\bar{v}(\theta x-y)\right|e^{-\rho\bar{v}(y)}\mathrm{d}y.\end{aligned}\quad(19)$$

One can easily see that the second term is of $o(\rho^{-(\alpha-d+2)/2\alpha})$ by using Lemma 2-(iii). Furthermore, it also follows that the variance of the first term on the right hand side of (19) is of $O(\rho^{-(\alpha+d+6)/2\alpha})$. Then we can conclude by use of Chebyshev's inequality that

$$\begin{aligned}&\tilde{\mathbb{P}}_\rho\left(\int_{\mathbb{R}^d\setminus B_{2M}(\rho)}\sup_{x\in B_M(\rho),\theta\in[0,1]}\left|\frac{\mathrm{d}^3}{\mathrm{d}\theta^3}\bar{v}(\theta x-y)\right|\bar{\omega}_\rho(\mathrm{d}y)>\epsilon\rho^{-\frac{\alpha-d+2}{2\alpha}}\right)\\&=O\left(\rho^{\frac{\alpha-3d+2}{2\alpha}}\right)=o(1)\end{aligned}$$

as $\rho\to\infty$ and the proof of Lemma 3 is completed. □

This lemma implies that for any $\epsilon>0$ one can choose large $M>0$ so that

$$\mathbb{P}\left(\lambda_\omega^1(B_M(\rho))\le\frac{da_1}{\alpha}\rho^{-\frac{\alpha-d}{\alpha}}+(a_2+\epsilon)\rho^{-\frac{\alpha-d+2}{2\alpha}}\right)\ge\exp\left\{-a_1\frac{\alpha-d}{\alpha}\rho^{d/\alpha}\right\}.$$

holds for all sufficiently large ρ. Finally, if one choose ρ to be the solution to

$$\lambda=\frac{da_1}{\alpha}\rho^{-\frac{\alpha-d}{\alpha}}+(a_2+\epsilon)\rho^{-\frac{\alpha-d+2}{2\alpha}},$$

the right-hand side above becomes

$$\exp\left\{-c_1(d,\alpha)\lambda^{-\frac{\alpha}{\alpha-d}}-(c_2(d,\alpha,\kappa)+\epsilon')\lambda^{-\frac{\alpha+d-2}{2(\alpha-d)}}\right\}$$

for some ϵ' which goes to 0 as $\epsilon\to0$, as well as $B_M(\rho)\subset(-N,N)^d$. Therefore we arrive at the desired bound

$$\mathbb{P}\left(\lambda_\omega^1((-N,N)^d)\le\lambda\right)\ge\exp\left\{-c_1(d,\alpha)\lambda^{-\frac{\alpha}{\alpha-d}}-(c_2(d,\alpha,\kappa)+o(1))\lambda^{-\frac{\alpha+d-2}{2(\alpha-d)}}\right\}.$$

as $\lambda\downarrow0$. (Note that $\lambda\downarrow0$ implies $\rho\to\infty$.)

Remark 3. In the case $\alpha < 2$, the estimate of $\tilde{\mathbb{P}}_\rho(E_1)$ in the proof of Lemma 3 requires a local central limit theorem and we also need a finer estimate on $\tilde{\mathbb{P}}_\rho(E_2)$. As a result, only a modified version of Lemma 3 is proved in [4]. See Subsection 4.2 of [4] for detail.

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