We review results on operator monotone functions on $(0,1)$ and Kwong matrices. For details, we refer [10].

1 Operator monotone functions on $(0,1)$

Let $f$ be a real-valued $C^1$ function on an interval $(a, b)$. For $n$ distinct real numbers $t_1, \ldots, t_n \in (a, b)$ a Loewner (or Pick) matrix $L_f(t_1, \ldots, t_n)$ is defined as

$$L_f(t_1, \ldots, t_n) = \left[ \frac{f(t_i) - f(t_j)}{t_i - t_j} \right].$$

In the case where $(a, b) \subseteq (0, \infty)$, a Kwong (or an anti-Loewner) matrix $K_f(t_1, \ldots, t_n)$ is defined by

$$K_f(t_1, \ldots, t_n) = \left[ \frac{f(t_i) + f(t_j)}{t_i + t_j} \right].$$

In this paper we study positive operator monotone functions on $(0,1)$ to continue our preceding studies on Loewner and Kwong matrices [3, 4, 7, 8, 11]. We show that the similar results of Loewner/Kwong matrices do not hold in general. For basic facts on operator monotone functions, we refer the reader to [2, 5, 6].

The following is useful for our study.

Lemma 1.1.

1. $K_f(t_1, \ldots, t_n) + L_f(t_1, \ldots, t_n) = 2 \left[ \frac{t_i f(t_i) - t_j f(t_j)}{t_i^2 - t_j^2} \right]$

   $= 2 C \circ L_{f(t)}(t_1, \ldots, t_n)$

   $= 2 L_{\sqrt{f(t)}}(s_1, \ldots, s_n)$,
where $C$ is given as $C = \begin{bmatrix} \frac{1}{t_i + t_j} \end{bmatrix}$, $\circ$ stands for the Schur product and $s_i = t_i^2$.

(2) \[ K_f(t_1, \ldots, t_n) - L_f(t_1, \ldots, t_n) = 2 \left[ \frac{t_i f(t_j) - t_j f(t_i)}{t_i^2 - t_j^2} \right] \]
\[ = 2D \left[ \frac{t_i / f(t_i) - t_j / f(t_j)}{t_i^2 - t_j^2} \right] D \]
\[ = 2C \circ (DL_{t/f(t)}(t_1, \ldots, t_n)D) \]
\[ = 2DL_{\sqrt{t}/f(\sqrt{t})}(s_1, \ldots, s_n)D, \]
where $C$ and $s_i$ are the same as in (1) and $D$ is given as $D = \text{diag}(f(t_1), \ldots, f(t_n))$.

For our study we prepare the representation of positive operator monotone functions on $(0,1)$.

**Theorem 1.2** A positive operator monotone function $f(s)$ on $(0,1)$ is of the form
\[ f(s) = \int_{[0,1]} \frac{s}{s + \zeta - 2s\zeta} dm(\zeta), \]
where $m$ is a positive measure on $[0,1]$.

For $0 \leq \zeta \leq 1$, put
\[ f_\zeta(s) := \frac{s}{(1 - 2\zeta)s + \zeta} = \frac{s}{s + \zeta - 2s\zeta}. \quad (1.1) \]

**Theorem 1.3** Let $f_\zeta(s)$ be the function in (1.1). Then $s/f_\zeta(s)$ is operator monotone if and only if $\zeta \leq 1/2$.

**Corollary 1.4** Let $f(s)$ be a positive operator monotone function on $(0,1)$ which is of the form
\[ f(s) = \int_{[0,1/2]} f_\zeta(s) dm(\zeta) = \int_{[0,1/2]} \frac{s}{(1 - 2\zeta)s + \zeta} dm(\zeta), \quad (1.2) \]
where $m$ is a positive measure on $[0,1/2]$. Then $s/f(s)$ is operator monotone on $(0,1)$.

The following corresponds to Kwong [9].

**Theorem 1.5** If $f(s)$ is the operator monotone function in (1.2), then all Kwong matrices associated with $f$ are positive semidefinite.
**Theorem 1.6** Let $f_{\zeta}(s)$ be the function in (1.1). Then all Kwong matrices associated with $f_{\zeta}$ are positive semidefinite if and only if $\zeta \leq 1/2$.

The following is a counterpart to Audenaert [1].

**Theorem 1.7** Let $f(s)$ be a positive function on $(0,1)$. If $\sqrt{s}f(\sqrt{s})$ or $\sqrt{s}/f(\sqrt{s})$ is the operator monotone function in (1.2), then all Kwong matrices associated with $f$ are positive semidefinite.

For $0 \leq \zeta \leq 1$, let us consider the function on $(0,1)$

$$g_{\zeta}(s) := \frac{f_{\zeta}(s^{2})}{s} = \frac{s}{(1-2\zeta)s^{2} + \zeta}. \quad (1.3)$$

We note the following:

**Theorem 1.8** Let $g_{\zeta}(s)$ be the function in (1.3). Then $g_{\zeta}(s)$ is operator monotone if and only if $1/2 \leq \zeta$, and all Kwong matrices associated with $g_{\zeta}$ are positive semidefinite if and only if $\zeta \leq 1/2$.

**Proposition 1.9** Let $f(s)$ be the operator monotone function in (1.2). Then for any positive integer $m$,

$$\left[ \frac{f(s_{i})^{m} - f(s_{j})^{m}}{s_{i}^{m} - s_{j}^{m}} \right]$$

are positive semidefinite for all $n$ and $s_{1}, \ldots, s_{n}$ in $(0,1)$.

## 参考文献


