ON SOME INEQUALITIES WITH MATRIX MEANS

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ABSTRACT. Let $0 < m \leq A, B \leq M$ and $\sigma, \tau$ two arbitrary means between harmonic and arithmetic means. Then for every positive unital linear map $\Phi$,

$\Phi(A\sigma B) \leq K(h)\Phi(A\tau B),$

$\Phi(A\sigma B) \leq K(h)\Phi(A)\tau\Phi(B),$

and

$\Phi(A)\sigma\Phi(B) \leq K(h)\Phi(A\tau B),$

$\Phi(A)\sigma\Phi(B) \leq K(h)\Phi(A)\tau\Phi(B),$

where $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$ is the Kantorovich constant.

1. INTRODUCTION

The axiomatic theory for connections and means for pairs of positive matrices have been studied by Kybo and Ando [4]. A binary operation $\sigma$ define on the set of positive definite matrices is called a connection if

(i) $A \leq C, B \leq D$ implies $A\sigma B \leq B\sigma D$;

(ii) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$;

(iii) $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$.

If $I\sigma I = I$, then $\sigma$ is called a mean.

Many authors study matrix inequalities containing means and linear unital positive maps on matrix algebras. Such inequalities are interesting by themselves and have many applications in quantum information theory.

In [2], Lin proved the following Theorem.

Theorem 1.1 ([2]). Let $0 < m \leq A, B \leq M$. Then for every positive unital linear map $\Phi$,

(1) $\Phi^2(A\nabla B) \leq K^2(h)\Phi^2(A\# B),$

and

(2) $\Phi^2(A\nabla B) \leq K^2(h)(\Phi(A)\#\Phi(B))^2,$

where $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$ is the Kantorovich constant.

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It is well-known that the arithmetic mean $\nabla$ is the biggest among symmetric means (see [4]). A natural question is that is the theorem above still true if we replace the biggest means by a smaller one? In this note, we consider such inequalities for two different means with Kantorovich constant. In applications, we give an analogous result of Uchiyama and Yamazaki in [7]. This note is based on preprint [1].

2. MAIN RESULTS

Lemma 2.1. Let $0 < m \leq A, B \leq M$ and $\sigma, \tau$ two arbitrary means between harmonic and arithmetic means. Then for every positive unital linear map $\Phi$,

\[
\Phi(A\sigma B) + Mm\Phi^{-1}(A\tau B) \leq M + m, \tag{3}
\]
and

\[
\Phi(A)\sigma\Phi(B) + Mm\Phi^{-1}(A\tau B) \leq M + m. \tag{4}
\]

Proof. It is easy to see that

\[(M - A)(m - A)A^{-1} \leq 0,
\]
or
\[mMA^{-1} + A \leq M + m.\]

Consequently,
\[\Phi(A) + mM\Phi(A^{-1}) \leq M + m.\]

Similarly,
\[\Phi(B) + mM\Phi(B^{-1}) \leq M + m.\]

Summing up two above inequalities, we get
\[\Phi(A \nabla B) + mM\Phi((A\nabla B)^{-1}) \leq M + m.\]

Besides, from the general theory of matrix means we know that $\nabla \geq \sigma$ and $\tau \geq !$. Hence,
\[
\Phi(A\sigma B) + Mm\Phi^{-1}(A\tau B) \leq \Phi(A\sigma B) + Mm\Phi((A\tau B)^{-1}) \\
\leq \Phi(A \nabla B) + Mm\Phi((A\nabla B)^{-1}) \\
\leq M + m.
\]

By a similar argument, we can get inequality (4) with using the fact that
\[\Phi(A)\sigma\Phi(B) \leq \Phi(A) \nabla \Phi(B) = \Phi(A \nabla B).\]

\[\square\]

The following Proposition is a generalization of Lin’s result mentioned in Introduction.
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Proposition 2.1. Let $0 < m \leq A, B \leq M$ and $\sigma, \tau$ two arbitrary means between harmonic and arithmetic means. Then for every positive unital linear map $\Phi$,

\begin{align*}
\Phi^2(\sigma A B) &\leq K^2(h)\Phi^2(\tau A B), \\
\Phi^2(\sigma A B) &\leq K^2(h) (\Phi(A)\tau \Phi(B))^2, \\
(\Phi(A)\sigma \Phi(B))^2 &\leq K^2(h)\Phi^2(\tau A B),
\end{align*}

and

\begin{align*}
(\Phi(A)\sigma \Phi(B))^2 &\leq K^2(h)(\Phi(A)\tau \Phi(B))^2,
\end{align*}

where $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$ is the Kantorovich constant.

Proof. We prove (2.1). The inequality (2.1) is equivalent to the following

$$
\Phi^{-1}(\tau A B)\Phi^2(\sigma A B)\Phi^{-1}(\tau A B) \leq K^2(h),
$$

or

$$
\|\Phi(\sigma A B)\Phi^{-1}(\tau A B)\| \leq K(h).
$$

On the other hand, it is well known that [5, Theorem 1] for $A, B \geq 0$,

$$
\|AB\| \leq \frac{1}{4}\|A + B\|^2.
$$

So, it is necessary to prove that

$$
\frac{1}{4mM}\|\Phi(\sigma A B) + mM\Phi^{-1}(\tau A B)\|^2 \leq \frac{(M + m)^2}{4Mm},
$$

or,

$$
\|\Phi(\sigma A B) + mM\Phi^{-1}(\tau A B)\| \leq M + m.
$$

The last inequality follows from Lemma 2.1.

Remain inequalities in Proposition can be proved analogously. □

Remark 1. As we mentioned in the proof of Proposition 2.1 that for any positive matrices $A, B$, $\Phi(\sigma A B) \leq \Phi(\nabla A B)$. From that, it can rise a wrong intuition that the proof of Proposition 2.1 can be obtained easily from Theorem 1.1. Unfortunately, the last inequality could not be squared as it was shown in [2, Proposition 1.2].

Theorem 2.1. Let $0 < m \leq A, B \leq M$ and $\sigma, \tau$ are two arbitrary symmetric means. Then for every positive unital linear map $\Phi$,

$$
\Phi(\sigma A B) \leq K(h)\Phi(\tau A B), \\
\Phi(\sigma A B) \leq K(h) (\Phi(A)\tau \Phi(B)), \\
\Phi(A)\sigma \Phi(B) \leq K(h)\Phi(\tau A B),
$$

and

$$
\Phi(A)\sigma \Phi(B) \leq K(h)(\Phi(A)\tau \Phi(B)),
$$

where $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$ is the Kantorovich constant.
Proof. The proof follows from Proposition 2.1 and the fact that the function $f(t) = t^{1/2}$ is operator monotone on $[0, \infty)$.

Corollary 2.1. Let $f, g$ be symmetric operator monotone functions on $[0, \infty)$. Then for any pair $0 < m < M$,

$$
\max\{\frac{f(t)}{g(t)}, \frac{f(t)}{g(t)}\} \leq K(h) = \frac{(m + M)^2}{4mM}, \quad t \in [m, M].
$$

Proof. It is necessary to apply above Theorem for the symmetric matrix means $\sigma$ and $\delta$ corresponding to the functions $f$ and $g$, and definition of matrix means via its representation functions.

Inequality (9) is interesting by itself, and the authors do not know an elementary proof even in the case when $f(t) = \sqrt{t}$.

As an application, now we give a similar result as in [7]. Uchiyama and Yamazaki showed that for an operator monotone function $f$ on $[0, \infty)$ if $f(\lambda B + I)^{-1}f(\lambda A + I) \leq I$ for all sufficiently small $\lambda > 0$, then $f(\lambda A + I) \leq f(\lambda B + I)$ and $A \leq B$. By applying Theorem 2.1 we get a similar result for any symmetric means.

Corollary 2.2. Let $f$ be operator monotone function on $[0, \infty)$ and $\sigma$ an arbitrary mean between harmonic and arithmetic ones. If for a given pair of positive invertible matrices $A, B$,

$$
f(\lambda B + I)^{-1}\sigma f(\lambda A + I) \leq K
$$

for all sufficiently small $\lambda > 0$ (where $K$ is Kantorovich constant), then $f(\lambda A + I) \leq f(\lambda B + I)$ and $A \leq B$.

References

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