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Elementary proofs of operator monotonicity of certain functions

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1 Introduction

A (bounded linear) operator A acting on a Hilbert space H is said to be positive, denoted by $A\geq 0$, if $(Av, v)\geq 0$ for all $v\in H$. The definition of positivity induces the order $A\geq B$ for self-adjoint operators A and B on H. A real-valued function f on $(0, \infty)$ is *operator monotone,* if $0\leq f(A)\leq f(B)$ for operators A and B on H such that $0\leq A\leq B$. As a typical example, $x\mapsto x^{p}$ $(0\leq p\leq 1)$ is an operator monotone function, which is well-known as *Löwner-Heinz theorem* (LH).

T. Furuta ([9], [10]) presented very elementary proofs of operator monotonicity of some functions by using only (LH) repeatedly. Motivated his works, we try to show somewhat elementary proofs of operator monotonicity of certain functions with the auxiliary facts (I)-(III) stated afterward.

Recently, A. Besenyei and D. Petz [2] showed the following two theorems by Löwner's theory:

Theorem 1.1 (cf. [2], [14], [16]). *The function*

$$s_{p}(x)=\left(\frac{p(x-1)}{x^{p}-1}\right)^{\frac{1}{1-p}}, \ p\neq 0,1 \ \left(s_{0}(x)\left(=\lim_{p\rightarrow 0}s_{p}(x)\right)=\frac{x-1}{\log x}, \ s_{1}(x)=\frac{1}{e}x^{\frac{x}{x-1}}\right)$$

is operator monotone if $-2\leq p\leq 2$.

Theorem 1.2 (cf. [2], [4], [6], [7], [8], [11], [12], [14], [16]). *The function*

$$a_{p}(x)=\left(\frac{1+x^{p}}{2}\right)^{\frac{1}{p}}, \ p\neq 0 \quad \left(a_{0}(x)=x^{\frac{1}{2}}\right)$$

is operator monotone if (and only if) $-1\leq p\leq 1$.

Theorem 1.2 is already known well ([4], [6], [7], [8], [11], [12], [16]). We shall give a simple proof of this theorem by using (LH) and some elementary facts.

Now define

$$k_{p}(x)=\frac{p-1}{p}\cdot\frac{x^{p}-1}{x^{p-1}-1}, \ p\neq 0,1 \ \left(k_{0}(x)=\frac{x\log x}{x-1}, \ k_{1}(x)=\frac{x-1}{\log x}\right).$$

Using an integral representation of $k_{p}(x)$, F. Hiai and H. Kosaki [12], by Löwner's theory, showed:

Theorem 1.3 ([8], [12], [9], [4], [6], [14]). *$k_p(x)$ is operator monotone if $-1 \leq p \leq 2$.*

In [8] by J.I. Fujii and Y. Seo, this fact had been shown essentially in virtue of Bendat-Sherman theorem. This fact was also shown in [9] by T. Furuta with a very elementary method, and in [4] by J.I. Fujii ([6] by J.I. Fujii-M. Fujii) with the notion of the integral mean of operator monotone functions.

In this note, starting from the proof of Theorem 1.2, we give somewhat elementary proofs of Theorems 1.1, 1.3 and some other related results. As an application of Theorem 1.1, we give a proof of Petz-Hasegawa theorem [17], an elementary proof of which was presented by T. Furuta [10]. We show some further extensions and applications of those results. Making use of Löwner's integral representation of an operator monotone function f on $(0,\infty)$, we, furthermore, show alternative proofs of operator monotonicity of the function $x \mapsto (f(x^p))^{\frac{1}{p}}$ for $0 < p \leq 1$ ([1],[6],[18]) and that of the function $x \mapsto \frac{x-a}{f(x)-f(a)}$ for $a \geq 0$, that is, the (modified) Bendat-Sherman theorem ([16], [8], [19]).

We assume that all operator monotone functions f are defined on $(0,\infty)$ and strictly positive, and $f(0) = \lim_{x\to 0} f(x)$ if necessary.

2 Preliminaries

By Kubo-Ando theory [15], an operator mean σ is defined as a binary relation of positive operators, satisfying the following properties in common:

(monotonicity)	$A \leq C, B \leq D \Longrightarrow A\sigma B \leq C\sigma D$,
(transformer inequality)	$C(A\sigma B)C \leq (CAC)\sigma(CBC)$,
(normality)	$A\sigma A = A$,
(strong operator semi-continuity)	$A_n \downarrow A, B_n \downarrow B \Longrightarrow A_n\sigma B_n \downarrow A\sigma B$.

As the basic operator means, we define: For $A, B \geq 0$

$$\begin{aligned} &\textit{arithmetic mean}: \quad A \ \nabla\ B = (A+B)/2, \\ &\textit{harmonic mean}: \quad A \ !\ B = \left\{\left(A^{-1}+B^{-1}\right)/2\right\}^{-1} \quad \text{and} \\ &\textit{geometric mean}: \quad A\#B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}. \end{aligned}$$

Sometimes for the definition of an operator mean we must assume operators to be invertible, say, for harmonic or geometric mean. Without any assumption for invertibility every mean is well-defined as the (strong operator) limits of $(A+\varepsilon I)\sigma(B+\varepsilon I)$ as $\varepsilon \downarrow 0$ instead of $A\sigma B$. (I is the identity operator.) For simplicity of discussions, from now on we assume that all positive operators are *invertible*.

Every operator mean σ corresponds a unique operator monotone function, that is, its representing function f_σ which is defined by $f_\sigma(x) = 1\sigma x$. Conversely, if f is an operator monotone function with $f(1) = 1$, then the definition of the operator mean corresponding to f is given by

$$A\sigma B = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}$$

for positive invertible operators A and B.

For our discussion, we use the following basic facts:

(I) For an operator mean σ and for two operator monotone functions g and h, if we define $g\sigma h$ by

$$(g\sigma h)(x) = g(x) f_\sigma\left(\frac{h(x)}{g(x)}\right),$$

then $g\sigma h$ is operator monotone.

(II) For a strictly positive function f on $(0,\infty)$, define $f^\circ(x) := xf(1/x)$ (transpose), $f^*(x) := 1/f(1/x)$ (adjoint) and $f^\perp(x) := x/f(x)$ (dual), then the four functions $f, f^\circ, f^*, f^\perp$ are equivalent to one another with respect to operator monotonicity ([15], [13]).

(III) For a (continuous) path σ_t $(t \in [0,1])$ of operator means, its integral mean $\tilde{\sigma}$ is defined [4] ([6]) for positive operators A and B by

$$A\tilde{\sigma}B = \int_0^1 A\sigma_t B dt.$$

Correspondingly, for a path f_t of operator monotone functions, its integral mean $\tilde{f}$ can be defined by

$$\tilde{f}(x) = \int_0^1 f_t(x) dt,$$

which is an operator monotone function.

3 Main results

To prove Theorem 1.2, we use the following fact: For integers m, n, q, r with $1 \le m \le n$, $n = mq + r$, $0 \le r \le m-1$, and any $i = 1, 2, \ldots, q$,

$$(x^{\frac{m}{n}})^i (1 + x^{\frac{m}{n}})^{\frac{r}{m}} = (1 + x^{\frac{m}{n}})^{\frac{r}{m}} \#_{\frac{i}{q}} x^{\frac{qm}{n}} (1 + x^{\frac{m}{n}})^{\frac{r}{m}} \tag{3.1}$$

holds. Now we show a proof of Theorem 1.2, borrowing Furuta's method, or applying the theorem (LH) repeatedly (say, in [9], [10]):

Proof of Theorem 1.2. It suffices to show the proof when p is rational, $p \neq 0, 1, -1$. First we assume that $0 < p < 1$, so put $p = \frac{m}{n}$, m, n are integers with $(m,n) = 1$, $1 \le m < n$. Then $n = qm + r$ for some $1 \le r \le m-1$, and

$$a_p(x) = \left(\frac{1 + x^{\frac{m}{n}}}{2}\right)^{\frac{n}{m}} = \left(\frac{1}{2}\right)^{\frac{n}{m}} (1 + x^{\frac{m}{n}})^q (1 + x^{\frac{m}{n}})^{\frac{r}{m}} = \left(\frac{1}{2}\right)^{\frac{n}{m}} \sum_{i=0}^{q} {}_qC_i \phi_i(x).$$

Here $\phi_i(x) = x^{\frac{im}{n}}(1+x^{\frac{m}{n}})^{\frac{r}{m}}$. The notations ${}_qC_i$ for $i = 0, 1, \ldots, q$ denote the binomial coefficients, i.e., ${}_qC_i = \frac{q!}{i!(q-i)!}$.

First note that $\phi_0(x) = (1+x^{\frac{m}{n}})^{\frac{r}{m}}$ is, clearly, operator monotone (by (LH)). Next for the last term $\phi_q(x) = x^{\frac{qm}{n}}(1+x^{\frac{m}{n}})^{\frac{r}{m}}$, taking its transpose $\phi_q^\circ(x)$, we have

$$\phi_q^\circ(x) = x\phi_q(x^{-1}) = x \cdot x^{-\frac{qm}{n}}(1+x^{-\frac{m}{n}})^{\frac{r}{m}} = (1+x^{\frac{m}{n}})^{\frac{r}{m}} = \phi_0(x).$$

Hence $\phi_q^\circ(x)$, and $\phi_q(x)$ are both operator monotone by (II) and (LH). Now recall (3.1) stated before. For the general i-th term of the sum, we see:

$$\phi_i(x) = \phi_0(x) \#_{\frac{i}{q}} \phi_q(x).$$

Hence all of $\phi_i(x)$ are operator monotone, so that the proof for $0 < p < 1$ is completed.

For $-1 < p < 0$, notice that

$$a_p^*(x) = \left(\frac{1+x^{-p}}{2}\right)^{-\frac{1}{p}} = a_{-p}(x).$$

Hence we see that a_p^*, or, equivalently, a_p is operator monotone.

A property of an operator monotone function on $(0, \infty)$ is concavity [13] ([15]). If $p > 1$ then we can see $a_p''(x) > 0$, so that $a_p(x)$ is not concave, which implies that the function is not operator monotone. If $p < -1$ then since $a_p^*(x) = a_{-p}(x)$ is not operator monotone, so that $a_p(x)$ is not operator monotone. □

As a slight extension of Theorem 1.2, we can easily see the following:

Lemma 3.1 (cf. [16], [14]). *Let $f_1, \ldots, f_n$ be operator monotone. Then $(\sum_{i=1}^n f_i^p)^{\frac{1}{p}}$ is operator monotone for $-1 \le p \le 1$. In particular, $(\sum_{i=1}^n (\alpha_i + \beta_i x)^p)^{\frac{1}{p}}$ $(\alpha_i, \beta_i \ge 0)$ is operator monotone.*

Theorem 3.2 ([6], [4]). *For $-1 \le p \le 1$, $0 \le s \le 1$, the function*

(3.2)

$$u_{p,s}(x) = \frac{p}{p+s} \cdot \frac{x^{p+s}-1}{x^p - 1},\ p \neq 0, -s\ \left(u_{0,s}(x) = \frac{x^s - 1}{\log x^s},\ u_{-s,s}(x) = \frac{\log x^{-s}}{x^{-s}-1}\right)$$

is operator monotone.

Corollary 3.3 (cf. [9]). *For $-1 \le p \le 1$, the function*

$$u_p(x) = \frac{p(x-1)}{x^p - 1},\ p \neq 0\ \left(u_0(x) = \frac{x-1}{\log x}\right)$$

is operator monotone.

Applying Theorem 3.2, we show:

Proof of Theorem 1.3. For $0 \le p \le 2$, replace p by $p-1$ in (3.2) of Theorem 3.2 and further put $s = 1$. Then we obtain $g_p(x)$, so that $g_p(x)$ is operator monotone. For $-1 \le p < 0$, put $p = -q$, then $0 < q \le 1$, and

$$g_p(x) = \frac{-q-1}{-q} \cdot \frac{x^{-q}-1}{x^{-q-1}-1} = \frac{q+1}{q} \cdot \frac{x(x^q-1)}{x^{q+1}-1}.$$

Hence

$$g_p^{\perp}(x) = \frac{x}{g_p(x)} = \frac{q}{q+1} \cdot \frac{x^{q+1}-1}{x^q-1}.$$

By the previous paragraph (the proof for $0 \le p \le 2$), we then see that $g_p^{\perp}(x)$, and hence $g_p(x)$ are operator monotone. □

Using the above lemma, we also show:

Proof of Theorem 1.1. We may consider the case for $p \ne 1$. We can represent $f_p(x)$ as follows by using the integral:

$$f_p(x) = \left[\int_0^1 (1-t+tx)^{p-1} dt\right]^{\frac{1}{p-1}}.$$

First, we consider the case for $0 \le p \le 2$, or $-1 \le q := p-1 \le 1, (q \ne 0)$. Let

$$I(x) = \int_0^1 (1-t+tx)^{p-1} dt = \int_0^1 (1-t+tx)^q dt.$$

Then as its approximate sum, we have

$$\Sigma_n(x) := \sum_{i=1}^n (1-t_i+t_i x)^q \Delta t_i$$

$$\left(0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1,\ \Delta t_i = t_i - t_{i-1} < \frac{2}{n}\right).$$

From Lemma 3.1, $\{\Sigma_n(x)\}^{\frac{1}{q}}$ is operator monotone. Therefore $f_p(x) = I(x)^{\frac{1}{q}}$, as the limit of $\{\Sigma_n(x)\}^{\frac{1}{q}}$, is operator monotone.

Second, for $-2 \le p \le 0$, we put $q = -p$, so that $0 \le q \le 2$. (We may assume that $p \ne -1$ $(q \ne 1)$.) Then note that $\left(\frac{q(x-1)}{x^q-1}\right)^{\frac{1}{1-q}}$ is operator monotone from the previous argument. We now consider the following two cases:

(i) The case $0 < q < 1$ $(-1 < p < 0)$: We have

$$f_p(x) = f_{-q}(x) = \left(\frac{-q(x-1)}{x^{-q}-1}\right)^{\frac{1}{1+q}} = \left(\frac{q(x-1)x^q}{x^q-1}\right)^{\frac{1}{1+q}} = \left(\frac{q(x-1)}{x^q-1}\right)^{\frac{1}{1-q}} \#_{\frac{2q}{1+q}} x^{\frac{1}{2}}.$$

Hence $f_p(x)$ is operator monotone.

(ii) The case $1 < q \leq 2$ $(-2 \leq p < -1)$: We may show that the adjoint $f_p^*(x) = f_p(x^{-1})^{-1}$ of $f_p(x)$ is operator monotone. We see:

$$f_p^*(x) = f_{-q}^*(x) = \left(\frac{-q(x^{-1}-1)}{x^q-1}\right)^{-\frac{1}{1+q}} = \left(\frac{q(x-1)}{(x^q-1)x}\right)^{-\frac{1}{1+q}} = \left(\frac{q(x-1)}{x^q-1}\right)^{\frac{1}{1-q}} \#_{\frac{2}{1+q}} x^{\frac{1}{2}}.$$

Hence $f_p^*(x)$ is operator monotone. The proof is completed. □

As an application of Theorem 1.1, we show an alternative proof of the following result due to D. Petz and H. Hasegawa [17] ([10]):

Theorem 3.4. *For* $-1 \leq p \leq 2$

$$h_p(x) = p(1-p)\cdot\frac{(x-1)^2}{(x^p-1)(x^{1-p}-1)},\ p \neq 0,1\ \left(h_0(x) = h_1(x) = \frac{x-1}{\log x}\right)$$

is operator monotone.

Proof. It is sufficient to consider the case for $p \neq 0, \pm 1, 2$. First notice that

$$h_p(x) = \left(\frac{p(x-1)}{x^p-1}\right)^{\frac{1}{1-p}} \#_p \left(\frac{(1-p)(x-1)}{x^{1-p}-1}\right)^{\frac{1}{p}}.$$

(Here $\#_p$ also expresses an extended weighted mean if $p > 1$ or $p < 0$.) By Theorem 1.1, both $\left(\frac{p(x-1)}{x^p-1}\right)^{\frac{1}{1-p}}$ and $\left(\frac{(1-p)(x-1)}{x^{1-p}-1}\right)^{\frac{1}{p}}$ are operator monotone. Hence if $0 < p < 1$, then we, at once, see that $h_p(x)$ is operator monotone. Next if $1 < p < 2$, then putting $p = q+1$ $(0 < q < 1)$, we have

$$h_p(x) = h_{q+1}(x) = (-q)(q+1)\cdot\frac{(x-1)^2}{(x^{q+1}-1)(x^{-q}-1)} = \frac{q(q+1)x^q(x-1)^2}{(x^{q+1}-1)(x^q-1)}.$$

Now since $0 < q < 1$, we see that $\left(\frac{q(x-1)}{x^q-1}\right)^{\frac{1}{1-q}}$ is operator monotone by Theorem 1.1. Further, since $1 < q+1 < 2$, we see that

$$(\eta(x) :=)\ \left(\frac{(q+1)(x-1)}{x^{q+1}-1}\right)^{\frac{1}{1-(q+1)}} = \left(\frac{(q+1)(x-1)}{x^{q+1}-1}\right)^{-\frac{1}{q}}$$

is operator monotone by Theorem 1.1, so that its dual $(\eta^\perp(x) =)\ x\cdot\left(\frac{(q+1)(x-1)}{x^{q+1}-1}\right)^{\frac{1}{q}}$ is operator monotone. Hence

$$\left(\frac{q(x-1)}{x^q-1}\right)^{\frac{1}{1-q}} \#_q\ x\cdot\left(\frac{(q+1)(x-1)}{x^{q+1}-1}\right)^{\frac{1}{q}} = h_p(x)$$

is operator monotone. Finally, if $-1 < p < 0$, then putting $p = -q$ $(0 < q < 1)$, we have

$$h_p(x) = \frac{(-q)(q+1)(x-1)^2}{(x^{-q}-1)(x^{1+q}-1)} = \frac{q(q+1)x^q(x-1)^2}{(x^{q+1}-1)(x^q-1)}.$$

Hence $h_p(x)$ has the same expression as in case $1 < p < 2$, so that it is operator monotone. □

Remark 3.5. *For the (extended) weighted geometric mean, the identity*

$$A\#_\alpha(A\#_\beta B) = A\#_{\alpha\beta}B \qquad (\alpha, \beta : \textit{real})$$

holds for positive operators A and B. Using this formula, we can get a slight extension of Theorem 3.4: Let $0 \leq \alpha \leq 1$. Then

$$\left(\frac{p(x-1)}{x^p-1}\right)^{\frac{1}{1-p}} \#_{\alpha p} \left(\frac{(1-p)(x-1)}{x^{1-p}-1}\right)^{\frac{1}{p}}$$

$$= \left(\frac{p(x-1)}{x^p-1}\right)^{\frac{1}{1-p}} \#_{\alpha} \left(\left(\frac{p(x-1)}{x^p-1}\right)^{\frac{1}{1-p}} \#_p \left(\frac{(1-p)(x-1)}{x^{1-p}-1}\right)^{\frac{1}{p}}\right)$$

is operator monotone.

(If $0 < p < 1$, then it is clear that

$$\left(\frac{p(x-1)}{x^p-1}\right)^{\frac{1}{1-p}} \#_{\alpha} \left(\frac{(1-p)(x-1)}{x^{1-p}-1}\right)^{\frac{1}{p}}$$

is also operator monotone.)

4 Extensions and applications of Theorems

Let $\sigma_{a_p}, \sigma_{s_p}$ and σ_{k_p} be the operator means corresponding to the operator monotone functions a_p, s_p and k_p. Then, as state before, for (strictly positive) operator monotone functions f and g (defined on $(0,\infty)$), $f\sigma_{a_p}g = \left(\frac{f^p+g^p}{2}\right)^{\frac{1}{p}}$ is operator monotone for $-1 \leq p \leq 1$. Similarly we can obtain the following facts, which extend Theorems 1.1 and 1.3:

Theorem 4.1 (An extension of Theorem 1.1).

$$f\sigma_{s_p}g = \left(\frac{p(f-g)}{f^p-g^p}\right)^{\frac{1}{1-p}}, \quad p \neq 0, 1,$$

is operator monotone if $-2 \leq p \leq 2$.

Proof. By theorem 1.1, we recall that $s_p(x) = (\frac{p(x-1)}{x^p-1})^{\frac{1}{1-p}}$ is operator monotone for $-2 \leq p \leq 2$. For (strictly positive) operator monotone functions f and g, defined on $(0,\infty)$, we obtain

$$f\sigma_{s_p}g = f \cdot \left(1\sigma_{s_p}\frac{g}{f}\right) = f \cdot \left(\frac{p(\frac{g}{f}-1)}{(\frac{g}{f})^p-1}\right)^{\frac{1}{1-p}} = \left(\frac{p(f-g)}{f^p-g^p}\right)^{\frac{1}{1-p}}.$$

□

Similarly, we can show:

Theorem 4.2 (cf. [8], [12], [9], [4], [6], [14]). *For operator monotone functions f, g $(f \neq g)$, the function*

$$f\sigma_{k_p}g = \frac{p-1}{p}\cdot\frac{f^p - g^p}{f^{p-1} - g^{p-1}}, \quad p \neq 0, 1, \quad \left(f\sigma_{k_0}g = \frac{f(\log f - \log g)}{f - g}, \ f\sigma_{k_1}g = \frac{f-g}{\log f - \log g} \right)$$

is operator monotone if $-1 \le p \le 2$.

Recall that the following fact (Theorem 3.2) was shown, as an extension of Theorem 1.3: For $-1 \le p \le 1$, $0 \le s \le 1$, the function

$$u_{p,s}(x) = \frac{p}{p+s}\cdot\frac{x^{p+s} - 1}{x^p - 1}, \ p \neq 0, -s \ \left(u_{0,s}(x) = \frac{x^s - 1}{\log x^s}, \ u_{-s,s}(x) = \frac{\log x^{-s}}{x^{-s} - 1} \right)$$

is operator monotone.

For the operator mean corresponding to the function $u_{p,s}$, we can obtain the following theorem which is an extension of Theorem 4.2 (and also Theorem 3.2):

Theorem 4.3. *For operator monotone functions f, g $(f \neq g)$, and for $-1 \le p \le 1$, $0 \le s \le 1$, the function*

$$(4.1) \quad f\sigma_{u_{p,s}}g = \frac{p}{p+s}\cdot\frac{f^{p+s} - g^{p+s}}{f^p - g^p}, \ p \neq 0, -s$$

$$\left(f\sigma_{u_{0,s}}g = \frac{f^s - g^s}{\log f^s - \log g^s}, \ f\sigma_{u_{-s,s}}g = \frac{f^{-s} - g^{-s}}{\log f^{-s} - \log g^{-s}} \right)$$ *is operator monotone.*

Example 4.4 (cf. [19, Example 2.4]). *For $-1 \le p \le 1$, $0 \le q - p \le 1$, $p \neq 0$, $q \neq 0$ (and for $a \ge 0$),*

$$\frac{p}{q}\cdot\frac{x^q - a^q}{x^p - a^p}$$ *is operator monotone.*

We can obtain this fact, by putting $f = x$, $g = a$, and $q = p + s$ in (4.1).

As an application of Theorem 1.1, we showed an alternative proof of the following result (Theorem 3.4) due to D. Petz and H. Hasegawa before ([17], [9]): For $-1 \le p \le 2$

$$h_p(x) = \frac{p(1-p)(x-1)^2}{(x^p - 1)(x^{1-p} - 1)}, \ p \neq 0, 1 \ \left(h_0(x) = h_1(x) = \frac{x-1}{\log x} \right)$$

is operator monotone.

As an extension of this fact and an application of Theorem 3.2, though the range of p is reduced, we have:

Theorem 4.5. *If f, g, k, l $(f \neq g, k \neq l)$ are operator monotone functions, then for $0 < p < 1$,*

$$\frac{(f-g)(k-l)}{(f^p - g^p)(k^{1-p} - l^{1-p})}$$ *is operator monotone.*

Proof. Since $f\sigma_{s_p}g$ and $k\sigma_{s_{1-p}}l$ are operator monotone, we see $\frac{1}{p(1-p)}\cdot(f\sigma_{s_p}g)\natural_p(k\sigma_{s_{1-p}}l) = \frac{(f-g)(k-l)}{(f^p-g^p)(k^{1-p}-l^{1-p})}$ is operator monotone. $\square$

Example 4.6 (cf. [19, Theorem 2.7]). *Putting $f = k = x$ and $g = a$, $l = b$ $(a, b \geq 0)$, we see that $\frac{(x-a)(x-b)}{(x^p-a^p)(x^{1-p}-b^{1-p})}$ is operator monotone.*

Further, we have:

Theorem 4.7. *For $-1 \leq p \leq 2$, $a, b \geq 0$*

$$h_p(a,b;x) = \frac{p(1-p)(x-a)(x-b)}{(x^p-a^p)(x^{1-p}-b^{1-p})} \quad \textit{is operator monotone.} \tag{4.2}$$

Proof. We may prove the theorem for $p \neq 0, \pm1, 2$ and $a, b > 0$. For the case $0 < p < 1$, then (4.2) is clear. There remain the two cases:

(i) If $1 < p < 2$, then we put $p = q+1$, so that $0 < q < 1$. We have:

$$\begin{aligned} h_p(a,b;x) = h_{q+1}(a,b;x) &= (-q)(q+1)\cdot\frac{(x-a)(x-b)}{(x^{q+1}-a^{q+1})(x^{-q}-b^{-q})} \\ &= \frac{q(q+1)b^q x^q(x-a)(x-b)}{(x^{q+1}-a^{q+1})(x^q-b^q)}. \end{aligned}$$

Now since $0 < q < 1$, we see that $\left(\frac{q(x-b)}{x^q-b^q}\right)^{\frac{1}{1-q}}$ is operator monotone by Theorem 1.1. Further, since $1 < q+1 < 2$, we see that

$$(\eta(a,b;x) :=) \left(\frac{(q+1)(x-a)}{x^{q+1}-a^{q+1}}\right)^{\frac{1}{1-(q+1)}} = \left(\frac{(q+1)(x-a)}{x^{q+1}-a^{q+1}}\right)^{-\frac{1}{q}}$$

is operator monotone by Theorem 1.1, so that its dual $(\eta^{\perp}(a,b;x) =)\ x\cdot\left(\frac{(q+1)(x-a)}{x^{q+1}-a^{q+1}}\right)^{\frac{1}{q}}$ is operator monotone. Hence

$$\left\{\left(\frac{q(x-b)}{x^q-b^q}\right)^{\frac{1}{1-q}} \natural_q\ x\cdot\left(\frac{(q+1)(x-a)}{x^{q+1}-a^{q+1}}\right)^{\frac{1}{q}}\right\} \times b^q = h_p(a,b;x)$$

is operator monotone.

(ii) If $-1 < p < 0$, then putting $p = -q$, we can similarly prove (4.2). $\square$

The following theorem was shown first by T. Ando [1], next by Y. Nakamura [16], and recently by J.I. Fujii-M. Fujii [5], by T. Sano-S. Tachibana [18]:

Theorem 4.8. *For an operator monotone function f, the function $x \mapsto (f(x^p))^{\frac{1}{p}}$ for $0 < p \leq 1$ is operator monotone.*

Applying Löwner's integral representation of an operator monotone function f:

$$f(x) = \alpha + \beta x + \int_0^\infty \frac{x}{x+\lambda} d\mu(\lambda)$$

with nonnegative α, β and a positive measure μ on $(0, \infty)$, we show an alternative proof to the above theorem. (We show that *the theorem is valid for a wider interval* $-1 \leq p \leq 1,\ p \neq 0$.)

Proof. From the integral representation of $f(x)$, we have:

$$(f(x^p))^{\frac{1}{p}} = \left(\alpha + \beta x^p + \int_0^\infty \frac{x^p}{x^p+\lambda} d\mu(\lambda)\right)^{\frac{1}{p}} \quad (-1 \leq p \leq 1,\ p \neq 0).$$

Note that the integral $\int_0^\infty \frac{x}{x+\lambda} d\mu(\lambda)$ *is approximated* by $J_{\epsilon,E}(x) := \int_\epsilon^E \frac{x}{x+\lambda} d\mu(\lambda)$ for $0 < \epsilon < E < \infty$, so that $\int_0^\infty \frac{x^p}{x^p+\lambda} d\mu(\lambda)$ by $J_{\epsilon,E}(x^p) := \int_\epsilon^E \frac{x^p}{x^p+\lambda} d\mu(\lambda)$. Let

$$\Sigma_{\epsilon,E}(x^p) := \Sigma_{i=1}^n \frac{x^p}{x^p+\lambda_i} m_i \quad (\epsilon = \lambda_0 < \lambda_1 < \ldots < \lambda_n = E)$$

with $m_i = \mu((\lambda_{i-1}, \lambda_i])$ be an approximate sum of $J_{\epsilon,E}(x^p)$. Then we have to show that

$$\phi_n(x) := (\alpha + \beta x^p + \Sigma_{\epsilon,E}(x^p))^{\frac{1}{p}}$$

is operator monotone. Now if we put $f_{-1} = \alpha^{\frac{1}{p}}$, $f_0 = \beta^{\frac{1}{p}} x$ and $f_i = \frac{x}{(x^p+\lambda_i)^{\frac{1}{p}}} m_i^{\frac{1}{p}}$ for $i = 1, \ldots, n$, then all f_i $(-1 \leq i \leq n)$ are operator monotone and $\phi_n(x) = (\sum_{i=-1}^n f_i^p)^{\frac{1}{p}}$, so that from Lemma 3.1, we see that $\phi_n(x)$ is operator monotone. □

Assuming Löwner's integral representation of the operator monotone function again, similarly as above, by using the approximate sum $\Sigma_{\epsilon,E}(x)$ of the integral $J_{\epsilon,E}(x)$, we can show the following (modified) Bendat-Sherman theorem (cf. [16], [8], [19]):

Theorem 4.9. *If f is a (non-constant) operator monotone function, then $F(x) := \frac{x-a}{f(x)-f(a)}$ for $a \geq 0$ is operator monotone.*

Proof. If we put $\psi_n(x) := \alpha + \beta x + \Sigma_{\epsilon,E}(x)$ instead of $f(x)$ in the proof of Theorem 4.8, then we have

$$F_n(x) := \frac{x-a}{\psi_n(x) - \psi_n(a)} = \left(\beta + \sum_{i=1}^n \frac{\lambda_i m_i}{(a+\lambda_i)(x+\lambda_i)}\right)^{-1}.$$

This function is operator monotone since $F_n^{\perp}(x) = \beta x + \sum_{i=1}^n \frac{\lambda_i m_i x}{(a+\lambda_i)(x+\lambda_i)}$ is operator monotone. Hence the limit $F(x)$ of $F_n(x)$ is operator monotone. □

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