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A CONSTRUCTION ON HILBERT REPRESENTATIONS OF QUIVERS

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1. Introduction

This is a joint work with Yasuo Watatani. We aim to study relations between operator theory and Hilbert representations of quivers on infinite dimensional Hilbert spaces. In [4],we studied transitive Hilbert representations of quivers. In [4] we omitted proofs of some statements. In this paper we supply the detailed proof of them.

We shall explain some notions to describe our results. A family $\Gamma=(V, E, s, r)$ is called a quiver if V is a vertex set and E is an edge set and s, r are mappings from E to V such that for $\alpha\in E$, $s(\alpha)\in V$ is the initial point of α and $r(\alpha)\in V$ is the end point of α. A pair (H, f) is called a Hilbert representation of a quiver Γ if $H=(H_{v})_{v\in V}$ is a family of Hilbert spaces and $f=(f_{\alpha})_{\alpha\in E}$ is a family of bounded linear operators f_{α} from $H_{s(\alpha)}$ to $H_{r(\alpha)}$. For Hilbert representations (K, g) and (K', g') of a quiver Γ, we define the direct sum (H, f) by $H_{v}=K_{v}\oplus K_{v}', (v\in V)$, $f_{\alpha}=g_{\alpha}\oplus g_{\alpha}', (\alpha\in E)$. For Hilbert representations (H, f) and (K, g) of Γ, a homomorphism $\phi:(H, f)\rightarrow(K, g)$ is a family $\phi=(\phi_{v})_{v\in V}$ of bounded operators $\phi_{v}:H_{v}\rightarrow K_{v}$ satisfying, for any arrow $\alpha\in E$, $\phi_{r(\alpha)}f_{\alpha}=g_{\alpha}\phi_{s(\alpha)}$. Let $Hom((H, f), (K, g))$ denote the set of homomorphisms from (H, f) to (K, g). Let $End(H, f)$ denote $Hom((H, f), (H, f))$. Let $Idem(H, f)$ be the set of idempotents of $End(H, f)$. Hilbert representations (H, f) and (K, g) of Γ are called isomorphic if there exists an isomorphism $\phi:(H, f)\rightarrow(K, g)$, that is, there exists a family $\phi=(\phi_{v})_{v\in V}$ of bounded invertible operators $\phi_{v}\in B(H_{v}, K_{v})$ such that, for any arrow $\alpha\in E$, $\phi_{r(\alpha)}f_{\alpha}=g_{\alpha}\phi_{s(\alpha)}$. A Hilbert representation (H, f) of Γ is called indecomposable if it is not isomorphic to nontrivial direct sum of Hilbert representations of Γ. A Hilbert representation (H, f) of Γ, is called transitive if $End(H, f)=\mathbb{C}$. We note that a Hilbert representation (H, f) of Γ is indecomposable if and only if $Idem(H, f)=\{0,1\}$. Therefore transitive Hilbert representations are indecomposable.

Gabriel characterized a class of quivers whose indecomposable finite dimensional representations are finite. Gabriel's theorem says that a finite, connected quiver has only finitely many indecomposable finite dimensional representations if and only if the underlying undirected graph is one of Dynkin diagrams A_n, D_n, E_6, E_7, E_8. In [2], we showed a complement of Gabriel's theorem for Hilbert representations. We constructed some examples of indecomposable, infinite-dimensional Hilbert representations of quivers whose underlying undirected graphs are extended Dynkin diagrams $\tilde{A}_n$ $(n \geq 0)$,$\tilde{D}_n$ $(n \geq 4), \tilde{E}_6, \tilde{E}_7$ and $\tilde{E}_8$.

The following quiver K_2 is called the Kronecker quiver.

$$0 \bullet \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \bullet 1 \quad (K_2)$$

In [3], we showed that the Kronecker quiver K_2 has a transitive infinite dimensional Hilbert representation . We also showed that in general, the transitivity property of Hilbert representations is not preserved by orientation changing. We consider the cyclic quiver C_2 with length 2.

$$0 \bullet \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \bullet 1 \quad (C_2)$$

In [4], we showed that there exist no infinite dimensional transitive Hilbert representations of the cyclic quiver C_2 with length 2. Orientation changing may affect the transitivity property. For other quivers whose underlying undirected graphs are $\tilde{D}_n$,$\tilde{E}_6$,$\tilde{E}_7$, $\tilde{E}_8$, we showed the following theorem in [4].

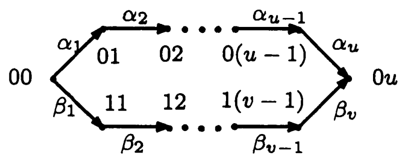
Theorem 1.1. *Let Γ be a finite, connected quiver. If the underlying undirected graph $|\Gamma|$ contains one of the extended Dynkin diagrams $\tilde{D}_n$ $(n \geq 4), \tilde{E}_6, \tilde{E}_7$ and $\tilde{E}_8$, then there exists an infinite-dimensional, transitive, Hilbert representation of Γ.*

In [5], C.M.Ringel considered some correspondences between finite dimensional representations of the Kronecker quiver and finite dimensional representations of the corresponding quivers whose underlying diagrams are extended Dynkin diagrams and P.Donovan and M.R.Freislich [1] also considered correspondences between finite dimensional representations of the quiver whose underlying diagram is $\widetilde{A_5}$ and finite dimensional representations of the corresponding quiver whose underlying diagram is $\widetilde{E_6}$. In [4] we studied isomorphisms between endomorphism algebras of these corresponding Hilbert representations of quivers. In [4] we omitted proofs of some statements for these correspondences. In this paper we supply the detailed proof of some statements in [4] to complete them.

2. Endomorphism algebras of Hilbert representations of quivers

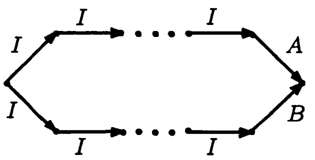
In [4] we investigated correspondences between Hilbert representations of several quivers which is originally given by C.M.Ringel [5] in the finite dimensional case . In [4] we studied isomorphisms between endomorphism algebras of the corresponding Hilbert representations of quivers. In particular the transitivity condition is preserved under these correspondences. In the following we describe our results given in [4] which we omitted the proof . Here we give the detailed proof for some statements for completeness. In [4] we gave the following result about the isomorphism of endomorphism algebras for the Hilbert representations constructed from extended Dynkin diagrams $\widetilde{A_1}$ and $\widetilde{A_n}$. For a subspace L, we denote by $L^m = L \oplus \cdots \oplus L$(m-times).

Theorem 2.1. *Let K_2 be the Kronecker quiver and Γ' be the quiver whose underlying diagram is an $\widetilde{A_n}$ diagram.*



(Γ')

Let (H, f) be a Hilbert representation of K_2 such that $f_\alpha = A, f_\beta = B$ for some $A, B \in B(H_0, H_1)$. For this Hilbert representation (H, f) of K_2, we put the associated Hilbert representation (K, g) of Γ' as follows. $K_{00} = K_{01} = \cdots = K_{0(u-1)} = H_0$, $K_{00} = K_{11} = \cdots = K_{1(v-1)} = H_0$, $K_{0u} = H_1$, $g_{\alpha_1} = \cdots = g_{\alpha_{u-1}} = I$, $g_{\beta_1} = \cdots = g_{\beta_{v-1}} = I$, $g_{\alpha_u} = A, g_{\beta_v} = B$.



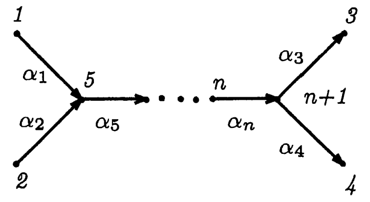
(K, g)

Then $End(H, f)$ is isomorphic to $End(K, g)$.

In [4] we gave the following result about the isomorphism of endomorphism algebras for the Hilbert representations constructed from extended Dynkin diagrams $\widetilde{A_1}$ and $\widetilde{D_n}$.

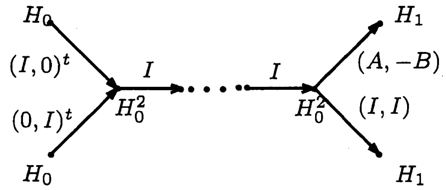
Theorem 2.2. *Let K_2 be the Kronecker quiver and Γ' be the quiver whose underlying diagram is a $\widetilde{D_n}$ diagram.*

$$0 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 1 \quad (K_2)$$



(Γ')

Let (H, f) be a Hilbert representation of K_2 such that $f_\alpha = A, f_\beta = B$ for some $A, B \in B(H_0, H_1)$. For this Hilbert representation (H, f) of K_2, we put the associated Hilbert representation (K, g) of Γ' as follows. $K_1 = K_2 = H_0$, $K_3 = K_4 = H_1$, $K_5 = K_6 = \cdots = K_{n+1} = H_0^2 = H_0 \oplus H_0$, $g_{\alpha_1} = (I, 0)^t, g_{\alpha_2} = (0, I)^t$, $g_{\alpha_3} = (A, -B), g_{\alpha_4} = (I, I), g_{\alpha_5} = g_{\alpha_6} = \cdots = g_{\alpha_n} = I$.

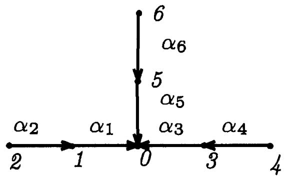


(K, g)

Then $End(H, f)$ is isomorphic to $End(K, g)$.

In [4] we gave the following result about the isomorphism of endomorphism algebras for Hilbert representations constructed from extended Dynkin diagrams $\widetilde{A_1}$ and $\widetilde{E_6}$.

Theorem 2.3. *Let K_2 be the Kronecker quiver and Γ' be the quiver whose underlying diagram is an $\widetilde{E_6}$ diagram.*



(Γ')

Let (H, f) be a Hilbert representation of K_2 such that $f_\alpha = A, f_\beta = B$ for some $A, B \in B(H_0, H_1)$. For this Hilbert representation (H, f) of K_2, we put the associated Hilbert representation (K, g) of Γ' as follows.

Put $K_0 = H_1^3, K_1 = H_1^2, K_2 = H_0, K_3 = H_1^2, K_4 = H_1, K_5 = H_1^2, K_6 = H_1$.

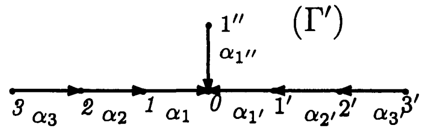
$$g_{\alpha_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \; g_{\alpha_2} = \begin{pmatrix} A \\ B \end{pmatrix}, \; g_{\alpha_3} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \; g_{\alpha_4} = \begin{pmatrix} I \\ I \end{pmatrix},$$

$$g_{\alpha_5} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \; g_{\alpha_6} = \begin{pmatrix} I \\ I \end{pmatrix},$$

Then $End(H, f)$ is isomorphic to $End(K, g)$.

In [4] we gave the following result about the isomorphism of endomorphism algebras for Hilbert representations constructed from extended Dynkin diagrams $\widetilde{A_1}$ and $\widetilde{E_7}$.

Theorem 2.4. *Let K_2 be the Kronecker quiver and Γ' be the quiver whose underlying diagram is an $\widetilde{E_7}$ diagram.*



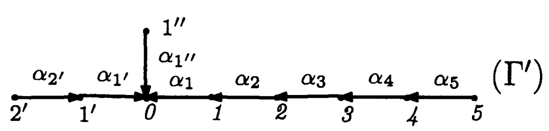
Let (H,f) be a Hilbert representation of K_2 such that $f_\alpha = A, f_\beta = B$ for some $A, B \in B(H_0, H_1)$. Then we put the Hilbert representation of Γ' as follows.

$K_0 = H_1^4, K_1 = H_1^3, K_2 = H_1^2, K_3 = H_0, K_{1'} = H_1^3, K_{2'} = H_1^2, K_{3'} = H_1, K_{1''} = H_1^2$, *and* $g_{\alpha_1} = \begin{pmatrix} 1_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $g_{\alpha_2} = \begin{pmatrix} 1_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, $g_{\alpha_3} = \begin{pmatrix} A \\ B \end{pmatrix}$, $g_{\alpha_{1'}} = \begin{pmatrix} 0 \\ 1_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $g_{\alpha_{2'}} = \begin{pmatrix} 0 \\ 1_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$, $g_{\alpha_{3'}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $g_{\alpha_{1''}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Then $End(H,f)$ is isomorphic to $End(K,g)$.

In [4] we gave the following result about the isomorphism between endomorphism algebras for the Hilbert representations constructed from extended Dynkin diagrams $\widetilde{A_1}$ and $\widetilde{E_8}$. In [4] we omitted proofs of these statements. Here we only give a proof for this last case ($\widetilde{E_8}$ case) because it is complicated. Other cases are similarly proved and so we omit here.

Theorem 2.5. *Let K_2 be the Kronecker quiver and Γ' be the quiver whose underlying diagram is an $\widetilde{E_8}$ diagram.*



Let (H,f) be a Hilbert representation of the Kronecker quiver K_2 such that such that $f_\alpha = A, f_\beta = B$ for some $A, B \in B(H_0, H_1)$. Let (K,g) be the associated Hilbert representation of Γ' as follows. $K_0 = H_1^6, K_1 = H_1^5, K_2 = H_1^4, K_3 = H_1^3, K_4 = H_1^2, K_5 = H_0, K_{1'} = H_1^4, K_{2'} = H_1^2, K_{1''} = H_1^3$, $g_{\alpha_1} = \begin{pmatrix} I_5 \\ 0 \end{pmatrix}$, $g_{\alpha_2} = \begin{pmatrix} I_4 \\ 0 \end{pmatrix}$, $g_{\alpha_3} = \begin{pmatrix} I_3 \\ 100 \end{pmatrix}$, $g_{\alpha_4} =$

$\begin{pmatrix} I_2 \\ 01 \end{pmatrix}$, $g_{\alpha_5} = \begin{pmatrix} B \\ -A \end{pmatrix}$, $g_{\alpha_{1'}} = \begin{pmatrix} 0 \\ 0 \\ I_4 \end{pmatrix}$, $g_{\alpha_{2'}} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$, $g_{\alpha_{1''}} =$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, *then* $End(H,f)$ *is isomorphic to* $End(K,g)$.

Proof. We take $T = (T_0, T_1, T_2, T_3, T_4, T_5, T_{1'}, T_{2'}, T_{1''}) \in End(K,g)$.

First we consider the following diagram.
$$\begin{tikzcd} K_1 \arrow[r, "g_{\alpha_1}"] \arrow[d, "T_1"'] & K_0 \arrow[d, "T_0"] \\ K_1 \arrow[r, "g_{\alpha_1}"'] & K_0 \end{tikzcd}$$

We put $T_0 = \begin{pmatrix} a_1 & \cdots & a_6 \\ b_1 & \cdots & b_6 \\ c_1 & \cdots & c_6 \\ d_1 & \cdots & d_6 \\ e_1 & \cdots & e_6 \\ f_1 & \cdots & f_6 \end{pmatrix}$, $T_1 = \begin{pmatrix} a_1^{(1)} & \cdots & a_5^{(1)} \\ \cdots & \cdots & \cdots \\ e_1^{(1)} & \cdots & e_5^{(1)} \end{pmatrix}$. Since

$T_0 g_{\alpha_1} = g_{\alpha_1} T_1$ and $T_0 \begin{pmatrix} I_5 \\ 0 \end{pmatrix} = \begin{pmatrix} I_5 \\ 0 \end{pmatrix} T_1$, $\begin{pmatrix} a_1 & \cdots & a_6 \\ b_1 & \cdots & b_6 \\ c_1 & \cdots & c_6 \\ d_1 & \cdots & d_6 \\ e_1 & \cdots & e_6 \\ f_1 & \cdots & f_6 \end{pmatrix} \begin{pmatrix} I_5 \\ 0 \end{pmatrix} =$
$\begin{pmatrix} I_5 \\ 0 \end{pmatrix} \begin{pmatrix} a_1^{(1)} & \cdots & a_5^{(1)} \\ \cdots & \cdots & \cdots \\ e_1^{(1)} & \cdots & e_5^{(1)} \end{pmatrix}$. $\begin{pmatrix} a_1 & \cdots & a_5 \\ \cdots & \cdots & \cdots \\ e_1 & \cdots & e_5 \\ f_1 & \cdots & f_5 \end{pmatrix} = \begin{pmatrix} a_1^{(1)} & \cdots & a_5^{(1)} \\ \cdots & \cdots & \cdots \\ e_1^{(1)} & \cdots & e_5^{(1)} \\ 0 & \cdots & 0 \end{pmatrix}$.

Hence $T_0 = \begin{pmatrix} a_1 & \cdots & a_5 & a_6 \\ \cdots & \cdots & \cdots & \cdots \\ e_1 & \cdots & e_5 & e_6 \\ 0 & \cdots & 0 & f_6 \end{pmatrix}$, $T_1 = \begin{pmatrix} a_1 & \cdots & a_5 \\ \cdots & \cdots & \cdots \\ e_1 & \cdots & e_5 \end{pmatrix}$.

Next we consider the following diagram.
$$\begin{tikzcd} K_2 \arrow[r, "g_{\alpha_2}"] \arrow[d, "T_2"'] & K_1 \arrow[d, "T_1"] \\ K_2 \arrow[r, "g_{\alpha_2}"'] & K_1 \end{tikzcd}$$

We put $T_2 = \begin{pmatrix} a_1^{(2)} & \cdots & a_4^{(2)} \\ \cdots & \cdots & \cdots \\ d_1^{(2)} & \cdots & d_4^{(2)} \end{pmatrix}$. Since $T_1 g_{\alpha_2} = g_{\alpha_2} T_2$ and $T_1 \begin{pmatrix} I_4 \\ 0 \end{pmatrix} = \begin{pmatrix} I_4 \\ 0 \end{pmatrix} T_2$, $\begin{pmatrix} a_1 & \cdots & a_5 \\ \cdots & \cdots & \cdots \\ e_1 & \cdots & e_5 \end{pmatrix} \begin{pmatrix} I_4 \\ 0 \end{pmatrix} = \begin{pmatrix} I_4 \\ 0 \end{pmatrix} \begin{pmatrix} a_1^{(2)} & \cdots & a_4^{(2)} \\ \cdots & \cdots & \cdots \\ d_1^{(2)} & \cdots & d_4^{(2)} \end{pmatrix}$.

Hence $\begin{pmatrix} a_1 & \cdots & a_4 \\ \cdots & \cdots & \cdots \\ e_1 & \cdots & e_4 \end{pmatrix} = \begin{pmatrix} a_1^{(2)} & \cdots & a_4^{(2)} \\ \cdots & \cdots & \cdots \\ d_1^{(2)} & \cdots & d_4^{(2)} \\ 0 & \cdots & 0 \end{pmatrix}$. Thus $T_2 = \begin{pmatrix} a_1 & \cdots & a_4 \\ \cdots & \cdots & \cdots \\ d_1 & \cdots & d_4 \end{pmatrix}$

, $T_0 = \begin{pmatrix} a_1 & \cdots & \cdots & a_5 & a_6 \\ b_1 & \cdots & \cdots & b_5 & b_6 \\ c_1 & \cdots & \cdots & c_5 & c_6 \\ d_1 & \cdots & \cdots & d_5 & d_6 \\ 0 & \cdots & 0 & e_5 & e_6 \\ 0 & \cdots & 0 & 0 & f_6 \end{pmatrix}$.

Next we consider the following diagram.

$$\begin{tikzcd} K_3 \arrow[r, "g_{\alpha_3}"] \arrow[d, "T_3"'] & K_2 \arrow[d, "T_2"] \\ K_3 \arrow[r, "g_{\alpha_3}"'] & K_2 \end{tikzcd}$$

Hence $T_2 g_{\alpha_3} = g_{\alpha_3} T_3$ and $T_2 \begin{pmatrix} I_3 \\ 100 \end{pmatrix} = \begin{pmatrix} I_3 \\ 100 \end{pmatrix} T_3$. Put $T_3 = \begin{pmatrix} a_1^{(3)} & a_2^{(3)} & a_3^{(3)} \\ b_1^{(3)} & b_2^{(3)} & b_3^{(3)} \\ c_1^{(3)} & c_2^{(3)} & c_3^{(3)} \end{pmatrix}$. We have $T_2 \begin{pmatrix} I_3 \\ 100 \end{pmatrix} = \begin{pmatrix} a_1 & \cdots & a_4 \\ \cdots & \cdots & \cdots \\ d_1 & \cdots & d_4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_1 + a_4 & a_2 & a_3 \\ b_1 + b_4 & b_2 & b_3 \\ c_1 + c_4 & c_2 & c_3 \\ d_1 + d_4 & d_2 & d_3 \end{pmatrix}$.

And $\begin{pmatrix} I_3 \\ 100 \end{pmatrix} T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1^{(3)} & a_2^{(3)} & a_3^{(3)} \\ b_1^{(3)} & b_2^{(3)} & b_3^{(3)} \\ c_1^{(3)} & c_2^{(3)} & c_3^{(3)} \end{pmatrix} = \begin{pmatrix} a_1^{(3)} & a_2^{(3)} & a_3^{(3)} \\ b_1^{(3)} & b_2^{(3)} & b_3^{(3)} \\ c_1^{(3)} & c_2^{(3)} & c_3^{(3)} \\ a_1^{(3)} & a_2^{(3)} & a_3^{(3)} \end{pmatrix}$.

Thus (Eq1)$a_1 + a_4 = d_1 + d_4, a_2 = d_2, a_3 = d_3$. Hence $T_3 = \begin{pmatrix} a_1 + a_4 & a_2 & a_3 \\ b_1 + b_4 & b_2 & b_3 \\ c_1 + c_4 & c_2 & c_3 \end{pmatrix}$.

Next we consider the following diagram. $\begin{tikzcd} K_4 \arrow[r, "g_{\alpha_4}"] \arrow[d, "T_4"'] & K_3 \arrow[d, "T_3"] \\ K_4 \arrow[r, "g_{\alpha_4}"'] & K_3 \end{tikzcd}$

$T_3 g_{\alpha_4} = g_{\alpha_4} T_4$ and $T_3 \begin{pmatrix} I_2 \\ 01 \end{pmatrix} = \begin{pmatrix} I_2 \\ 01 \end{pmatrix} T_4$. We put $T_4 = \begin{pmatrix} a_1^{(4)} & a_2^{(4)} \\ b_1^{(4)} & b_2^{(4)} \end{pmatrix}$.

$$T_3 \begin{pmatrix} I_2 \\ 01 \end{pmatrix} = \begin{pmatrix} a_1 + a_4 & a_2 & a_3 \\ b_1 + b_4 & b_2 & b_3 \\ c_1 + c_4 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1 + a_4 & a_2 + a_3 \\ b_1 + b_4 & b_2 + b_3 \\ c_1 + c_4 & c_2 + c_3 \end{pmatrix}.$$

$$\begin{pmatrix} I_2 \\ 01 \end{pmatrix} T_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1^{(4)} & a_2^{(4)} \\ b_1^{(4)} & b_2^{(4)} \end{pmatrix} = \begin{pmatrix} a_1^{(4)} & a_2^{(4)} \\ b_1^{(4)} & b_2^{(4)} \\ b_1^{(4)} & b_2^{(4)} \end{pmatrix}.$$

Hence $T_4 = \begin{pmatrix} a_1 + a_4 & a_2 + a_3 \\ b_1 + b_4 & b_2 + b_3 \end{pmatrix}$, (Eq2) $b_1 + b_4 = c_1 + c_4, b_2 + b_3 = c_2 + c_3$.

Next we consider the following diagram. $\begin{tikzcd} K_5 \arrow[r, "g_{\alpha_5}"] \arrow[d, "T_5"'] & K_4 \arrow[d, "T_4"] \\ K_5 \arrow[r, "g_{\alpha_5}"'] & K_4 \end{tikzcd}$

Since $T_4 g_{\alpha_5} = g_{\alpha_5} T_5$ and $T_4 \begin{pmatrix} B \\ -A \end{pmatrix} = \begin{pmatrix} B \\ -A \end{pmatrix} T_5$, $\begin{pmatrix} a_1 + a_4 & a_2 + a_3 \\ b_1 + b_4 & b_2 + b_3 \end{pmatrix} \begin{pmatrix} B \\ -A \end{pmatrix} = \begin{pmatrix} BT_5 \\ -AT_5 \end{pmatrix}$.

Hence (Eq3) $(a_1+a_4)B-(a_2+a_3)A = BT_5$, $(b_1+b_4)B-(b_2+b_3)A = -AT_5$.

Next we consider the following diagram. $\begin{tikzcd} K_{1'} \arrow[r, "g_{\alpha_{1'}}"] \arrow[d, "T_{1'}"'] & K_0 \arrow[d, "T_0"] \\ K_{1'} \arrow[r, "g_{\alpha_{1'}}"'] & K_0 \end{tikzcd}$

$T_0 g_{\alpha_{1'}} = g_{\alpha_{1'}} T_{1'}$ and $T_0 \begin{pmatrix} 0 \\ 0 \\ I_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ I_4 \end{pmatrix} T_{1'}$. We put $T_{1'} = \begin{pmatrix} a_1^{(1')} & \cdots & a_4^{(1')} \\ \cdots & \cdots & \cdots \\ d_1^{(1')} & \cdots & d_4^{(1')} \end{pmatrix}$.

$$T_0\begin{pmatrix}0\\0\\I_4\end{pmatrix}=\begin{pmatrix}a_1&\cdots&\cdots&a_5&a_6\\b_1&\cdots&\cdots&b_5&b_6\\c_1&\cdots&\cdots&c_5&c_6\\d_1&\cdots&\cdots&d_5&d_6\\0&\cdots&0&e_5&e_6\\0&\cdots&0&0&f_6\end{pmatrix}\begin{pmatrix}0\\0\\I_4\end{pmatrix}=\begin{pmatrix}a_3&a_4&a_5&a_6\\b_3&b_4&b_5&b_6\\c_3&c_4&c_5&c_6\\d_3&d_4&d_5&d_6\\0&0&e_5&e_6\\0&0&0&f_6\end{pmatrix}$$

$$=\begin{pmatrix}0\\0\\I_4\end{pmatrix}T_{1'}=\begin{pmatrix}0&\cdots&0\\0&\cdots&0\\a_1^{(1')}&\cdots&a_4^{(1')}\\\cdots&\cdots&\cdots\\d_1^{(1')}&\cdots&d_4^{(1')}\end{pmatrix}.$$ Hence (Eq4)$a_3=a_4=a_5=a_6=0, b_3=b_4=b_5=b_6=0$. $T_{1'}=\begin{pmatrix}a_1^{(1')}&\cdots&a_4^{(1')}\\\cdots&\cdots&\cdots\\d_1^{(1')}&\cdots&d_4^{(1')}\end{pmatrix}=$

$$\begin{pmatrix}c_3&c_4&c_5&c_6\\d_3&d_4&d_5&d_6\\0&0&e_5&e_6\\0&0&0&f_6\end{pmatrix},\ T_0=\begin{pmatrix}a_1&a_2&0&0&0&0\\b_1&b_2&0&0&0&0\\c_1&c_2&c_3&c_4&c_5&c_6\\d_1&d_2&d_3&d_4&d_5&d_6\\0&0&0&0&e_5&e_6\\0&0&0&0&0&f_6\end{pmatrix}.$$ Since $b_1+b_4=c_1+c_4$ and $b_2+b_3=c_2+c_3$, (Eq5)$b_1=c_1+c_4$ and $b_2=c_2+c_3$.

Next we consider the following diagram. Put $T_{2'}=\begin{pmatrix}a_1^{(2')}&a_2^{(2')}\\b_1^{(2')}&b_2^{(2')}\end{pmatrix}$.

$$\begin{tikzcd}
K_{2'} \arrow[r, "g_{\alpha_{2'}}"] \arrow[d, "T_{2'}"'] & K_{1'} \arrow[d, "T_{1'}"] \\
K_{2'} \arrow[r, "g_{\alpha_{2'}}"'] & K_{1'}
\end{tikzcd}$$

We have $T_{1'}g_{\alpha_{2'}}=g_{\alpha_{2'}}T_{2'}$ and $T_{1'}\begin{pmatrix}1&0\\1&0\\0&1\\1&0\end{pmatrix}=\begin{pmatrix}1&0\\1&0\\0&1\\1&0\end{pmatrix}T_{2'}$.

$$\begin{pmatrix}c_3&c_4&c_5&c_6\\d_3&d_4&d_5&d_6\\0&0&e_5&e_6\\0&0&0&f_6\end{pmatrix}\begin{pmatrix}1&0\\1&0\\0&1\\1&0\end{pmatrix}=\begin{pmatrix}c_3+c_4+c_6&c_5\\d_3+d_4+d_6&d_5\\e_6&e_5\\f_6&0\end{pmatrix}$$ and $$\begin{pmatrix}1&0\\1&0\\0&1\\1&0\end{pmatrix}T_{2'}=\begin{pmatrix}1&0\\1&0\\0&1\\1&0\end{pmatrix}\begin{pmatrix}a_1^{(2')}&a_2^{(2')}\\b_1^{(2')}&b_2^{(2')}\end{pmatrix}=\begin{pmatrix}a_1^{(2')}&a_2^{(2')}\\a_1^{(2')}&a_2^{(2')}\\b_1^{(2')}&b_2^{(2')}\\a_1^{(2')}&a_2^{(2')}\end{pmatrix}.$$

Therefore $c_3 + c_4 + c_6 = a_1^{(2')} = d_3 + d_4 + d_6 = f_6$, $a_2^{(2')} = c_5 = d_5 = 0, b_1^{(2')} = e_6, b_2^{(2')} = e_5$. Hence (Eq6)$c_3 + c_4 + c_6 = d_3 + d_4 + d_6 = f_6$, $c_5 = d_5 = 0$. Thus $T_{2'} = \begin{pmatrix} c_3 + c_4 + c_6 & 0 \\ e_6 & e_5 \end{pmatrix}$.

$$T_0 = \begin{pmatrix} a_1 & a_2 & 0 & 0 & 0 & 0 \\ b_1 & b_2 & 0 & 0 & 0 & 0 \\ c_1 & c_2 & c_3 & c_4 & 0 & c_6 \\ d_1 & d_2 & d_3 & d_4 & 0 & d_6 \\ 0 & 0 & 0 & 0 & e_5 & e_6 \\ 0 & 0 & 0 & 0 & 0 & f_6 \end{pmatrix}.$$

Next we consider the following diagram. Put $T_{1''} = \begin{pmatrix} a_1^{(1'')} & a_2^{(1'')} & a_3^{(1'')} \\ b_1^{(1'')} & b_2^{(1'')} & b_3^{(1'')} \\ c_1^{(1'')} & c_2^{(1'')} & c_3^{(1'')} \end{pmatrix}$.

$$\begin{tikzcd} K_{1''} \arrow[r, "g_{\alpha_{1''}}"] \arrow[d, "T_{1''}"'] & K_0 \arrow[d, "T_0"] \\ K_{1''} \arrow[r, "g_{\alpha_{1''}}"'] & K_0 \end{tikzcd}$$

Then $T_0 g_{\alpha_{1''}} = g_{\alpha_{1''}} T_{1''}$ and $T_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} T_{1''}$.

$$T_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & 0 & 0 & 0 & 0 \\ b_1 & b_2 & 0 & 0 & 0 & 0 \\ c_1 & c_2 & c_3 & c_4 & 0 & c_6 \\ d_1 & d_2 & d_3 & d_4 & 0 & d_6 \\ 0 & 0 & 0 & 0 & e_5 & e_6 \\ 0 & 0 & 0 & 0 & 0 & f_6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \\ c_1 & c_2 & c_6 \\ d_1 & d_2 & d_6 \\ 0 & e_5 & e_6 \\ 0 & 0 & f_6 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} T_{1''} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1^{(1'')} & a_2^{(1'')} & a_3^{(1'')} \\ b_1^{(1'')} & b_2^{(1'')} & b_3^{(1'')} \\ c_1^{(1'')} & c_2^{(1'')} & c_3^{(1'')} \end{pmatrix} = \begin{pmatrix} a_1^{(1'')} & a_2^{(1'')} & a_3^{(1'')} \\ b_1^{(1'')} & b_2^{(1'')} & b_3^{(1'')} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_1^{(1'')} & b_2^{(1'')} & b_3^{(1'')} \\ c_1^{(1'')} & c_2^{(1'')} & c_3^{(1'')} \end{pmatrix}.$$

Then we have $a_1 = a_1^{(1'')}, a_2 = a_2^{(1'')}, 0 = a_3^{(1'')}$, $b_1 = b_1^{(1'')}, b_2 = b_2^{(1'')}, 0 = b_3^{(1'')}$, $c_1 = c_2 = c_6 = 0$, $d_1 = d_2 = d_6 = 0$, $= b_1 = b_1^{(1'')} = 0, b_2^{(1'')} = b_2 = e_5, b_3^{(1'')} = e_6$, $c_1^{(1'')} = 0, c_2^{(1'')} = 0, c_3^{(1'')} = f_6$. Hence (Eq7)

$b_1 = c_1 = c_2 = c_6 = d_1 = d_2 = d_6 = e_6 = 0$, $b_2 = e_5$, $T_{1''} = \begin{pmatrix} a_1 & a_2 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & f_6 \end{pmatrix}, T_0 = \begin{pmatrix} a_1 & a_2 & 0 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & c_4 & 0 & 0 \\ 0 & 0 & d_3 & d_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_5 & e_6 \\ 0 & 0 & 0 & 0 & 0 & f_6 \end{pmatrix}$.

The equation $a_2 = 0$ is implied by (Eq1) and (Eq7). The equation $c_4 = 0$ is implied by (Eq5) and (Eq7). The equation $d_3 = 0$ is implied by (Eq1) and (Eq4). The equation $e_6 = 0$ is implied by (Eq7). The equation $a_1 = d_4$ is implied by (Eq1),(Eq4) and (Eq7). The equation $b_2 = c_3 = e_5$ is implied by (Eq2),(Eq4) and (Eq7). The equation $c_3 = f_6$ is implied by (Eq5) and (Eq7). The equation $d_4 = f_6$ is implied by (Eq1) and (Eq7). Hence we have $a_1 = b_2 = c_3 = d_4 = e_5 = f_6$.

Therefore $T_0 = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_1 \end{pmatrix}$.

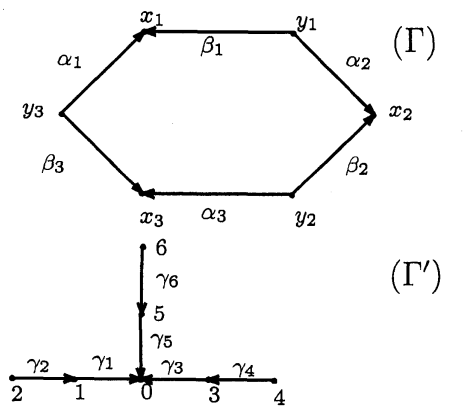
Thus $T_0 = a_1 \oplus a_1 \oplus a_1 \oplus a_1 \oplus a_1 \oplus a_1$. Therefore $T_1 = a_1 \oplus a_1 \oplus a_1 \oplus a_1 \oplus a_1, T_2 = a_1 \oplus a_1 \oplus a_1 \oplus a_1$, $T_3 = \begin{pmatrix} a_1 + a_4 & a_2 & a_3 \\ b_1 + b_4 & b_2 & b_3 \\ c_1 + c_4 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 + 0 & 0 & 0 \\ 0+0 & a_1 & 0 \\ 0+0 & 0 & a_1 \end{pmatrix} = a_1 \oplus a_1 \oplus a_1$,

$T_4 = \begin{pmatrix} a_1 + a_4 & a_2 + a_3 \\ b_1 + b_4 & b_2 + b_3 \end{pmatrix} = \begin{pmatrix} a_1 + 0 & 0+0 \\ 0+0 & b_2 + 0 \end{pmatrix} = a_1 \oplus a_1$, $T_5 \in B(H_0), T_{1'} = \begin{pmatrix} c_3 & c_4 & c_5 & c_6 \\ d_3 & d_4 & d_5 & d_6 \\ 0 & 0 & e_5 & e_6 \\ 0 & 0 & 0 & f_6 \end{pmatrix} = a_1 \oplus a_1 \oplus a_1 \oplus a_1, T_{2'} = \begin{pmatrix} c_3 + c_4 + c_6 & 0 \\ e_6 & e_5 \end{pmatrix} = \begin{pmatrix} c_3 + 0 + 0 & 0 \\ 0 & e_5 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_1 \end{pmatrix} = a_1 \oplus a_1$, $T_{1''} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & f_6 \end{pmatrix} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_1 \end{pmatrix} = a_1 \oplus a_1 \oplus a_1$. Next we consider the relations for A and B. $(a_1 + a_4)B - (a_2 + a_3)A = (a_1 + 0)B - (0+0)A = BT_5$, $(b_1 + b_4)B - (b_2 + b_3)A = (0+0)B - (b_2 + 0)A = -AT_5$. Thus we have $a_1 B = BT_5, b_2 A = AT_5$. Since $a_1 = b_2$, we have the relations $a_1 B = BT_5, a_1 A = AT_5$,where $a_1 \in B(H_1), T_5 \in B(H_0)$. Combining above considerations, we make a correspondence φ from $End(H, f)$ to

$End(K,g)$ by the following. For $S = (S_0, S_1) \in End(H,f)$, we put $\varphi(S) = T = (T_0, T_1, T_2, T_3, T_4, T_5, T_{1'}, T_{2'}, T_{1''})$ by $T_0 = S_1 I_6, T_1 = S_1 I_5, T_2 = S_1 I_4, T_3 = S_1 I_3, T_4 = S_1 I_2, T_5 = S_0, T_{1'} = S_1 I_4, T_{2'} = S_1 I_2, T_{1''} = S_1 I_3$. Since $S = (S_0, S_1) \in End(H,f)$, we have $S_1 A = AS_0, S_1 B = BS_0$. From this, we have $\varphi(S) = T \in End(K,g)$. Next we consider the reverse correspondence ψ from $End(K,g)$ to $End(H,f)$. For $T = (T_0, T_1, T_2, T_3, T_4, T_5, T_{1'}, T_{2'}, T_{1''}) \in End(K,g)$, we put $\psi(T) = S = (S_0, S_1)$ by $S_0 = T_5, S_1 = (T_0)_{1,1}$. Then it is easy to show that $S \in End(H,f)$. Thus $End(H,f)$ are isomorphic to $End(K,g)$ by the relation $\psi\varphi = I$. $\square$

3. $\widetilde{A_5}$ DIAGRAM AND $\widetilde{E_6}$ DIAGRAM

Let Γ be the following quiver whose underlying diagram is an $\widetilde{A_5}$ diagram and Γ' be the following quiver whose underlying diagram is an $\widetilde{E_6}$ diagram.



In [4] we considered the correspondence by P.Donovan and M.R.Freislich [1] and we gave the following result about the isomorphism of endomorphism algebras of the corresponding Hilbert representations. In [4] we omitted the proof. For completeness we give the proof here.

Theorem 3.1. *Let (H,f) be a Hilbert representation of Γ by $H_{x_1} = X_1, H_{x_2} = X_2,\ H_{x_3} = X_3, H_{y_1} = Y_1, H_{y_2} = Y_2, H_{y_3} = Y_3,\ f_{\alpha_1} = A_1, f_{\beta_1} = B_1, f_{\alpha_2} = A_2, f_{\beta_2} = B_2, f_{\alpha_3} = A_3, f_{\beta_3} = B_3$. Let (K,g) be the associated Hilbert representation of Γ' which is given by the following. $K_0 = (X_1 \oplus X_2 \oplus X_3),\ K_1 = (X_1 \oplus X_2),\ K_2 = Y_1,\ K_3 = (X_2 \oplus X_3),\ K_4 = Y_2,\ K_5 = (X_1 \oplus X_3),\ K_6 = Y_3,\ g_{\gamma_1}(x_1, x_2) = (x_1, x_2, 0)$ for $x_1 \in X_1, x_2 \in X_2,\ g_{\gamma_2}(y_1) = (B_1 y_1, A_2 y_1)$ for $y_1 \in Y_1,\ g_{\gamma_3}(x_2, x_3) = (0, x_2, x_3)$ for $x_2 \in X_2, x_3 \in X_3,\ g_{\gamma_4}(y_2) = (B_2 y_2, A_3 y_2)$ for $y_2 \in Y_2,\ g_{\gamma_5}(x_1, x_3) = (x_1, 0, x_3)$ for $x_1 \in X_1, x_3 \in X_3,\ g_{\gamma_6}(y_3) = (A_1 y_3, B_3 y_3)$ for $y_3 \in Y_3$. Then $End(H,f)$ is isomorphic to $End(K,g)$.*

Proof. Take $T \in End(K,g)$. Then T has the form $T = (T_0, ..., T_6)$. Since $T_1 = T_0 \mid_{K_1} = T_0 \mid_{X_1 \oplus X_2}, T_0(X_1 \oplus X_2) \subset X_1 \oplus X_2$. Since $T_3 = T_0 \mid_{K_3} = T_0 \mid_{X_2 \oplus X_3},\ T_0(X_2 \oplus X_3) \subset X_2 \oplus X_3$. Since $T_5 = T_0 \mid_{K_5} =$

$T_0 |_{X_1 \oplus X_3}$, $T_0(X_1 \oplus X_3) \subset X_1 \oplus X_3$. By $(X_1 \oplus X_2) \cap (X_1 \oplus X_3) = X_1$, we have $T_0(X_1) \subset X_1$. By $(X_1 \oplus X_2) \cap (X_2 \oplus X_3) = X_2$, we have $T_0(X_2) \subset X_2$. By $(X_1 \oplus X_3) \cap (X_2 \oplus X_3) = X_3$, we have $T_0(X_3) \subset X_3$. From this, we may assume that $T_0 = R_1 \oplus R_2 \oplus R_3$. where $R_i : X_i \to X_i (i = 1, 2, 3)$. And $T_1 = R_1 \oplus R_2, T_3 = R_2 \oplus R_3, T_5 = R_1 \oplus R_3$. Next we consider the compatibility condition from $T \in End(K, g)$. Since $T_1 g_{\gamma_2}(y_1) = g_{\gamma_2} T_2(y_1)$, $T_1 g_{\gamma_2}(y_1) = (R_1 \oplus R_2)(B_1 y_1, A_2 y_1) = (R_1 B_1 y_1, R_2 A_2 y_1)$ and $g_{\gamma_2} T_2(y_1) = (B_1 T_2 y_1, A_2 T_2 y_1)$, hence $R_1 B_1 = B_1 T_2, R_2 A_2 = A_2 T_2$. Since $T_3 g_{\gamma_4}(y_2) = g_{\gamma_4} T_4(y_2)$, $T_3 g_{\gamma_4}(y_2) = (R_2 \oplus R_3)(B_2 y_2, A_3 y_2) = (R_2 B_2 y_2, R_3 A_3 y_2)$ and $g_{\gamma_4} T_4(y_2) = (B_2 T_4 y_2, A_3 T_4 y_2)$, hence $R_2 B_2 = B_2 T_4, R_3 A_3 = A_3 T_4$. Since $T_5 g_{\gamma_6}(y_3) = g_{\gamma_6} T_6(y_3)$, $T_5 g_{\gamma_6}(y_3) = (R_1 \oplus R_3)(A_1 y_3, B_3 y_3) = (R_1 A_1 y_3, R_3 B_3 y_3)$ and $g_{\gamma_6} T_6(y_3) = (A_1 T_6 y_3, B_3 T_6 y_3)$, hence $R_1 A_1 = A_1 T_6, R_3 B_3 = B_3 T_6$. Therefore $T = (T_0, T_1, \cdots, T_6)$ has the following property. $R_1 B_1 = B_1 T_2, R_2 A_2 = A_2 T_2$. $R_2 B_2 = B_2 T_4$, $R_3 A_3 = A_3 T_4$. $R_1 A_1 = A_1 T_6$, $R_3 B_3 = B_3 T_6$. Next we take the map $Z = (Z_1, Z_2, Z_3, W_1, W_2, W_3) \in End(H, f)$ such that $Z_i \in B(X_i), W_i \in B(Y_i), i = 1, 2, 3$. Compatibility condition is $Z_i B_i = B_i W_i, i = 1, 2, 3$, and $Z_2 A_2 = A_2 W_1, Z_3 A_3 = A_3 W_2, Z_1 A_1 = A_1 W_3$. For $T \in End(K, g)$ and $Z \in End(H, f)$, its relation is $R_i = Z_i, T_2 = W_1, T_6 = W_3, T_4 = W_2$. By this correspondence, $End(K, g))$ is isomorphic to $End(H, f)$, $\square$

References

[1] P.Donovan and M.R.Freislich: Indecomposable representations of graphs and algebras. Carleton Math.Lecture Notes 5(1973).

[2] M.Enomoto and Y.Watatani: Indecomposable representations of quivers on infinite-dimensional Hilbert spaces, Journal of Functional Analysis **256**(2009),959-991.

[3] M.Enomoto and Y.Watatani: Strongly irreducible operators and indecomposable representations of quivers on infinite-dimensional Hilbert spaces, arXiv:1303.2485.

[4] M.Enomoto and Y.Watatani: Transitive Hilbert representations of quivers on infinite-dimensional Hilbert spaces.

[5] C.M.Ringel:The rational invariants of the tame quivers, inventiones math.58,217-239(1980).