1. INTRODUCTION

This is a joint work with Yasuo Watatani. We aim to study relations between operator theory and Hilbert representations of quivers on infinite dimensional Hilbert spaces. In [4], we studied transitive Hilbert representations of quivers. In [4] we omitted proofs of some statements. In this paper we supply the detailed proof of them.

We shall explain some notions to describe our results. A family $\Gamma=(V, E, s, r)$ is called a quiver if $V$ is a vertex set and $E$ is an edge set and $s, r$ are mappings from $E$ to $V$ such that for $\alpha \in E, s(\alpha) \in V$ is the initial point of $\alpha$ and $r(\alpha) \in V$ is the end point of $\alpha$. A pair $(H, f)$ is called a Hilbert representation of a quiver $\Gamma$ if $H = (H_v)_{v \in V}$ is a family of Hilbert spaces and $f = (f_{\alpha})_{\alpha \in E}$ is a family of bounded linear operators $f_{\alpha}$ from $H_{s(\alpha)}$ to $H_{r(\alpha)}$. For Hilbert representations $(K, g)$ and $(K', g')$ of a quiver $\Gamma$, we define the direct sum $(H, f)$ by $H_v = K_v \oplus K'_v, (v \in V), f_{\alpha} = g_{\alpha} \oplus g'_{\alpha}, (\alpha \in E)$. For Hilbert representations $(H, f)$ and $(K, g)$ of $\Gamma$, a homomorphism $\phi : (H, f) \to (K, g)$ is a family $\phi = (\phi_v)_{v \in V}$ of bounded operators $\phi_v : H_v \to K_v$ satisfying, for any arrow $\alpha \in E, \phi_{r(\alpha)}f_{\alpha} = g_{\alpha}\phi_{s(\alpha)}$. Let $\text{Hom}((H, f), (K, g))$ denote the set of homomorphisms from $(H, f)$ to $(K, g)$. Let $\text{End}(H, f)$ denote $\text{Hom}((H, f), (H,f))$. Let $\text{Idem}(H, f)$ be the set of idempotents of $\text{End}(H, f)$. Hilbert representations $(H, f)$ and $(K, g)$ of $\Gamma$ are called isomorphic if there exists an isomorphism $\phi : (H, f) \to (K, g)$, that is, there exists a family $\phi = (\phi_v)_{v \in V}$ of bounded invertible operators $\phi_v \in B(H_v, K_v)$ such that, for any arrow $\alpha \in E, \phi_{r(\alpha)}f_{\alpha} = g_{\alpha}\phi_{s(\alpha)}$. A Hilbert representation $(H, f)$ of $\Gamma$ is called indecomposable if it is not isomorphic to nontrivial direct sum of Hilbert representations of $\Gamma$. A Hilbert representation $(H, f)$ of $\Gamma$, is called transitive if $\text{End}(H, f) = C$. We note that a Hilbert representation $(H, f)$ of $\Gamma$ is indecomposable if and only if $\text{Idem}(H, f) = \{0, 1\}$. Therefore transitive Hilbert representations are indecomposable.

Gabriel characterized a class of quivers whose indecomposable finite dimensional representations are finite. Gabriel's theorem says that a finite, connected quiver has only finitely many indecomposable
finite dimensional representations if and only if the underlying undirected graph is one of Dynkin diagrams $A_n, D_n, E_6, E_7, E_8$. In [2], we showed a complement of Gabriel's theorem for Hilbert representations. We constructed some examples of indecomposable, infinite-dimensional Hilbert representations of quivers whose underlying undirected graphs are extended Dynkin diagrams $A_n$ ($n \geq 0$), $D_n$ ($n \geq 4$), $E_6$, $E_7$ and $E_8$.

The following quiver $K_2$ is called the Kronecker quiver.

\[ 0 \xrightarrow{\alpha} \beta \rightarrow 1 \ (K_2) \]

In [3], we showed that the Kronecker quiver $K_2$ has a transitive infinite dimensional Hilbert representation. We also showed that in general, the transitivity property of Hilbert representations is not preserved by orientation changing. We consider the cyclic quiver $C_2$ with length 2.

\[ 0 \xrightarrow{\alpha} \beta \rightarrow 1 \ (C_2) \]

In [4], we showed that there exist no infinite dimensional transitive Hilbert representations of the cyclic quiver $C_2$ with length 2. Orientation changing may affect the transitivity property. For other quivers whose underlying undirected graphs are $\tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$, we showed the following theorem in [4].

**Theorem 1.1.** Let $\Gamma$ be a finite, connected quiver. If the underlying undirected graph $|\Gamma|$ contains one of the extended Dynkin diagrams $\tilde{D}_n$ ($n \geq 4$), $\tilde{E}_6, \tilde{E}_7$ and $\tilde{E}_8$, then there exists an infinite-dimensional, transitive, Hilbert representation of $\Gamma$.

In [5], C.M.Ringel considered some correspondences between finite dimensional representations of the Kronecker quiver and finite dimensional representations of the corresponding quivers whose underlying diagrams are extended Dynkin diagrams and P.Donovan and M.R.Freislich [1] also considered correspondences between finite dimensional representations of the quiver whose underlying diagram is $\overline{A}_5$ and finite dimensional representations of the corresponding quiver whose underlying diagram is $\overline{E}_6$. In [4] we studied isomorphisms between endomorphism algebras of these corresponding Hilbert representations of quivers. In [4] we omitted proofs of some statements for these correspondences. In this paper we supply the detailed proof of some statements in [4] to complete them.
2. Endomorphism algebras of Hilbert representations of quivers

In [4] we investigated correspondences between Hilbert representations of several quivers which is originally given by C.M.Ringel [5] in the finite dimensional case. In [4] we studied isomorphisms between endomorphism algebras of the corresponding Hilbert representations of quivers. In particular the transitivity condition is preserved under these correspondences. In the following we describe our results given in [4] which we omitted the proof. Here we give the detailed proof for some statements for completeness. In [4] we gave the following result about the isomorphism of endomorphism algebras for the Hilbert representations constructed from extended Dynkin diagrams $\overline{A}_1$ and $D_n$.

**Theorem 2.1.** Let $K_2$ be the Kronecker quiver and $\Gamma'$ be the quiver whose underlying diagram is an $\overline{A}_n$ diagram.

Let $(H, f)$ be a Hilbert representation of $K_2$ such that $f_\alpha = A, f_\beta = B$ for some $A, B \in B(H_0, H_1)$. For this Hilbert representation $(H, f)$ of $K_2$, we put the associated Hilbert representation $(K, g)$ of $\Gamma'$ as follows.

$K_{00} = K_{01} = \cdots = K_{0(u-1)} = H_0, K_{00} = K_{11} = \cdots = K_{1(v-1)} = H_0,$

$K_{0u} = H_1, g_{\alpha_1} = \cdots = g_{\alpha_{u-1}} = I, g_{\beta_1} = \cdots = g_{\beta_{v-1}} = I, g_{\alpha_u} = A, g_{\beta_v} = B.$

$(K, g)$

Then $\text{End}(H, f)$ is isomorphic to $\text{End}(K, g)$.

In [4] we gave the following result about the isomorphism of endomorphism algebras for the Hilbert representations constructed from extended Dynkin diagrams $\overline{A}_1$ and $\overline{D}_n$.

**Theorem 2.2.** Let $K_2$ be the Kronecker quiver and $\Gamma'$ be the quiver whose underlying diagram is a $\overline{D}_n$ diagram.

$(K_2)$
Let \((H, f)\) be a Hilbert representation of \(K_2\) such that \(f_\alpha = A, f_\beta = B\) for some \(A, B \in B(H_0, H_1)\). For this Hilbert representation \((H, f)\) of \(K_2\), we put the associated Hilbert representation \((K, g)\) of \(\Gamma'\) as follows.

\[ K_1 = K_2 = H_0, \quad K_3 = K_4 = H_1, \quad K_5 = K_6 = \cdots = K_{n+1} = H_0^2 = H_0 \oplus H_0, \]

\[ g_{\alpha_1} = (I, 0)^t, \quad g_{\alpha_2} = (0, I)^t, \quad g_{\alpha_3} = (A, -B), \quad g_{\alpha_4} = (I, I), \quad g_{\alpha_5} = \cdots = g_{\alpha_n} = I. \]

Then \(\text{End}(H, f)\) is isomorphic to \(\text{End}(K, g)\).

In [4] we gave the following result about the isomorphism of endomorphism algebras for Hilbert representations constructed from extended Dynkin diagrams \(\widetilde{A}_1\) and \(\widetilde{E}_6\).

**Theorem 2.3.** Let \(K_2\) be the Kronecker quiver and \(\Gamma'\) be the quiver whose underlying diagram is an \(\widetilde{E}_6\) diagram.

Let \((H, f)\) be a Hilbert representation of \(K_2\) such that \(f_\alpha = A, f_\beta = B\) for some \(A, B \in B(H_0, H_1)\). For this Hilbert representation \((H, f)\) of \(K_2\), we put the associated Hilbert representation \((K, g)\) of \(\Gamma'\) as follows.

Put \(K_0 = H_0^3, K_1 = H_1^2, K_2 = H_0, K_3 = H_1^2, K_4 = H_1, K_5 = H_0^2, K_6 = H_1.\)

\[ g_{\alpha_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad g_{\alpha_2} = \begin{pmatrix} A \\ B \end{pmatrix}, \quad g_{\alpha_3} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_{\alpha_4} = \begin{pmatrix} I \\ I \end{pmatrix}, \]

\[ g_{\alpha_5} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_{\alpha_6} = \begin{pmatrix} I \\ I \end{pmatrix}. \]

Then \(\text{End}(H, f)\) is isomorphic to \(\text{End}(K, g)\).

In [4] we gave the following result about the isomorphism of endomorphism algebras for Hilbert representations constructed from extended Dynkin diagrams \(\widetilde{A}_1\) and \(\widetilde{E}_7\).
\textbf{Theorem 2.4.} Let $K_2$ be the Kronecker quiver and $\Gamma'$ be the quiver whose underlying diagram is an $E_7$ diagram.

Let $(H, f)$ be a Hilbert representation of $K_2$ such that $f_\alpha = A, f_\beta = B$ for some $A, B \in B(H_0, H_1)$. Then we put the Hilbert representation of $\Gamma'$ as follows.

\begin{align*}
K_0 &= H_1^4, K_1 = H_1^3, K_2 = H_1^3, K_3 = H_0, K_1' = H_1^3, K_2' = H_1^2, K_3' = H_1, K_0' = H_1^6, K_1' = H_1^5, K_2' = H_1^4, K_3' = H_1^3, K_4' = H_1^2, K_5' = H_1^1, K_1'' = H_1^3, K_2'' = H_1^2, K_3'' = H_1^1, K_4'' = H_1^0.
\end{align*}

Let $(H, f)$ be a Hilbert representation of $K_2$ such that $f_\alpha = A, f_\beta = B$ for some $A, B \in B(H_0, H_1)$. Then we put the Hilbert representation of $\Gamma'$ as follows.

\begin{align*}
K_0 &= H_1^4, K_1 = H_1^3, K_2 = H_1^3, K_3 = H_0, K_1' = H_1^3, K_2' = H_1^2, K_3' = H_1, K_0' = H_1^6, K_1' = H_1^5, K_2' = H_1^4, K_3' = H_1^3, K_4' = H_1^2, K_5' = H_1^1, K_1'' = H_1^3, K_2'' = H_1^2, K_3'' = H_1^1, K_4'' = H_1^0.
\end{align*}

Theorem 2.5. Let $K_2$ be the Kronecker quiver and $\Gamma'$ be the quiver whose underlying diagram is an $E_8$ diagram.

Let $(H, f)$ be a Hilbert representation of the Kronecker quiver $K_2$ such that $f_\alpha = A, f_\beta = B$ for some $A, B \in B(H_0, H_1)$. Let $(K, g)$ be the associated Hilbert representation of $\Gamma'$ as follows. $K_0 = H_1^8, K_1 = H_1^7, K_2 = H_1^6, K_3 = H_1^5, K_4 = H_1^4, K_5 = H_1^3, K_0' = H_1^4, K_1' = H_1^3, K_2' = H_1^2, K_3' = H_1^1, g_{\alpha_1} = (I_5, 0), g_{\alpha_2} = (I_4, 0), g_{\alpha_3} = (I_3, 0), g_{\alpha_4} =
\[
\begin{pmatrix}
  I_2 \\
  0_1
\end{pmatrix},
\ g_{\alpha_5} = \begin{pmatrix}
  B \\
  -A
\end{pmatrix},
\ g_{\alpha_1'} = \begin{pmatrix}
  0 \\
  I_4
\end{pmatrix},
\ g_{\alpha_2'} = \begin{pmatrix}
  1 & 0 \\
  1 & 0 \\
  0 & 1 \\
  1 & 0
\end{pmatrix},
\ g_{\alpha_{1''}} = \begin{pmatrix}
  1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

then \( End(H, f) \) is isomorphic to \( End(K, g) \).

**Proof.** We take \( T = (T_0, T_1, T_2, T_3, T_4, T_5, T_{1'}, T_{2'}, T_{1''}) \in End(K, g) \).

First we consider the following diagram.

\[
\begin{array}{c}
K_1 \\
T_1 \\
K_1 \\
\end{array}
\xrightarrow{g_{\alpha_1}}
\begin{array}{c}
K_0 \\
T_0 \\
K_0 \\
\end{array}
\]

We put \( T_0 = \begin{pmatrix}
  a_1 & \cdots & a_6 \\
  b_1 & \cdots & b_6 \\
  c_1 & \cdots & c_6 \\
  d_1 & \cdots & d_6 \\
  e_1 & \cdots & e_6 \\
  f_1 & \cdots & f_6
\end{pmatrix},
\ T_1 = \begin{pmatrix}
  a_1^{(1)} & \cdots & a_5^{(1)} \\
  \cdots & \cdots & \cdots \\
  e_1^{(1)} & \cdots & e_5^{(1)}
\end{pmatrix}.
\]

Since \( T_0 g_{\alpha_1} = g_{\alpha_1} T_1 \) and \( T_0 \begin{pmatrix}
  I_5 \\
  0
\end{pmatrix} = \begin{pmatrix}
  I_5 \\
  0
\end{pmatrix} T_1, \)

\[
\begin{pmatrix}
  I_5 \\
  0
\end{pmatrix} = \begin{pmatrix}
  a_1^{(1)} & \cdots & a_5^{(1)} \\
  \cdots & \cdots & \cdots \\
  e_1^{(1)} & \cdots & e_5^{(1)}
\end{pmatrix} = \begin{pmatrix}
  a_1 & \cdots & a_5 \\
  b_1 & \cdots & b_6 \\
  c_1 & \cdots & c_6 \\
  d_1 & \cdots & d_6 \\
  e_1 & \cdots & e_6 \\
  f_1 & \cdots & f_6
\end{pmatrix} = \begin{pmatrix}
  a_1 & \cdots & a_6 \\
  b_1 & \cdots & b_6 \\
  c_1 & \cdots & c_6 \\
  d_1 & \cdots & d_6 \\
  e_1 & \cdots & e_6 \\
  f_1 & \cdots & f_6
\end{pmatrix}.
\]

Hence \( T_0 = \begin{pmatrix}
  a_1 & \cdots & a_5 & a_6 \\
  \cdots & \cdots & \cdots & \cdots \\
  e_1 & \cdots & e_5 & e_6 \\
  0 & \cdots & 0 & f_6
\end{pmatrix},
\ T_1 = \begin{pmatrix}
  a_1 & \cdots & a_5 \\
  \cdots & \cdots & \cdots \\
  e_1 & \cdots & e_5
\end{pmatrix}.
\]

Next we consider the following diagram.
We put $T_2 = \begin{pmatrix} a_1^{(2)} & \cdots & a_4^{(2)} \\ \cdots & \cdots & \cdots \\ d_1^{(2)} & \cdots & d_4^{(2)} \end{pmatrix}$. Since $T_1 g_{\alpha_2} = g_{\alpha_2} T_2$ and $T_1 \begin{pmatrix} I_4 \\ 0 \end{pmatrix} = \begin{pmatrix} I_4 \\ 0 \end{pmatrix} T_2$,

\[
\begin{pmatrix} I_4 \\ 0 \end{pmatrix} T_2, \quad \begin{pmatrix} a_1 & \cdots & a_5 \\ \cdots & \cdots & \cdots \\ e_1 & \cdots & e_5 \end{pmatrix} \begin{pmatrix} I_4 \\ 0 \end{pmatrix} = \begin{pmatrix} I_4 \\ 0 \end{pmatrix} \begin{pmatrix} a_1^{(2)} & \cdots & a_4^{(2)} \\ \cdots & \cdots & \cdots \\ d_1^{(2)} & \cdots & d_4^{(2)} \end{pmatrix}.
\]

Hence

\[
\begin{pmatrix} a_1 & \cdots & a_4 \\ \cdots & \cdots & \cdots \\ e_1 & \cdots & e_4 \end{pmatrix} = \begin{pmatrix} a_1^{(2)} & \cdots & a_4^{(2)} \\ \cdots & \cdots & \cdots \\ d_1^{(2)} & \cdots & d_4^{(2)} \end{pmatrix}.
\]

Thus $T_2 = \begin{pmatrix} a_1 & \cdots & a_4 \\ \cdots & \cdots & \cdots \\ d_1 & \cdots & d_4 \end{pmatrix}$.

Next we consider the following diagram.}

Hence $T_2 g_{\alpha_3} = g_{\alpha_3} T_2$ and $T_2 \begin{pmatrix} I_3 \\ 100 \end{pmatrix} = \begin{pmatrix} I_3 \\ 100 \end{pmatrix} T_3$. Put $T_3 = \begin{pmatrix} a_1^{(3)} & a_2^{(3)} & a_3^{(3)} \\ b_1^{(3)} & b_2^{(3)} & b_3^{(3)} \\ c_1^{(3)} & c_2^{(3)} & c_3^{(3)} \end{pmatrix}$. We have $T_2 \begin{pmatrix} I_3 \\ 100 \end{pmatrix} = \begin{pmatrix} a_1 & \cdots & a_4 \\ \cdots & \cdots & \cdots \\ d_1 & \cdots & d_4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. And $T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Thus (Eq1) $a_1 + a_4 = d_1 + d_4, a_2 = d_2, a_3 = d_3$. Hence $T_3 = \begin{pmatrix} a_1 + a_4 & a_2 & a_3 \\ b_1 + b_4 & b_2 & b_3 \\ c_1 + c_4 & c_2 & c_3 \end{pmatrix}$.  

Next we consider the following diagram.

Hence $T_2 g_{\alpha_3} = g_{\alpha_3} T_2$ and $T_2 \begin{pmatrix} I_3 \\ 100 \end{pmatrix} = \begin{pmatrix} I_3 \\ 100 \end{pmatrix} T_3$. Put $T_3 = \begin{pmatrix} a_1^{(3)} & a_2^{(3)} & a_3^{(3)} \\ b_1^{(3)} & b_2^{(3)} & b_3^{(3)} \\ c_1^{(3)} & c_2^{(3)} & c_3^{(3)} \end{pmatrix}$. We have $T_2 \begin{pmatrix} I_3 \\ 100 \end{pmatrix} = \begin{pmatrix} a_1 & \cdots & a_4 \\ \cdots & \cdots & \cdots \\ d_1 & \cdots & d_4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. And $T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Thus (Eq1) $a_1 + a_4 = d_1 + d_4, a_2 = d_2, a_3 = d_3$. Hence $T_3 = \begin{pmatrix} a_1 + a_4 & a_2 & a_3 \\ b_1 + b_4 & b_2 & b_3 \\ c_1 + c_4 & c_2 & c_3 \end{pmatrix}$.
Next we consider the following diagram. 

\[ T_3 g_{\alpha} = g_{\alpha} T_4 \text{ and } T_3 \left( \begin{array}{l} I_2 \\ 01 \end{array} \right) = \left( \begin{array}{l} I_2 \\ 01 \end{array} \right) T_4. \text{ We put } T_4 = \left( \begin{array}{ll} a_1^{(4)} & a_2^{(4)} \\ b_1^{(4)} & b_2^{(4)} \end{array} \right). \]

\[ T_3 \left( \begin{array}{l} I_2 \\ 01 \end{array} \right) = \left( \begin{array}{cc} a_1 + a_4 & a_2 \\ b_1 + b_4 & b_2 \\ c_1 + c_4 & c_2 \end{array} \right) \left( \begin{array}{l} 1 \\ 01 \end{array} \right) = \left( \begin{array}{ll} a_1 + a_4 & a_2 + a_3 \\ b_1 + b_4 & b_2 + b_3 \\ c_1 + c_4 & c_2 + c_3 \end{array} \right). \]

\[ \left( \begin{array}{l} I_2 \\ 01 \end{array} \right) T_4 = \left( \begin{array}{ll} 1 & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{ll} a_1^{(4)} & a_2^{(4)} \\ b_1^{(4)} & b_2^{(4)} \end{array} \right) = \left( \begin{array}{ll} a_1 & a_2^{(4)} \\ b_1^{(4)} & b_2^{(4)} \\ a_2 & a_3 \end{array} \right). \]

Hence \[ T_4 = \left( \begin{array}{ll} a_1 + a_4 & a_2 + a_3 \\ b_1 + b_4 & b_2 + b_3 \end{array} \right), \text{ (Eq2) } b_1 + b_4 = c_1 + c_4, b_2 + b_3 = c_2 + c_3. \]

Next we consider the following diagram.

Since \[ T_4 g_{\alpha_5} = g_{\alpha_5} T_5 \] and \[ T_4 \left( \begin{array}{l} B \\ -A \end{array} \right) = \left( \begin{array}{l} B \\ -A \end{array} \right) T_5, \left( \begin{array}{ll} a_1 + a_4 & a_2 + a_3 \\ b_1 + b_4 & b_2 + b_3 \end{array} \right) \left( \begin{array}{l} B \\ -A \end{array} \right) = \left( \begin{array}{l} BT_5 \\ -AT_5 \end{array} \right). \]

Hence (Eq3) \[ (a_1 + a_4)B - (a_2 + a_3)A = BT_5, (b_1 + b_4)B - (b_2 + b_3)A = -AT_5. \]

Next we consider the following diagram.

\[ T_0 g_{\alpha_1'} = g_{\alpha_1'} T_1' \text{ and } T_0 \left( \begin{array}{l} 0 \\ I_4 \end{array} \right) = \left( \begin{array}{l} 0 \\ I_4 \end{array} \right) T_1'. \text{ We put } T_1' = \left( \begin{array}{llll} a_1^{(1')} & \cdots & a_4^{(1')} \\ \cdots & \cdots & \cdots & \cdots \\ a_1^{(1')} & \cdots & a_4^{(1')} \end{array} \right). \]
\[
T_0 \begin{pmatrix} 0 & 0 \\ I_4 & 0 \end{pmatrix} = \begin{pmatrix} a_1 & \cdots & a_5 & a_6 \\ b_1 & \cdots & b_5 & b_6 \\ c_1 & \cdots & c_5 & c_6 \\ d_1 & \cdots & d_5 & d_6 \\ 0 & \cdots & 0 & e_5 \\ 0 & \cdots & 0 & f_6 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ I_4 & 0 \end{pmatrix} \begin{pmatrix} a_3 & a_4 & a_5 & a_6 \\ b_3 & b_4 & b_5 & b_6 \\ c_3 & c_4 & c_5 & c_6 \\ d_3 & d_4 & d_5 & d_6 \\ 0 & 0 & e_5 & e_6 \\ 0 & 0 & 0 & f_6 \end{pmatrix} = \begin{pmatrix} a_3 & a_4 & a_5 & a_6 \\ b_3 & b_4 & b_5 & b_6 \\ c_3 & c_4 & c_5 & c_6 \\ d_3 & d_4 & d_5 & d_6 \\ 0 & 0 & e_5 & e_6 \\ 0 & 0 & 0 & f_6 \end{pmatrix}.
\]

Hence (Eq 4)

\[
a_3 = a_4 = a_5 = a_6 = 0, b_3 = b_4 = b_5 = b_6 = 0.
\]

\[
T_1' = \begin{pmatrix} a_1' & \cdots & a_4' \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ a_1' & \cdots & a_4' \\ \cdots & \cdots & \cdots \\ d_1' & \cdots & d_4' \end{pmatrix}.
\]

\[
(T_0)\begin{pmatrix} 1 0 1 0 0 0 \\ 0 1 0 0 0 0 \\ 0 0 1 0 0 0 \\ 0 0 0 0 0 0 \\ 0 0 0 0 0 0 \end{pmatrix} = \begin{pmatrix} 1 0 1 0 0 0 \\ 0 1 0 0 0 0 \\ 0 0 1 0 0 0 \\ 0 0 0 0 0 0 \\ 0 0 0 0 0 0 \end{pmatrix} T_2'.
\]

We have $T_1' g_{\alpha_2'} = g_{\alpha_2'} T_2'$ and $T_1' \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} T_2'$. 

Next we consider the following diagram. Put $T_2' = \begin{pmatrix} a_1^{(2')} & a_2^{(2')} \\ b_1^{(2')} & b_2^{(2')} \end{pmatrix}$. 

\[
\begin{pmatrix} c_3 & c_4 & c_5 & c_6 \\ d_3 & d_4 & d_5 & d_6 \\ 0 & 0 & e_5 & e_6 \\ 0 & 0 & 0 & f_6 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c_3 + c_4 + c_6 & c_5 \\ d_3 + d_4 + d_6 & d_5 \\ e_6 & e_5 \\ f_6 & 0 \end{pmatrix} T_2' = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} T_2'.
\]
Therefore \( c_3 + c_4 + c_6 = a_1^{(2')} = d_3 + d_4 + d_6 = f_6, \ a_2^{(2')} = c_5 = d_5 = 0, \ b_1^{(2')} = e_6, \ b_2^{(2')} = e_5. \) Hence (Eq6) \( c_3 + c_4 + c_6 = d_3 + d_4 + d_6 = f_6, \ c_5 = d_5 = 0. \) Thus \( T_{2'} = \begin{pmatrix} c_3 + c_4 + c_6 & 0 \\ e_6 & e_5 \end{pmatrix}. \)

\[
T_0 = \begin{pmatrix}
  a_1 & a_2 & 0 & 0 & 0 & 0 \\
  b_1 & b_2 & 0 & 0 & 0 & 0 \\
  c_1 & c_2 & c_3 & c_4 & 0 & c_6 \\
  d_1 & d_2 & d_3 & d_4 & 0 & d_6 \\
  0 & 0 & 0 & 0 & e_5 & e_6 \\
  0 & 0 & 0 & 0 & 0 & f_6
\end{pmatrix}
\]

Next we consider the following diagram. Put \( T_{1''} = \begin{pmatrix} a_1^{(1'')} & a_2^{(1'')} & a_3^{(1'')} \\ b_1^{(1'')} & b_2^{(1'')} & b_3^{(1'')} \\ c_1^{(1'')} & c_2^{(1'')} & c_3^{(1'')} \end{pmatrix}. \)

\[
K_1'' \xrightarrow{g_{\alpha_{1''}}} K_0 \quad T_{1''} \quad T_0 \quad K_0 \xleftarrow{g_{\alpha_{1''}}} K_1''
\]

Then \( T_0 g_{\alpha_{1''}} = g_{\alpha_{1''}} T_{1''} \) and \( T_0 \begin{pmatrix} 1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0 \end{pmatrix} T_{1''}. \)

\[
T_0 = \begin{pmatrix}
  1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & c_1 & c_2 & c_3 \\
  0 & 0 & 0 & 0 & c_4 & c_5 \\
  0 & 0 & 0 & 0 & 0 & f_6
\end{pmatrix}
\]

Then we have \( a_1 = a_1^{(1''}), a_2 = a_2^{(1'')}, 0 = a_3^{(1'')}, b_1 = b_1^{(1'')}, b_2 = b_2^{(1'')}, 0 = b_3^{(1'')}, c_1 = c_2 = c_6 = 0, d_1 = d_2 = d_6 = 0, = b_1^{(1''')} = 0, b_2^{(1'')} = b_2 = e_5, b_3^{(1''')} = e_6, c_1^{(1''')} = 0, c_2^{(1'')} = 0, c_3^{(1'')} = f_6. \) Hence (Eq7)
\( b_1 = c_1 = c_2 = c_6 = d_1 = d_2 = d_4 = e_6 = 0, \)
\( b_2 = e_5, T_1^\prime = \begin{pmatrix} a_1 & a_2 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 & 0 \\ 0 & 0 & c_3 & c_4 & 0 \\ 0 & 0 & d_3 & d_4 & 0 \\ 0 & 0 & 0 & e_5 & e_6 \\ 0 & 0 & 0 & 0 & f_6 \end{pmatrix}, T_0 = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 0 & a_1 \end{pmatrix}. \]

The equation \( a_2 = 0 \) is implied by (Eq1) and (Eq7). The equation \( c_4 = 0 \) is implied by (Eq5) and (Eq7). The equation \( d_3 = 0 \) is implied by (Eq2) and (Eq4). The equation \( e_6 = 0 \) is implied by (Eq7). The equation \( a_1 = d_4 \) is implied by (Eq1), (Eq4) and (Eq7). The equation \( b_2 = c_3 = e_5 \) is implied by (Eq2), (Eq4) and (Eq7). The equation \( c_3 = f_6 \) is implied by (Eq5) and (Eq7). The equation \( d_4 = f_6 \) is implied by (Eq1) and (Eq7). Hence we have \( a_1 = b_2 = c_3 = d_4 = e_5 = f_6. \)

Therefore \( T_0 = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 0 & a_1 \end{pmatrix}. \)

Thus \( T_0 = a_1 \oplus a_1 \oplus a_1 \oplus a_1 \oplus a_1 \oplus a_1. \) Therefore \( T_1 = a_1 \oplus a_1 \oplus a_1 \oplus a_1 \oplus a_1, T_2 = a_1 \oplus a_1 \oplus a_1 \oplus a_1, T_3 = \begin{pmatrix} a_1 + a_4 & a_2 & a_3 \\ b_1 + b_4 & b_2 & b_3 \\ c_1 + c_4 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 + 0 & 0 & 0 \\ 0 + 0 & a_1 & 0 \\ 0 + 0 & 0 & a_1 \end{pmatrix} = a_1 \oplus a_1 \oplus a_1, T_5 \in B(H_0), T_1^\prime = \begin{pmatrix} c_3 & c_4 & c_5 & c_6 \\ d_3 & d_4 & d_5 & d_6 \\ 0 & 0 & e_5 & e_6 \\ 0 & 0 & 0 & f_6 \end{pmatrix} = a_1 \oplus a_1 \oplus a_1 \oplus a_1, T_2^\prime = \begin{pmatrix} c_3 + c_4 + c_6 \end{pmatrix} \begin{pmatrix} 0 \\ e_5 \end{pmatrix} = a_1 \oplus a_1, T_1^\prime = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & f_6 \end{pmatrix} = a_1 \oplus a_1 \oplus a_1. \)

Next we consider the relations for \( A \) and \( B. \) \( (a_1 + a_4)B - (a_2 + a_3)A = (a_1 + 0)B - (0 + 0)A = BT_5, \)
\( (b_1 + b_4)B - (b_2 + b_3)A = (0 + 0)B - (b_2 + 0)A = -AT_5. \) Thus we have \( a_1 B = BT_5, b_2 A = AT_5. \) Since \( a_1 = b_2, \) we have the relations \( a_1 B = BT_5, a_1 A = AT_5, \) where \( a_1 \in B(H_1), T_5 \in B(H_0). \) Combining above considerations, we make a correspondence \( \varphi \) from \( \text{End}(H, f) \) to
$\text{End}(K, g)$ by the following. For $S = (S_0, S_1) \in \text{End}(H, f)$, we put $\varphi(S) = T = (T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{10})$ by $T_0 = S_1 T_0, T_1 = S_1 T_1, T_2 = S_1 T_2, T_3 = S_1 T_3, T_4 = S_1 T_4, T_5 = S_1 T_5, T_6 = S_1 T_6, T_7 = S_1 T_7, T_8 = S_1 T_8, T_9 = S_1 T_9, T_{10} = S_1 T_{10}$. Since $S = (S_0, S_1) \in \text{End}(H, f)$, we have $S_1 A = AS_0, S_1 B = BS_0$. From this, we have $\varphi(S) = T \in \text{End}(K, g)$. Next we consider the reverse correspondence $\psi$ from $\text{End}(K, g)$ to $\text{End}(H, f)$. For $T = (T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{10}) \in \text{End}(K, g)$, we put $\psi(T) = S = (S_0, S_1)$ by $S_0 = T_5, S_1 = (T_0)_{1,1}$. Then it is easy to show that $S \in \text{End}(H, f)$. Thus $\text{End}(H, f)$ are isomorphic to $\text{End}(K, g)$ by the relation $\psi \varphi = I$.

3. $\widetilde{A}_5$ Diagram and $\widetilde{E}_6$ Diagram

Let $\Gamma$ be the following quiver whose underlying diagram is an $\widetilde{A}_5$ diagram and $\Gamma'$ be the following quiver whose underlying diagram is an $\widetilde{E}_6$ diagram.

![Diagram](image)

In [4] we considered the correspondence by P. Donovan and M. R. Freislich [1] and we gave the following result about the isomorphism of endomorphism algebras of the corresponding Hilbert representations. In [4] we omitted the proof. For completeness we give the proof here.

**Theorem 3.1.** Let $(H, f)$ be a Hilbert representation of $\Gamma$ by $H_{x_1} = X_1, H_{x_2} = X_2, H_{x_3} = X_3, H_{y_1} = Y_1, H_{y_2} = Y_2, H_{y_3} = Y_3$, $f_{\alpha_1} = A_1, f_{\beta_1} = B_1, f_{\alpha_2} = A_2, f_{\beta_2} = B_2, f_{\alpha_3} = A_3, f_{\beta_3} = B_3$. Let $(K, g)$ be the associated Hilbert representation of $\Gamma'$ which is given by the following. $K_0 = (X_1 \oplus X_2 \oplus X_3), K_1 = (X_1 \oplus X_2), K_2 = Y_1, K_3 = (X_2 \oplus X_3), K_4 = Y_2, K_5 = (X_1 \oplus X_3), K_6 = Y_3, g_{\gamma_1}(x_1, x_2) = (x_1, x_2, 0)$ for $x_1 \in X_1, x_2 \in X_2, g_{\gamma_2}(y_1) = (B_1 y_1, A_2 y_1)$ for $y_1 \in Y_1, g_{\gamma_3}(x_1, x_2, x_3) = (0, x_2, x_3)$ for $x_2 \in X_2, x_3 \in X_3, g_{\gamma_4}(y_2) = (B_2 y_2, A_3 y_2)$ for $y_2 \in Y_2, g_{\gamma_5}(x_1, x_3) = (x_1, 0, x_3)$ for $x_1 \in X_1, x_3 \in X_3, g_{\gamma_6}(y_3) = (A_1 y_3, B_3 y_3)$ for $y_3 \in Y_3$. Then $\text{End}(H, f)$ is isomorphic to $\text{End}(K, g)$.

**Proof.** Take $T \in \text{End}(K, g)$. Then $T$ has the form $T = (T_0, \ldots, T_6)$. Since $T_1 = T_0 |_{K_1} = T_0 |_{X_1 \oplus X_2}, T_0 (X_1 \oplus X_2) \subset X_1 \oplus X_2$. Since $T_3 = T_0 |_{K_3} = T_0 |_{X_2 \oplus X_3}, T_0 (X_2 \oplus X_3) \subset X_2 \oplus X_3$. Since $T_5 = T_0 |_{K_5} = T_0 |_{X_1 \oplus X_3}, T_0 (X_1 \oplus X_3) \subset X_1 \oplus X_3$.
$T_0 |_{X_1 \oplus X_3}, T_0(X_1 \oplus X_3) \subset X_1 \oplus X_3$. By $(X_1 \oplus X_2) \cap (X_1 \oplus X_3) = X_1$, we have $T_0(X_1) \subset X_1$. By $(X_1 \oplus X_2) \cap (X_2 \oplus X_3) = X_2$, we have $T_0(X_2) \subset X_2$. By $(X_1 \oplus X_3) \cap (X_2 \oplus X_3) = X_3$, we have $T_0(X_3) \subset X_3$. From this, we may assume that $T_0 = R_1 \oplus R_2 \oplus R_3$. where $R_i : X_i \to X_i (i = 1, 2, 3)$. And $T_1 = R_1 \oplus R_2, T_3 = R_2 \oplus R_3, T_5 = R_1 \oplus R_3$. Next we consider the compatibility condition from $T \in \text{End}(K,g)$. Since $T_1 g_{\gamma_2}(y_1) = g_{\gamma_2} T_2(y_1), T_3 g_{\gamma_4}(y_2) = g_{\gamma_2} T_4(y_2), T_5 g_{\gamma_6}(y_3) = g_{\gamma_2} T_6(y_3)$, hence $R_1 A_1 = A_1 T_6, R_3 A_3 = B_3 T_6$. Therefore $T = (T_0, T_1, \cdots, T_6)$ has the following property. $R_1 B_1 = B_1 T_2, R_2 A_2 = A_2 T_2$. Since $T_3 g_{\gamma_4}(y_2) = g_{\gamma_4} T_4(y_2) = (R_2 B_2 y_2, A_2 T_2 y_2)$, hence $R_2 B_2 = B_2 T_4, R_3 A_3 = A_3 T_4$. Since $T_5 g_{\gamma_6}(y_3) = g_{\gamma_6} T_6(y_3)$, hence $R_1 A_1 = A_1 T_6, R_3 B_3 = B_3 T_6$. For $T \in \text{End}(K,g)$ and $Z \in \text{End}(H,f)$, its relation is $R_i = Z_i, T_2 = W_1, T_6 = W_3, T_4 = W_2$. By this correspondence, $\text{End}(K,g)$ is isomorphic to $\text{End}(H,f)$. 

REFERENCES


