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AN OPERATOR INEQUALITY FOR OPERATOR MONOTONE FUNCTIONS

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ABSTRACT. We give a characterization of convex functions in terms of difference among values of a function. As an application, we propose an estimation of operator monotone functions: If $A>B\geq 0$ and f is operator monotone on $(0, \infty)$, then

$$f(A)-f(B)\geq f(\Vert B\Vert+\epsilon)-f(\Vert B\Vert)>0,$$

where $\epsilon=\Vert(A-B)^{-1}\Vert^{-1}$. As a consequance, we give a refined estimation of Löwner-Heinz inequality under the assumption $A>B\geq 0$. Moreover it gives a simple proof to Furuta's theorem: If $\log A>\log B$ for $A,\ B>0$ and f is operator monotone on $(0, \infty)$, then there exists a $\beta>0$ such that

$$f(A^{\alpha})>f(B^{\alpha}) \text{ for all } 0<\alpha\leq\beta.$$

Finally we discuss strict positivity of Furuta inequality which is a beautiful extension of Löwner-Heinz inequality.

1. INTRODUCTION

For a twice differentiable real-valued function f, its convexity is characterized by $f''\geq 0$. Since there are many non-differentiable convex functions, we consider a characterization of general convex functions. We cannot use the differentiation, but the average rate of change is available. Roughly speaking, we claim that the convexity of a function is characterized by the non-decreasingness of average rate of change. It seems to be natural as a generalization of the condition $f''\geq 0$. Actually it will be formulated as Lemma 1 in the next section.

To explain operator monotone functions, we introduce the operator order $A\geq B$ among selfadjoint operators A, B on a Hilbert space H by $(Ax, x)\geq(Bx, x)$ for all $x\in H$. In particular, A is positive if $A\geq 0$, i.e., $(Ax, x)\geq 0$ for all $x\in H$. Next, a positive operator A is said to be strictly positive, denoted by $A>0$, if $A\geq c$ for some constant $c>0$. So $A>B$ means that $A-B>0$.

A real-valued continuous function f defined on $[0, \infty)$ is called operator monotone if it preserves the operator order, i.e., $f(A)\geq f(B)$ for $A\geq B\geq 0$. One of the most important examples is the power function $t\mapsto t^{p}$ for $0\leq p\leq 1$ (Löwner-Heinz inequality). In general, f is called operator monotone on an interval J if $f(A)\geq f(B)$ for $A\geq B$ whose spectra contained in J. For this, we pose $\log t$ as a fundamental example of an operator monotone function on $(0, \infty)$.

Very recently, Moslehian and Najafi [13] proposed an excellent extension of the Löwner-Heinz inequality as follows:

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Theorem MN. *If* $A > B \geq 0$ *and* $0 < r \leq 1$*, then* $A^r - B^r \geq \|A\|^r - (\|A\| - \epsilon)^r > 0$*, and* $\log A - \log B \geq \log \|A\| - \log(\|A\| - \epsilon) > 0$*, where* $\epsilon = \|(A-B)^{-1}\|^{-1}$.

In this note, we apply our characterization of concave functions and give an improvement and a generalization of Theorem MN (Theorem 5). As another application, we can give a short proof to a recent result due to Furuta [9, Theorem 2.1], which is an operator inequality related to operator monotone functions and chaotic order, i.e., the order defined by $\log A \geq \log B$ among positive invertible operators.

Incidentally, this note is based on our paper [5].

2. A characterization of convex functions

In this section, we propose an elementary characterization of convex functions. We essentially use average rate of change.

Lemma 2.1. *A real valued continuous function* f *on an interval* $J = [a, b)$ *with* $b \in (-\infty, +\infty]$ *is convex (resp. concave) if and only if, for each* $0 < \epsilon < b - a$, $D_\epsilon(t) = f(t+\epsilon) - f(t)$ *is non-decreasing (resp. non-increasing) on* $[a, b-\epsilon)$.

Proof. Suppose that f is convex on J. Take $s, t \in J$ with $s < t$ and $t + \epsilon \in J$. We may assume that $t - s < \epsilon$. Let $y = L(t)$ be the linear function through $(s, f(s))$ and $(s+\epsilon, f(s+\epsilon))$. Then we have

$$L(t) \geq f(t) \text{ and } L(t+\epsilon) \leq f(t+\epsilon)$$

by the convexity of f. Hence it implies that

$$\begin{aligned} D_\epsilon(t) &= f(t+\epsilon) - f(t) \\ &\geq L(t+\epsilon) - L(t) \\ &= L(s+\epsilon) - L(s) \quad \text{by the linearity of } L \\ &= f(s+\epsilon) - f(s) \\ &= D_\epsilon(s), \end{aligned}$$

as desired.

Conversely suppose that $D_\epsilon(t)$ is non-decreasing. Take t, $s \in J$ with $s < t = s + 2\epsilon$. Since $D_\epsilon(s) \leq D_\epsilon(s+\epsilon)$, we have

$$2f\left(\frac{s+t}{2}\right) = 2f(s+\epsilon) \leq f(s+2\epsilon) + f(s) = f(t) + f(s).$$

So f is convex. □

Corollary 2.2. *If* f *is strictly increasing and concave on an interval* $[a, b+\delta]$ *in* $\mathbb{R}$ *for some* $\delta > 0$*, then for each* $0 < \epsilon \leq \delta$, $D_\epsilon(t) \geq D_\epsilon(b) > 0$ *for all* $t \in [a, b]$.

Remark 2.3. *Analogous argument on convexity of functions as above has been done in* [12, page 2].

3. Applications to Operator monotone functions

As an application of Corollary 2.2, we give an estimation of operator monotone functions.

Lemma 3.1. *If* f *is non-constant and operator monotone on the interval* $\mathbb{R}_+ = [0, \infty)$*, then* f *is strictly increasing.*

Proof. First of all, we note that f is non-decreasing. Next we suppose that $f'(c) = 0$ for some $c > 0$. Noting that the Löwner matrix

$$\begin{pmatrix} f'(c) & f^{[1]}(c,d) \\ f^{[1]}(d,c) & f'(d) \end{pmatrix}$$

is positive semidefinite for any $d > 0$ by the operator monotonicity of f, where $f^{[1]}(c,d) = \frac{f(c)-f(d)}{c-d}$ is the devided difference.

Therefore its determinant is nonnegative, so that $f^{[1]}(c,d) = 0$ for any $d > 0$. This means that f is constant, which is a contradiction. Consequently we have $f' > 0$. □

Lemma 3.2. *If $C \geq 0$ and f is a concave and strictly increasing function on an interval $[a,d)$ containing the spectrum of C, then for each $0 < \epsilon < d - \|C\|$, $f(C+\epsilon) \geq f(C) + D_\epsilon(\|C\|)$.*

Proof. We first note that for a given $0 < \epsilon < d - \|C\|$, we can take $c > 0$ satisfying $0 < c < d$ and $\epsilon < c - \|C\|$. Applying Corollary 2.2 to $b = \|C\|$ and $\delta = c - \|C\|$, it follows that

$$f(C+\epsilon) - f(C) \geq D_\epsilon(\|C\|).$$

□

We here give a precise estimation of [9, Theorem 2.1] and [12, Proposition 2.2], cf. [13].

Theorem 3.3. *If $A > B \geq 0$ and f is non-constant operator monotone on $[0,\infty)$, then $f(A) - f(B) \geq f(\|B\| + \epsilon) - f(\|B\|) > 0$, where $\epsilon = \|(A-B)^{-1}\|^{-1}$.*

Proof. Since $A \geq B + \epsilon$ for $\epsilon = \|(A-B)^{-1}\|^{-1} > 0$, we have

$$f(A) \geq f(B+\epsilon).$$

Furthermore Lemmas 3.1 and 3.2 imply that

$$f(B+\epsilon) \geq f(B) + D_\epsilon(\|B\|).$$

Hence we have

$$f(A) - f(B) \geq D_\epsilon(\|B\|) = f(\|B\| + \epsilon) - f(\|B\|) > 0.$$

□

As a consequence, we have an improvement of the estimation due to Moslehian and Najafi [13]:

Corollary 3.4. *If $A > B \geq 0$ and $0 < r \leq 1$, then $A^r - B^r \geq (\|B\| + \epsilon)^r - (\|B\|)^r > 0$, and $\log A - \log B \geq \log(\|B\| + \epsilon) - \log\|B\| > 0$, where $\epsilon = \|(A-B)^{-1}\|^{-1}$.*

Remark 3.5. *We note that Corollary 3.4 actually improves Theorem MN. Since $\|A\| - (\|A\| - \epsilon) = \epsilon = (\|B\| + \epsilon) - \|B\|$ and the function $t \mapsto t^r$ is strictly concave, it follows that*

$$\|A\|^r - (\|A\| - \epsilon)^r \leq (\|B\| + \epsilon)^r - \|B\|^r.$$

We here pose an example:

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $A - B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \geq 1$ *and so* $\epsilon = 1$. *Hence we have*

$$\|A\|^r - (\|A\| - \epsilon)^r = 4^r - 3^r < (\|B\| + \epsilon)^r - \|B\|^r = 3^r - 2^r.$$

Now Theorem 3.3 can be regarded as a difference version. So we give a ratio version of it. It is obtained by Theorem 3.3 itself:

Corollary 3.6. *If* $A > B > 0$ *and* f *is non-constant operator monotone on* $(0, \infty)$, *then*

$$f(B)^{-\frac{1}{2}} f(A) f(B)^{-\frac{1}{2}} \geq 1 + (f(\|B\| + \epsilon) - f(\|B\|))\|f(B)\|^{-1},$$

where $\epsilon = \|(A - B)^{-1}\|^{-1}$.

Proof. Put $\delta = f(\|B\| + \epsilon) - f(\|B\|)$. It follows from Theorem 3.3 that

$$\begin{aligned} f(B)^{-\frac{1}{2}} f(A) f(B)^{-\frac{1}{2}} &\geq f(B)^{-\frac{1}{2}}(f(B) + \delta) f(B)^{-\frac{1}{2}} \\ &= 1 + \delta f(B)^{-1} \geq 1 + \delta \|f(B)\|^{-1}. \end{aligned}$$

□

As another application of Theorem 3.3, we need the chaotic order: For $A > 0$, we can define the selfadjoint operator $\log A$. So a weaker order than the operator order appears by $\log A \geq \log B$ for $A, B > 0$. We call it the chaotic order. The chaotic order plays an substantial role in operator inequalities. Among others, it brightens the Furuta inequality [7], [3], [4], [1], [6], [10] and recent development of Karcher mean theory [16].

Now we give a simple and elementary proof to the following recent theorem [9, Theorem 2.1] due to Furuta, in which we don't use any integral representation of operator monotone functions.

Theorem 3.7. *If* $\log A > \log B$ *for* A, $B > 0$ *and* f *is operator monotone on* $(0, \infty)$, *then there exists* $\beta > 0$ *such that*

$$f(A^\alpha) > f(B^\alpha) \quad \text{for all } 0 < \alpha \leq \beta.$$

Proof. Since $\log A > \log B$, it is known that there exists $\beta > 0$ such that

$$A^\alpha > B^\alpha \quad \text{for all } 0 < \alpha \leq \beta.$$

Therefore it follows from Theorem 3.3 that, for each fixed $\alpha \in (0, \beta]$,

$$f(A^\alpha) > f(B^\alpha),$$

as desired. □

4. Furuta inequality.

First of all, we cite the Furuta inequality (FI) in [7], see also [2], [8], [11] and [14] for the best possibility of it.

The Furuta inequality. If $A \geq B \geq 0$, then for each $r \geq 0$,

$$A^{\frac{p+r}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

holds for $p \geq 0$, $q \geq 1$ with

$$(1 + r)q \geq p + r.$$

To extend Corollary 3.4, we remark that the case $r = 0$ in (FI) is just the Löwner–Heinz inequality. Now we introduce a constant $k(b, m, p, q, r)$ for $b, m, p, q, r \geq 0$ by

$$k(b, m, p, q, r) = (b + m)^{\frac{p+r}{q} - r} - b^{\frac{p+r}{q} - r}.$$

As a matter of fact, we have an extension of Corollary 3.4 in the form of Furuta inequality as follows:

Theorem 4.1. *Let A and B be invertible positive operators with $A - B \geq m > 0$. Then for $0 < r \leq 1$,*

$$A^{\frac{p+r}{q}} - (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq k(\|B\|, m, p, q, r)(\|B^{-1}\|^{-1} + m)^r$$

holds for $p \geq 0,\ q \geq 1$ with $(1 + r)q \geq p + r \geq qr$.

Proof. We note that $q \geq 1$ and $(1 + r)q \geq p + r \geq qr$ assure the exponent $\frac{p+r}{q} - r$ in the constant k belongs to $[0, 1]$. Since $0 \leq r \leq 1$, it follows from Theorem B that

$$\begin{aligned}
A^{\frac{p+r}{q}} - (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} &= A^{\frac{p+r}{q}} - A^{\frac{r}{2}} B^{\frac{p}{2}} (B^{\frac{p}{2}} A^r B^{\frac{p}{2}})^{\frac{1}{q} - 1} B^{\frac{p}{2}} A^{\frac{r}{2}} \\
&= A^{\frac{p+r}{q}} - A^{\frac{r}{2}} B^{\frac{p}{2}} (B^{\frac{-p}{2}} A^{-r} B^{\frac{-p}{2}})^{1 - \frac{1}{q}} B^{\frac{p}{2}} A^{\frac{r}{2}} \\
&\geq A^{\frac{p+r}{q}} - A^{\frac{r}{2}} B^{\frac{p}{2}} (B^{-\frac{p}{2}} B^{-r} B^{-\frac{p}{2}})^{1 - \frac{1}{q}} B^{\frac{p}{2}} A^{\frac{r}{2}} \\
&= A^{\frac{p+r}{q}} - A^{\frac{r}{2}} B^{p - (p+r)(1 - \frac{1}{q})} A^{\frac{r}{2}} \\
&= A^{\frac{r}{2}} (A^{\frac{p+r}{q} - r} - B^{\frac{p+r}{q} - r}) A^{\frac{r}{2}} \\
&\geq k(\|B\|, m, p, q, r) A^r \\
&\geq k(\|B\|, m, p, q, r)(B + m)^r \\
&\geq k(\|B\|, m, p, q, r)(\|B^{-1}\|^{-1} + m)^r.
\end{aligned}$$

□

For a general case on r, we have the following estimation of Furuta inequality by repeating method as in a proof of Furuta inequality.

Theorem 4.2. *Let A and B be invertible positive operators with $A - B \geq m > 0$ and $r = n + s$ for some natural number n and $0 < s \leq 1$. Then*

$$A^{\frac{p+r}{q}} - (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq k(\|B_n\|^{\frac{1}{q}}, m_{n-1}, p, q, r)(\|B^{-1}\|^{-1} + m)^s$$

holds for $p \geq 1,\ q \geq 1$ with $p + 1 \geq q \geq \frac{p+1}{2}$, where $B_n = A^{\frac{n}{2}} B^p A^{\frac{n}{2}}$,

$$m_n = k(\|B_n\|^{\frac{1}{q}}, m_{n-1}, p, q, 1)(\|B^{-1}\|^{-1} + m) \quad \text{for } n \geq 1$$

and $m_0 = k(\|B\|, m, p, q, s)(\|B^{-1}\|^{-1} + m)^s$.

Proof. Taking $r = 1$ in the above theorem, we have

$$A^{\frac{p+1}{q}} - (A^{\frac{1}{2}} B^p A^{\frac{1}{2}})^{\frac{1}{q}} \geq k(\|B\|, m, p, q, 1)(\|B^{-1}\|^{-1} + m) := m_1.$$

Next we put $C = A^{\frac{1}{2}} B^p A^{\frac{1}{2}}$. Since $A \geq C^{\frac{1}{p+1}}$ and $0 \leq s \leq 1$, it follows that

$$\begin{aligned}
(A^{\frac{1+s}{2}} B^p A^{\frac{1+s}{2}})^{\frac{1}{q}} &= (A^{\frac{s}{2}} C A^{\frac{s}{2}})^{\frac{1}{q}} \\
&= A^{\frac{s}{2}} C^{\frac{1}{2}} (C^{\frac{1}{2}} A^s C^{\frac{1}{2}})^{\frac{1-q}{q}} C^{\frac{1}{2}} A^{\frac{s}{2}}
\end{aligned}$$

$$\leq A^{\frac{s}{2}}C^{\frac{1}{2}}(C^{-\frac{1}{2}}C^{\frac{-s}{p+1}}C^{-\frac{1}{2}})^{\frac{q-1}{q}}C^{\frac{1}{2}}A^{\frac{s}{2}}$$
$$= A^{\frac{s}{2}}(C^{\frac{1}{q}})^{\frac{p+1-(q-1)s}{p+1}}A^{\frac{s}{2}}.$$

Consequently we have

$$A^{\frac{p+1+s}{q}} - (A^{\frac{1+s}{2}}B^pA^{\frac{1+s}{2}})^{\frac{1}{q}}$$
$$\geq A^{\frac{s}{2}}((A^{\frac{p+1}{q}})^{\frac{p+1-(q-1)s}{p+1}} - (C^{\frac{1}{q}})^{\frac{p+1-(q-1)s}{p+1}})A^{\frac{s}{2}}$$
$$\geq ((\|C^{\frac{1}{q}}\| + m_1)^{\frac{p+1-(q-1)s}{p+1}} - \|C^{\frac{1}{q}}\|^{\frac{p+1-(q-1)s}{p+1}})(\|B^{-1}\|^{-1} + m)^s.$$

Taking $s = 1$ in the above, we have

$$A^{\frac{p+2}{q}} - (AB^pA)^{\frac{1}{q}} \geq ((\|C^{\frac{1}{q}}\| + m_1)^{\frac{p+2-q}{p+1}} - \|C^{\frac{1}{q}}\|^{\frac{p+2-q}{p+1}})(\|B^{-1}\|^{-1} + m)^s := m_2.$$

Inductively we put $D = AB^pA$ and then we have

$$(A^{\frac{2+s}{2}}B^pA^{\frac{2+s}{2}})^{\frac{1}{q}} = (A^{\frac{s}{2}}DA^{\frac{s}{2}})^{\frac{1}{q}} \leq A^{\frac{s}{2}}(D^{\frac{1}{q}})^{\frac{p+2-(q-1)s}{p+2}}A^{\frac{s}{2}}$$

and so

$$A^{\frac{p+2+s}{q}} - (A^{\frac{2+s}{2}}B^pA^{\frac{2+s}{2}})^{\frac{1}{q}}$$
$$\geq A^{\frac{s}{2}}((A^{\frac{p+2}{q}})^{\frac{p+2-(q-1)s}{p+2}} - (D^{\frac{1}{q}})^{\frac{p+2-(q-1)s}{p+2}})A^{\frac{s}{2}}$$
$$\geq ((\|D^{\frac{1}{q}}\| + m_2)^{\frac{p+2-(q-1)s}{p+2}} - \|D^{\frac{1}{q}}\|^{\frac{p+2-(q-1)s}{p+2}})(\|B^{-1}\|^{-1} + m)^s.$$

Repeating this, we obtain the conclusion. □

In the Furuta inequality, the optimal case where $p \geq 1$ and $(1+r)q = p+r$ is the most important by virtue of the Löwner–Heinz inequality. So we would like to mention the following result:

Corollary 4.3. *Let A and B be invertible positive operators with $A - B \geq m > 0$. Then*

$$A^{1+r} - (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geq m(\|B^{-1}\|^{-1} + m)^r$$

holds for $p \geq 1$ and $r \geq 0$.

Proof. First of all, we note that if $q = \frac{p+r}{1+r}$ for $p \geq 1$ and $r \geq 0$, then for each $M > 0$, $k(b, M, p, q, r) = M$ for arbitrary $b > 0$. Hence we have the conclusion for $0 < r \leq 1$ by Theorem 2.1.

Next, if $r > 1$, that is, $r = n + s$ for some natural number n and $0 < s \leq 1$, then Theorem 2.2 implies that

$$A^{1+r} - (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geq m_{n-1}(\|B^{-1}\|^{-1} + m)^s,$$

where m_{n-1} is the constant defined in Theorem 2.2. On the other hand, since

$$m_{n-1} = m_{n-2}(\|B^{-1}\|^{-1} + m) = m_{n-2}(\|B^{-1}\|^{-1} + m) = \cdots$$
$$= m_0(\|B^{-1}\|^{-1} + m)^{n-1} = m(\|B^{-1}\|^{-1} + m)^n,$$

we get the desired lower bound. □

5. Concluding remarks.

We now pose a proof of Theorem 3.3 by the use of integral representation for operator monotone functions.

Proof of Theorem 3.3. We first prepare the basic tool: If $A > B > 0$ and $m = \|(A - B)^{-1}\|^{-1}$, then

$$B^{-1} - A^{-1} \geq \frac{m}{(\|B\| + m)\|B\|}. \tag{5.1}$$

It is shown by

$$B^{-1} - A^{-1} \geq B^{-1} - (B + m)^{-1} = mB^{-1}(B + m)^{-1} \geq \frac{m}{\|B\|(\|B\| + m)}$$

because of $A - B \geq m$. Note that f admits the integral representation:

$$f(t) = a + bt + \int_{-\infty}^{0} \frac{1 + ts}{s - t} dm(s) = a + bt + \int_{-\infty}^{0} (-s - \frac{1 + s^2}{t - s}) dm(s)$$

where $b \geq 0$ and $m(s)$ is a positive measure. Hence it follows that

$$\begin{aligned} f(A) - f(B) &= b(A - B) + \int_{-\infty}^{0} (1 + s^2)((B - s)^{-1} - (A - s)^{-1}) dm(s) \\ &\geq bm + \int_{-\infty}^{0} (1 + s^2) \left(\frac{1}{\|B\| - s} - \frac{1}{\|B\| - s + m} \right) dm(s) \\ &= f(\|B\| + m) - f(\|B\|) \ (> 0). \end{aligned}$$

□

Finally we discuss an operator extension of Lemma 2.1. Namely we may expect the following conjecture:
A real valued function f on an interval $J = (a, b)$ with $b \in (-\infty, +\infty]$ is operator convex if and only if, for each $0 < \epsilon < b - a$, $D_\epsilon(t)$ is operator monotone on $(a, b - \epsilon)$.

Unfortunately we have a negative answer as follows: We choose the function $f(t) = \frac{1}{t}$ on $(0, \infty)$. It is a typical example of operator convex functions. Nevertheless, $D_1(t) = -\frac{1}{t(t+1)}$ is not operator monotone. As a matter of fact, we take two 2×2 matrices A and B:

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that $D_1(A) \geq D_1(B)$ if and only if $A(A + 1) \geq B(B + 1)$. Clearly $A \geq B$, but

$$A(A + 1) - B(B + 1) = \begin{pmatrix} 13 & 6 \\ 6 & 7 \end{pmatrix} - \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ 6 & 5 \end{pmatrix} \not\geq 0.$$

This is a counterexample.

Incidentally, the operator convexity of the function $\frac{1}{t}$ is easily shown as follows: It is enough to prove the inequality

$$\left(\frac{A + B}{2} \right)^{-1} \leq \frac{1}{2}(A^{-1} + B^{-1}).$$

And it is simplified by putting $C = A^{\frac{1}{2}} B^{-1} A^{\frac{1}{2}}$ that

$$4(1 + C^{-1})^{-1} \leq 1 + C,$$

which follows from the numerical inequality $4 \leq (1+x^{-1})(1+x)$.

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