AN OPERATOR INEQUALITY FOR OPERATOR MONOTONE FUNCTIONS

Masatoshi Fujii  
Osaka Kyoiku University  
mfujii@cc.osaka-kyoiku.ac.jp

Young Ok Kim  
Suwon University  
evergreen1317@gmail.com

Ritsuo Nakamoto  
Ibaraki University  
r-naka@net1.jway.ne.jp

Abstract. We give a characterization of convex functions in terms of difference among values of a function. As an application, we propose an estimation of operator monotone functions: If $A > B \geq 0$ and $f$ is operator monotone on $(0,\infty)$, then

$$f(A) - f(B) \geq f(||B|| + \epsilon) - f(||B||) > 0,$$

where $\epsilon = \|(A - B)^{-1}\|^{-1}$. As a consequence, we give a refined estimation of Löwner-Heinz inequality under the assumption $A > B \geq 0$. Moreover it gives a simple proof to Furuta’s theorem: If $\log A > \log B$ for $A, B > 0$ and $f$ is operator monotone on $(0,\infty)$, then there exists a $\beta > 0$ such that

$$f(A^\alpha) > f(B^\alpha)$$

for all $0 < \alpha \leq \beta$.

Finally we discuss strict positivity of Furuta inequality which is a beautiful extension of Löwner-Heinz inequality.

1. Introduction

For a twice differentiable real-valued function $f$, its convexity is characterized by $f'' \geq 0$. Since there are many non-differentiable convex functions, we consider a characterization of general convex functions. We cannot use the differentiation, but the average rate of change is available. Roughly speaking, we claim that the convexity of a function is characterized by the non-decreasingness of average rate of change. It seems to be natural as a generalization of the condition $f'' \geq 0$. Actually it will be formulated as Lemma 1 in the next section.

To explain operator monotone functions, we introduce the operator order $A \geq B$ among selfadjoint operators $A, B$ on a Hilbert space $H$ by $(Ax, x) \geq (Bx, x)$ for all $x \in H$. In particular, $A$ is positive if $A \geq 0$, i.e., $(Ax, x) \geq 0$ for all $x \in H$. Next, a positive operator $A$ is said to be strictly positive, denoted by $A > 0$, if $A \geq c$ for some constant $c > 0$. So $A > B$ means that $A - B > 0$.

A real-valued continuous function $f$ defined on $[0,\infty)$ is called operator monotone if it preserves the operator order, i.e., $f(A) \geq f(B)$ for $A \geq B \geq 0$. One of the most important examples is the power function $t \mapsto t^p$ for $0 \leq p \leq 1$ (Löwner-Heinz inequality). In general, $f$ is called operator monotone on an interval $J$ if $f(A) \geq f(B)$ for $A \geq B$ whose spectra contained in $J$. For this, we pose $\log t$ as a fundamental example of an operator monotone function on $(0,\infty)$.

Very recently, Moslehian and Najafi [13] proposed an excellent extension of the Löwner-Heinz inequality as follows:

2010 Mathematics Subject Classification. Primary 47A63; Secondary 47B10, 47BA30.

Key words and phrases. convex function, operator monotone function, Löwner-Heinz inequality, Furuta inequality and chaotic order.
Theorem MN. If $A > B \geq 0$ and $0 < r \leq 1$, then $A^r - B^r \geq \|A\|^r - (\|A\| - \epsilon)^r > 0$, and $\log A - \log B \geq \log \|A\| - \log (\|A\| - \epsilon) > 0$, where $\epsilon = \|(A - B)^{-1}\|^{-1}$.

In this note, we apply our characterization of concave functions and give an improvement and a generalization of Theorem MN (Theorem 5). As another application, we can give a short proof to a recent result due to Furuta [9, Theorem 2.1], which is an operator inequality related to operator monotone functions and chaotic order, i.e., the order defined by $\log A \geq \log B$ among positive invertible operators.

Incidentally, this note is based on our paper [5].

2. A CHARACTERIZATION OF CONVEX FUNCTIONS

In this section, we propose an elementary characterization of convex functions. We essentially use average rate of change.

Lemma 2.1. A real valued continuous function $f$ on an interval $J = [a, b)$ with $b \in (-\infty, +\infty]$ is convex (resp. concave) if and only if, for each $0 < \epsilon < b - a$, $D_\epsilon(t) = f(t + \epsilon) - f(t)$ is non-decreasing (resp. non-increasing) on $[a, b - \epsilon]$.

Proof. Suppose that $f$ is convex on $J$. Take $s, t \in J$ with $s < t$ and $t + \epsilon \in J$. We may assume that $t - s < \epsilon$. Let $y = L(t)$ be the linear function through $(s, f(s))$ and $(s + \epsilon, f(s + \epsilon))$. Then we have

$$L(t) \geq f(t) \quad \text{and} \quad L(t + \epsilon) \leq f(t + \epsilon)$$

by the convexity of $f$. Hence it implies that

$$D_\epsilon(t) = f(t + \epsilon) - f(t)$$

$$\geq L(t + \epsilon) - L(t)$$

$$= L(s + \epsilon) - L(s) \quad \text{by the linearity of} \ L$$

$$= f(s + \epsilon) - f(s)$$

$$= D_\epsilon(s),$$

as desired.

Conversely suppose that $D_\epsilon(t)$ is non-decreasing. Take $t, s \in J$ with $s < t = s + 2\epsilon$. Since $D_\epsilon(s) \leq D_\epsilon(s + \epsilon)$, we have

$$2f \left( \frac{s + t}{2} \right) = 2f(s + \epsilon) \leq f(s + 2\epsilon) + f(s) = f(t) + f(s).$$

So $f$ is convex. \(\square\)

Corollary 2.2. If $f$ is strictly increasing and concave on an interval $[a, b + \delta]$ in $\mathbb{R}$ for some $\delta > 0$, then for each $0 < \epsilon \leq \delta$, $D_\epsilon(t) \geq D_\epsilon(b) > 0$ for all $t \in [a, b]$.

Remark 2.3. Analogous argument on convexity of functions as above has been done in [12, page 2].

3. APPLICATIONS TO OPERATOR MONOTONE FUNCTIONS

As an application of Corollary 2.2, we give an estimation of operator monotone functions.

Lemma 3.1. If $f$ is non-constant and operator monotone on the interval $\mathbb{R}_+ = [0, \infty)$, then $f$ is strictly increasing.
Proof. First of all, we note that \( f \) is non-decreasing. Next we suppose that \( f'(c) = 0 \) for some \( c > 0 \). Noting that the Löwner matrix

\[
\begin{pmatrix}
  f'(c) & f^{[1]}(c, d)
  \\
  f^{[1]}(d, c) & f'(d)
\end{pmatrix}
\]

is positive semidefinite for any \( d > 0 \) by the operator monotonicity of \( f \), where \( f^{[1]}(c, d) = \frac{f(c) - f(d)}{c - d} \) is the divided difference.

Therefore its determinant is nonnegative, so that \( f^{[1]}(c, d) = 0 \) for any \( d > 0 \). This means that \( f \) is constant, which is a contradiction. Consequently we have \( f' > 0 \). \( \square \)

Lemma 3.2. If \( C \geq 0 \) and \( f \) is a concave and strictly increasing function on an interval \([a, d)\) containing the spectrum of \( C \), then for each \( 0 < \epsilon < d - \|C\| \), \( f(C + \epsilon) \geq f(C) + D_\epsilon(\|C\|) \).

Proof. We first note that for a given \( 0 < \epsilon < d - \|C\| \), we can take \( c > 0 \) satisfying \( 0 < c < d \) and \( \epsilon < c - \|C\| \). Applying Corollary 2.2 to \( b = \|C\| \) and \( \delta = c - \|C\| \), it follows that

\[
f(C + \epsilon) - f(C) \geq D_\epsilon(\|C\|) \]

\( \square \)

We here give a precise estimation of [9, Theorem 2.1] and [12, Proposition 2.2], cf. [13].

Theorem 3.3. If \( A > B \geq 0 \) and \( f \) is non-constant operator monotone on \([0, \infty)\), then \( f(A) - f(B) \geq f(\|B\| + \epsilon) - f(\|B\|) > 0 \), where \( \epsilon = \|(A - B)^{-1}\|^{-1} \).

Proof. Since \( A \geq B + \epsilon \) for \( \epsilon = \|(A - B)^{-1}\|^{-1} > 0 \), we have

\[
f(A) \geq f(B + \epsilon) \]

Furthermore Lemmas 3.1 and 3.2 imply that

\[
f(B + \epsilon) \geq f(B) + D_\epsilon(\|B\|) \]

Hence we have

\[
f(A) - f(B) \geq D_\epsilon(\|B\|) = f(\|B\| + \epsilon) - f(\|B\|) > 0 \]

\( \square \)

As a consequence, we have an improvement of the estimation due to Moslehian and Najafi [13]:

Corollary 3.4. If \( A > B \geq 0 \) and \( 0 < r \leq 1 \), then \( A^r - B^r \geq (\|B\| + \epsilon)^r - (\|B\|)^r > 0 \), and \( \log A - \log B \geq \log(\|B\| + \epsilon) - \log \|B\| > 0 \), where \( \epsilon = \|(A - B)^{-1}\|^{-1} \).

Remark 3.5. We note that Corollary 3.4 actually improves Theorem MN. Since \( \|A\| - (\|A\| - \epsilon) = \epsilon = (\|B\| + \epsilon) - \|B\| \) and the function \( t \mapsto t^r \) is strictly concave, it follows that

\[
\|A\|^r - (\|A\| - \epsilon)^r \leq (\|B\| + \epsilon)^r - \|B\|^r \]

We here pose an example:

\[
A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Then \( A - B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \geq 1 \) and so \( \epsilon = 1 \). Hence we have
\[
\|A\|^r - (\|A\| - \epsilon)^r = 4^r - 3^r < (\|B\| + \epsilon)^r - \|B\|^r = 3^r - 2^r.
\]

Now Theorem 3.3 can be regarded as a difference version. So we give a ratio version of it. It is obtained by Theorem 3.3 itself:

**Corollary 3.6.** If \( A > B > 0 \) and \( f \) is non-constant operator monotone on \((0, \infty)\), then
\[
f(B)^{-\frac{1}{2}}f(A)f(B)^{-\frac{1}{2}} \geq (f(\|B\| + \epsilon) - f(\|B\|))\|f(B)\|^{-1},
\]
where \( \epsilon = \|(A - B)^{-1}\|^{-1} \).

**Proof.** Put \( \delta = f(\|B\| + \epsilon) - f(\|B\|) \). It follows from Theorem 3.3 that
\[
f(B)^{-\frac{1}{2}}f(A)f(B)^{-\frac{1}{2}} \geq f(B)^{-\frac{1}{2}}(f(B) + \delta)f(B)^{-\frac{1}{2}} = 1 + \delta\|f(B)\|^{-1}.
\]
\( \square \)

As another application of Theorem 3.3, we need the chaotic order: For \( A > 0 \), we can define the selfadjoint operator \( \log A \). So a weaker order than the operator order appears by \( \log A \geq \log B \) for \( A, B > 0 \). We call it the chaotic order. The chaotic order plays an substantial role in operator inequalities. Among others, it brightens the Furuta inequality [7], [3], [4], [1], [6], [10] and recent development of Karcher mean theory [16].

Now we give a simple and elementary proof to the following recent theorem [9, Theorem 2.1] due to Furuta, in which we don’t use any integral representation of operator monotone functions.

**Theorem 3.7.** If \( \log A > \log B \) for \( A, B > 0 \) and \( f \) is operator monotone on \((0, \infty)\), then there exists \( \beta > 0 \) such that
\[
f(A^\alpha) > f(B^\alpha) \quad \text{for all } 0 < \alpha \leq \beta.
\]

**Proof.** Since \( \log A > \log B \), it is known that there exists \( \beta > 0 \) such that
\[
A^\alpha > B^\alpha \quad \text{for all } 0 < \alpha \leq \beta.
\]
Therefore it follows from Theorem 3.3 that, for each fixed \( \alpha \in (0, \beta] \),
\[
f(A^\alpha) > f(B^\alpha),
\]
as desired. \( \square \)

4. Furuta inequality.

First of all, we cite the Furuta inequality (FI) in [7], see also [2], [8], [11] and [14] for the best possibility of it.

**The Furuta inequality.** If \( A \geq B \geq 0 \), then for each \( r \geq 0 \),
\[
A^{\frac{p+r}{q}} \geq (A^\frac{r}{q}B^pA^\frac{r}{q})^\frac{1}{q}
\]
holds for \( p \geq 0, q \geq 1 \) with
\[
(1 + r)q \geq p + r.
\]
To extend Corollary 3.4, we remark that the case $r = 0$ in (F1) is just the Löwner–Heinz inequality. Now we introduce a constant $k(b, m, p, q, r)$ for $b, m, p, q, r \geq 0$ by

\[ k(b, m, p, q, r) = (b + m)^{\frac{p+r}{q}} - b^{\frac{p+r}{q}}. \]

As a matter of fact, we have an extension of Corollary 3.4 in the form of Furuta inequality as follows:

**Theorem 4.1.** Let $A$ and $B$ be invertible positive operators with $A - B \geq m > 0$. Then for $0 < r \leq 1$,

\[ A^{\frac{p+r}{q}} - (A^{\frac{1}{2}} B^{p} A^{\frac{1}{2}})^{\frac{1}{q}} \geq k(\|B\|, m, p, q, r)(\|B^{-1}\|^{-1} + m)^{r} \]

holds for $p \geq 0$, $q \geq 1$ with $(1 + r)q \geq p + r \geq qr$.

**Proof.** We note that $q \geq 1$ and $(1 + r)q \geq p + r \geq qr$ assure the exponent $\frac{p+r}{q} - r$ in the constant $k$ belongs to $[0, 1]$. Since $0 \leq r \leq 1$, it follows from Theorem B that

\[ A^{\frac{p+r}{q}} - (A^{\frac{1}{2}} B^{p} A^{\frac{1}{2}})^{\frac{1}{q}} = A^{\frac{p+r}{q}} - A^{\frac{1}{2}} B^{p} (B^{q} A^{p} B^{q})^{\frac{1}{q}} A^{\frac{1}{2}} \]

\[ = A^{\frac{p+r}{q}} - A^{\frac{1}{2}} B^{p} (B^{q} A^{p} B^{q})^{\frac{1}{q}} A^{\frac{1}{2}} \]

\[ \geq A^{\frac{p+r}{q}} - A^{\frac{1}{2}} B^{p} (B^{q} A^{p} B^{q})^{\frac{1}{q}} A^{\frac{1}{2}} \]

\[ = A^{\frac{p+r}{q}} - A^{\frac{1}{2}} B^{p} (B^{q} A^{p} B^{q})^{\frac{1}{q}} A^{\frac{1}{2}} \]

\[ \geq k(\|B\|, m, p, q, r) A^{r} \]

\[ \geq k(\|B\|, m, p, q, r) (B + m)^{r} \]

\[ \geq k(\|B\|, m, p, q, r) (\|B^{-1}\|^{-1} + m)^{r}. \]

\[ \square \]

For a general case on $r$, we have the following estimation of Furuta inequality by repeating method as in a proof of Furuta inequality.

**Theorem 4.2.** Let $A$ and $B$ be invertible positive operators with $A - B \geq m > 0$ and $r = n + s$ for some natural number $n$ and $0 < s \leq 1$. Then

\[ A^{\frac{p+r}{q}} - (A^{\frac{1}{2}} B^{p} A^{\frac{1}{2}})^{\frac{1}{q}} \geq k(\|B_{n}\|^{\frac{1}{q}}, m, n-1, p, q, r)(\|B^{-1}\|^{-1} + m)^{s} \]

holds for $p \geq 1$, $q \geq 1$ with $p + 1 \geq q \geq \frac{p+1}{2}$, where $B_{n} = A^{n} B^{p} A^{\frac{1}{2}}$,

\[ m_{n} = k(\|B_{n}\|^{\frac{1}{q}}, m, n-1, p, q, 1)(\|B^{-1}\|^{-1} + m) \quad \text{for } n \geq 1 \]

and $m_{0} = k(\|B\|, m, p, q, s)(\|B^{-1}\|^{-1} + m)^{s}$.

**Proof.** Taking $r = 1$ in the above theorem, we have

\[ A^{\frac{p+1}{q}} - (A^{\frac{1}{2}} B^{p} A^{\frac{1}{2}})^{\frac{1}{q}} \geq k(\|B\|, m, p, q, 1)(\|B^{-1}\|^{-1} + m) := m_{1}. \]

Next we put $C = A^{\frac{1}{2}} B^{p} A^{\frac{1}{2}}$. Since $A \geq C^{\frac{1}{p+1}}$ and $0 \leq s \leq 1$, it follows that

\[ (A^{\frac{1}{2}} B^{p} A^{\frac{1}{2}})^{\frac{1}{q}} = (A^{\frac{1}{2}} C A^{\frac{1}{2}})^{\frac{1}{q}} \]

\[ = A^{\frac{1}{2}} C^{\frac{1}{2}} (C^{\frac{1}{2}} A^{\frac{1}{2}} C^{\frac{1}{2}})^{\frac{1-s}{s}} C^{\frac{1}{2}} A^{\frac{1}{2}} \]

\[ \square \]
\[ \leq A^\frac{1}{2}C^\frac{1}{2}(C^{-\frac{1}{2}}C^{\frac{1}{2}}C^{-\frac{1}{2}})^{\frac{r-1}{r}}C^\frac{1}{2}A^\frac{1}{2} = A^\frac{1}{2}(C^\frac{1}{2})\frac{p+1-(q-1)s}{p+1}A^\frac{1}{2}. \]

Consequently we have
\[
A^\frac{p+1-r}{4} - (A^\frac{1}{2}B^pA^\frac{1}{2})^\frac{1}{4} \\
\geq A^\frac{1}{2}((A^\frac{p+1}{4})\frac{p+1-(q-1)s}{p+1} - (C^\frac{1}{4})\frac{p+1-(q-1)s}{p+1})A^\frac{1}{2} \\
\geq ((||C^\frac{1}{4}|| + m_1)\frac{p+1-(q-1)s}{p+1} - ||C^\frac{1}{4}||\frac{p+1-(q-1)s}{p+1})(||B^{-1}||^{-1} + m)^s.
\]

Taking \( s = 1 \) in the above, we have
\[
A^\frac{p+2}{4} - (AB^pA)^\frac{1}{4} \geq ((||C^\frac{1}{4}|| + m_1)\frac{p+2-q}{p+1} - ||C^\frac{1}{4}||\frac{p+2-q}{p+1})(||B^{-1}||^{-1} + m)^s := m_2.
\]

Inductively we put \( D = AB^pA \) and then we have
\[
(A^\frac{2+\epsilon}{4}B^pA^\frac{2+\epsilon}{4})^\frac{1}{4} = (A^\frac{1}{2}DA^\frac{1}{2})^\frac{1}{4} \leq A^\frac{1}{2}(D^\frac{1}{4})\frac{p+2-(q-1)s}{p+2}A^\frac{1}{2}
\]
and so
\[
A^\frac{p+2+\epsilon}{4} - (A^\frac{1}{2}B^pA^\frac{1}{2})^\frac{1}{4} \\
\geq A^\frac{1}{4}((A^\frac{p+2}{4})\frac{p+2-(q-1)s}{p+2} - (D^\frac{1}{4})\frac{p+2-(q-1)s}{p+2})A^\frac{1}{4} \\
\geq ((||D^\frac{1}{4}|| + m_2)\frac{p+2-(q-1)s}{p+2} - ||D^\frac{1}{4}||\frac{p+2-(q-1)s}{p+2})(||B^{-1}||^{-1} + m)^s.
\]

Repeating this, we obtain the conclusion. \( \square \)

In the Furuta inequality, the optimal case where \( p \geq 1 \) and \( (1+r)q = p + r \) is the most important by virtue of the Löwner–Heinz inequality. So we would like to mention the following result:

**Corollary 4.3.** Let \( A \) and \( B \) be invertible positive operators with \( A - B \geq m > 0 \). Then
\[
A^{1+r} - (A^{\frac{1}{2}}B^pA^{\frac{1}{2}})^{\frac{1+r}{p+r}} \geq m(||B^{-1}||^{-1} + m)^r
\]
holds for \( p \geq 1 \) and \( r \geq 0 \).

**Proof.** First of all, we note that if \( q = \frac{p+r}{1+r} \) for \( p \geq 1 \) and \( r \geq 0 \), then for each \( M > 0 \), \( k(b, M, p, q, r) = M \) for arbitrary \( b > 0 \). Hence we have the conclusion for \( 0 < r \leq 1 \) by Theorem 2.1.

Next, if \( r > 1 \), that is, \( r = n + s \) for some natural number \( n \) and \( 0 < s \leq 1 \), then Theorem 2.2 implies that
\[
A^{1+r} - (A^{\frac{1}{2}}B^pA^{\frac{1}{2}})^{\frac{1+r}{p+r}} \geq m_{n-1}(||B^{-1}||^{-1} + m)^s
\]
where \( m_{n-1} \) is the constant defined in Theorem 2.2. On the other hand, since
\[
m_{n-1} = m_{n-2}(||B^{-1}||^{-1} + m) = m_{n-2}(||B^{-1}||^{-1} + m) = \cdots = m_0(||B^{-1}||^{-1} + m)^{n-1} = m(||B^{-1}||^{-1} + m)^n,
\]
we get the desired lower bound. \( \square \)
5. Concluding remarks.

We now pose a proof of Theorem 3.3 by the use of integral representation for operator monotone functions.

Proof of Theorem 3.3. We first prepare the basic tool: If $A > B > 0$ and $m = \|(A - B)^{-1}\|^{-1}$, then

\[ B^{-1} - A^{-1} \geq \frac{m}{\|B\| + m}\|B\|. \]  

(5.1)

It is shown by

\[ B^{-1} - A^{-1} \geq B^{-1} - (B + m)^{-1} = mB^{-1}(B + m)^{-1} \geq \frac{m}{\|B\| + m}. \]

because of $A - B \geq m$. Note that $f$ admits the integral representation:

\[ f(t) = a + bt + \int_{-\infty}^{0} \frac{1 + ts}{s - t} dm(s) = a + bt + \int_{-\infty}^{0} (-s - \frac{1 + s^2}{t - s}) dm(s) \]

where $b \geq 0$ and $m(s)$ is a positive measure. Hence it follows that

\[ f(A) - f(B) = b(A - B) + \int_{-\infty}^{0} (1 + s^2)((B - s)^{-1} - (A - s)^{-1}) dm(s) \]

\[ \geq bm + \int_{-\infty}^{0} (1 + s^2) \left( \frac{1}{\|B\| - s} - \frac{1}{\|B\| - s + m} \right) dm(s) \]

\[ = f(\|B\| + m) - f(\|B\|) > 0. \]

Finally we discuss an operator extension of Lemma 2.1. Namely we may expect the following conjecture:

A real valued function $f$ on an interval $J = (a, b)$ with $b \in (-\infty, +\infty]$ is operator convex if and only if, for each $0 < \epsilon < b - a$, $D_\epsilon(t)$ is operator monotone on $(a, b - \epsilon)$.

Unfortunately we have a negative answer as follows: We choose the function $f(t) = \frac{1}{t}$ on $(0, \infty)$. It is a typical example of operator convex functions. Nevertheless, $D_1(t) = -\frac{1}{t(t+1)}$ is not operator monotone. As a matter of fact, we take two $2 \times 2$ matrices $A$ and $B$:

\[ A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \] and \[ B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}. \]

Note that $D_1(A) \geq D_1(B)$ if and only if $A(A + 1) \geq B(B + 1)$. Clearly $A \geq B$, but

\[ A(A + 1) - B(B + 1) = \begin{pmatrix} 13 & 6 \\ 6 & 7 \end{pmatrix} - \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ 6 & 5 \end{pmatrix} \not\geq 0. \]

This is a counterexample.

Incidentally, the operator convexity of the function $\frac{1}{t}$ is easily shown as follows: It is enough to prove the inequality

\[ \left( \frac{A + B}{2} \right)^{-1} \leq \frac{1}{2}(A^{-1} + B^{-1}). \]

And it is simplified by putting $C = A^\frac{1}{2}B^{-1}A^\frac{1}{2}$ that

\[ 4(1 + C^{-1})^{-1} \leq 1 + C, \]
which follows from the numerical inequality $4 \leq (1 + x^{-1})(1 + x)$.

REFERENCES

[7] T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1 + 2r)q \geq p + 2r$, Proc. Amer. Math. Soc. 101 (1987), 85–88.