数理解析研究所講究録
第 1895 巻 2014 年 12-15

P_λ-FILTERS AND REGULAR EMBEDDINGS OF BOOLEAN ALGEBRAS

ALEKSANDER BŁASZCZYK

Institute of Mathematics, University of Silesia

ABSTRACT. This is a résumé of previously published results on an analogue of Mathias forcing associated to the tree $\omega^{<\omega}$ endowed with a topology appointed by a set of filters.

1. INTRODUCTION

We shall consider the set

$$\mathrm{Seq} = \omega^{<\omega} = \bigcup\{{}^n\omega : n < \omega\}$$

of all finite sequences of natural numbers. The set Seq is a tree with a natural order defined as follows: for $s, t \in \mathrm{Seq}$ we set

$$s \leq t \Longleftrightarrow t \restriction \mathrm{dom}(s) = s.$$

The set of all immediate successors of an element $s \in \mathrm{Seq}$ is denoted by

$$\mathrm{succ}(s) = \{t \in \mathrm{Seq}\colon t \text{ is minimal in } \{t \in \mathrm{Seq}\colon t > s\}\}.$$

Hence $\mathrm{succ}(s) = \{s^\frown n\colon n \in \omega\}$, where $s^\frown k$ denotes the concatenation of sequence $s \in {}^n\omega$ by a number k, i.e. $s^\frown k = s \cup \{(n, k)\}$ is the sequence that extends s and whose last term is k.

Definition 1. *Assume that $\mathfrak{F} = (\mathcal{F}_t\colon t \in \mathrm{Seq})$, where $\mathcal{F}_t \subseteq \mathcal{P}(\mathrm{succ}(t))$ is a collection of free filters. A set $U \subseteq \mathrm{Seq}$ is open in the $\mathfrak{F}$-topology on* Seq *whenever*

$$(\forall s \in U)(\exists F \in \mathcal{F}_s)(F \subseteq U).$$

Then $\mathrm{Seq}(\mathfrak{F})$ *denotes the set* Seq *endowed with the $\mathfrak{F}$-topology.*

The idea of $\mathfrak{F}$-topology on Seq has been given by Szymański [10] and, independently, by Trnkova [11]. Later on it was developed by several other authors. A review of $\mathfrak{F}$-topologies and their generalizations can be found in [2]. In this paper, in particular, one can find a proof of the following theorem:

Theorem 1. *For every $\mathfrak{F} = (\mathcal{F}_t\colon t \in \mathrm{Seq})$ the space $\mathrm{Seq}(\mathfrak{F})$ is a zero-dimensional, nowhere compact Hausdorff space. Moreover, $\mathrm{Seq}(\mathfrak{F})$ is extremally disconnected iff all the filters in $\mathfrak{F}$ are ultrafilters.*

Here, nowhere compact means that every compact subset of the space is nowhere dense. In the sequel we shall discuss the Boolean algebra $\mathbb{C}\mathrm{lop}(\mathrm{Seq}(\mathfrak{F}))$ of all the clopen (=closed and open) subsets of the space $\mathrm{Seq}(\mathfrak{F})$. Clearly, $\mathbb{C}\mathrm{lop}(\mathrm{Seq}(\mathfrak{F}))$ is in fact a field of subsets of Seq. Theorem 1 immediately implies the following:

Corollary 1. *The Boolean algebra $\mathbb{C}\mathrm{lop}(\mathrm{Seq}(\mathfrak{F}))$ is complete iff all the filters in $\mathfrak{F}$ are ultrafilters.*

The space $\mathrm{Seq}(\mathfrak{F})$ was used in a construction of a complete rigid Boolean algebra. Namely, it was proved by Dow, Gubi and Szymański [7] that if $\mathfrak{F}$ consists of one weak P-ultrafilter, then the Boolean algebra $\mathbb{C}\mathrm{lop}(\mathrm{Seq}(\mathfrak{F}))$ is complete and rigid.

Let us recollect some well-known cardinal numbers connected to the set of all the functions from ω to ω ordered by the relation $\leq^*$ defined as follows:

$$f \leq^* g \Longleftrightarrow (\exists n < \omega)(\forall k > n)(f(k) \leq g(k)).$$

This relation leads to the following cardinal characteristics:

1. The *dominating number* $\mathfrak{d}$ is defined as follows:

$$\mathfrak{d} = \min\{|D|\colon (\forall f \in {}^{\omega}\omega)(\exists g \in D)(f \leq^* g)\}.$$

2. The *boundedness* $\mathfrak{b}$ denotes the minimal cardinality of an unbounded subset of ${}^{\omega}\omega$, i.e.

$$\mathfrak{b} = \min\{|U|\colon (\forall f \in {}^{\omega}\omega)(\exists g \in U)(|\{n \in \omega\colon f(n) < g(n)\}| \geq \omega)\}.$$

3. The *cardinal number* $\mathfrak{p}$ is defined as

$$\mathfrak{p} = \min\{|\mathcal{U}|\colon \mathcal{U} \subseteq \mathrm{Clop}(\mathcal{P}(\omega^*)) \text{ is centered and } \mathrm{Int}\bigcap\mathcal{U} = \emptyset\}.$$

It is well-known that:

$$\omega < \mathfrak{p} \leq \mathfrak{b} \leq \mathfrak{d} \leq 2^{\omega}.$$

The character of a (free) filter denotes the character of the corresponding subset of ω^*, i.e. for every (free) filter $\mathcal{F} \subseteq \mathcal{P}(\omega)$ there is

$$\chi(\mathcal{F}) = \chi(A_{\mathcal{F}}, \omega^*),$$

where $A_{\mathcal{F}} = \bigcap\{\mathrm{cl}_{\beta\mathbb{N}}\, U\colon U \in \mathcal{F}\}$.

Using the cardinal $\mathfrak{d}$ one can calculate the character of a space $\mathrm{Seq}(\mathfrak{F})$ at every point of Seq.

Proposition 1. *For every $\mathfrak{F} = (\mathcal{F}_t\colon t \in \mathrm{Seq})$ and every $s \in \mathrm{Seq}$ we have*

$$\chi(s, \mathrm{Seq}(\mathfrak{F})) = \mathfrak{d} + \chi(\mathcal{F}_s).$$

2. P_λ-FILTERS AND P_λ-SETS

Since for every $s \in \mathrm{Seq}$ the set $\mathrm{succ}(s)$ of all the successors of s is countable, in the definition of $\mathfrak{F}$-topology on Seq instead of filters on $\mathcal{P}(\mathrm{succ}(s))$ one can consider filters on $\mathcal{P}(\omega)$. Let us recall that all filters considered here are assumed to be free filters.

Definition 2. *A filter $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called a P_λ-filter whenever for every family $\mathcal{R} \subseteq \mathcal{F}$ of size less than λ there exists $F \in \mathcal{F}$ such that $F \subseteq^* U$ for every $U \in \mathcal{F}$. A P_{ω_1}-filter is simply called a P-filter.*

Here, as usual, $F \subseteq^* U$ means that the set $F \setminus U$ is finite. In the virtue of the well-known result of Shelah, P-ultrafilters do not exists in ZFC. However, it is quite easy to construct a P-filter. In fact, if $\{U_n\colon n \in \omega\}$ consists of infinite subsets of ω and $U_{n+1} \subseteq^* U_n$ for every $n \in \omega$, then there exists an infinite set $V \subseteq \omega$ such that $V \subseteq^* U_n$ for every $n < \omega$. Hence, by transfinite induction one can construct a sequence $\{U_\alpha\colon \alpha < \omega_1\} \subseteq \mathcal{P}(\omega)$ of infinite sets such that $U_\alpha \subseteq^* U_\beta$ for all $\beta < \alpha < \omega_1$. Clearly, the family $\{U_\alpha\colon \alpha < \omega_1\}$ generates a P-filter.

For topological spaces we have an analogous definition.

Definition 3. *For $\lambda \geq \omega$ and a topological space X, a set $S \subseteq X$ is called a P_λ-set provided that S is contained in the interior of the intersection of every family of size less than λ consisting of open neighborhoods of S. Also, a P-set is just a P_{ω_1}-set.*

Let us note that if a Tychonoff space X is nowhere compact then X is simultaneously dense and boundary in the Čech–Stone compactification βX. Hence, by Theorem 1, $\mathrm{Seq}(\mathfrak{F})$ is dense and boundary in $\beta\,\mathrm{Seq}(\mathfrak{F})$. The next theorem has been recently proved in [3].

Theorem 2 (Błaszczyk and Brzeska [3]). *If $\mathfrak{F} = (\mathcal{F}_s\colon s \in \mathrm{Seq})$ is a collection of P_λ-filters and $\omega < \lambda \leq \mathfrak{b}$ then the space $\mathrm{Seq}(\mathfrak{F})$ is a P_λ-set in $\beta\,\mathrm{Seq}(\mathfrak{F})$.*

From Theorem 2 we immediately obtain the following:

Corollary 2 (Simon [9]). *If every filter in $\mathfrak{F} = (\mathcal{F}_s\colon s \in \mathrm{Seq})$ is a P-filter, then $\mathrm{Seq}(\mathfrak{F})$ is a P-set in $\beta\,\mathrm{Seq}(\mathfrak{F})$.*

Corollary 3 (Juhász and Szymański [8]). *If $\omega < \lambda \leq \mathfrak{b}$ and $\mathcal{F}_s = \mathcal{F}$ for every $s \in \mathrm{Seq}$, where $\mathcal{F}$ is a P_λ-ultrafilter, then* $\mathrm{Seq}(\mathfrak{F})$ *is a P_λ-set in* $\beta\,\mathrm{Seq}(\mathfrak{F})$.

The theorem of Simon answers a question of Arhangel'skiĭ whereas Juhász–Szymański's theorem leads to some constructions in the theory of calibers and tightness in compact spaces.

3. Regular embeddings

In this section we shall outline some further applications of Theorem 2 in the theory of Boolean algebras. First, we shall recall some definitions. Let $\mathbb{B}$ be a Boolean algebra. A subalgebra $\mathbb{A} \subseteq \mathbb{B}$ is *regular* if for every set $X \subseteq \mathbb{A}$ there is

$$\sup_{\mathbb{A}} X = \mathbf{1} \Longrightarrow \sup_{\mathbb{B}} X = \mathbf{1}.$$

A Boolean algebra $\mathbb{A}$ is *regularly embedded* in a Boolean algebra $\mathbb{B}$ provided that there exists a monomorphism of $\mathbb{A}$ into $\mathbb{B}$ such that the image of $\mathbb{A}$ is a regular subalgebra of $\mathbb{B}$.

The symbol ${}^\omega 2$ in the next definition, given by Baumgartner [1], denotes the Cantor set.

Definition 4. *A filter $\mathfrak{F} \subseteq \mathcal{P}(\omega)$ is nowhere dense if for every $f\colon \omega \to {}^\omega 2$ there exists a set $A \in \mathfrak{F}$ such that $f[A]$ is a nowhere dense subset of the Cantor set.*

The next theorem gives us an unexpected interrelation between nowhere dense ultrafilters and Boolean algebras. Let us recall that a Boolean algebra $\mathbb{B}$ is σ-centered if it is the union of countably many ultrafilters. An element $b \in \mathbb{B} \setminus \{0\}$ is an *atom* if there is no $a \in \mathbb{B}$ such that $0 < a < b$. A Boolean algebra is *atomless* if it has no atoms and it is *atomic* whenever below every element of $\mathbb{B} \setminus \{0\}$ there is an atom.

Theorem 3 (Błaszczyk and Shelah [4]). *There exists a σ-centered, atomless, complete Boolean algebra which does not contain any atomless, countable, regular subalgebra iff there exists a nowhere dense ultrafilter.*

Easier part of Theorem 3 can be derived from the next theorem. A proof of this theorem can be obtained by some modification of Theorem 17 from [5].

Theorem 4. *Assume that $\mathfrak{F} = (\mathcal{F}_s\colon s \in \mathrm{Seq})$ is a collection of filters. If $\mathcal{F}_s = \mathcal{F}$ for every $s \in \mathrm{Seq}$, then the countable free algebra $\mathrm{Fr}(\omega)$ can be embedded in $\mathrm{Clop}(\mathrm{Seq}(\mathfrak{F}))$ as a regular subalgebra iff the filter $\mathcal{F}$ is not nowhere dense.*

The next theorem says that from the point of view of regular embeddings atomlessness is a very essential requirement. For a Boolean algebra $\mathbb{B}$, $\pi(\mathbb{B})$ denotes the minimal size of a dense subset of $\mathbb{B}$.

Theorem 5. *Assume that $\mathfrak{F} = (\mathcal{F}_s\colon s \in \mathrm{Seq})$ is a collection of P_λ-filters where $\omega < \lambda \leq \mathfrak{b}$. If $\mathbb{B} \subseteq \mathrm{Clop}(\mathrm{Seq}(\mathfrak{F}))$ is a regular subalgebra and $\pi(\mathbb{B}) < \lambda$, then $\mathbb{B}$ is an atomic algebra.*

A topological version of this theorem has been proved in [3]; see Theorem 3.6.

4. Skeletal mappings

A continuous mapping $f\colon X \to Y$ is called *skeletal* whenever for every open and dense set $G \subseteq Y$ the set $f^{-1}[G]$ is dense in X. Skeletal maps are also known as *semi-open* mappings. The mapping f is semi-open whenever for every non-empty open set $U \subseteq X$, the image $f[U]$ has a non-empty interior. It appears that skeletal mappings of topological spaces correspond to regular embeddings of Boolean algebras.

Proposition 2. *An embedding of Boolean algebras is regular iff the continuous surjection of the corresponding mapping of Stone spaces is skeletal.*

The following example shows that skeletal mappings are easy to construct.

Example 1. *If $\mathcal{U}$ is an infinite, maximal disjoint family of clopen subsets of a zero-dimensional compact space X, then the quotient mapping determined by the closed partition $\{\{U\}: U \in \mathcal{U}\} \cup \{X \setminus \bigcup\mathcal{U}\}$ is a skeletal mapping onto the one-point compactification of the discrete space of cardinality $|\mathcal{U}|$.*

Clearly, if a continuous surjection is skeletal and the set of values is dense in itself, then the domain has to be dense in itself as well. However, the above example shows that a skeletal surjection can map a dense in itself compact space onto a space with a dense set of isolated points. The next theorem, proved in [3] says a bit more about it.

Theorem 6. *Assume that $\mathfrak{F} = (\mathcal{F}_s : s \in \mathrm{Seq})$ is a collection of P_λ-filters where $\omega < \lambda \leq \mathfrak{b}$. If X is a Hausdorff space with $\pi w(X) < \lambda$ and a continuous surjection $f: \beta\,\mathrm{Seq}(\mathfrak{F}) \to X$ is skeletal then the set of isolated points in X is dense.*

In connection to skeletal mappings Burke [6] has introduced the following notion:

Definition 5. *A continuous mapping $f: X \to Y$ is called* nowhere constant *if $f^{-1}(y)$ is nowhere dense for every $y \in Y$.*

Clearly, if Y is a T_1-space, a mapping $f: X \to Y$ is skeletal and the set $f[X]$ is dense in itself, then f is nowhere constant. Example 1 shows that in general the converse is not true: skeletal mappings do not have to be nowhere constant. However, the following theorem of Burke shows an interesting connection between nowhere constant and skeletal mappings.

Theorem 7 (Burke [6]). *If X is Tychonoff and there is a nowhere constant continuous function from X into $\mathbb{R}$, and $\pi w(X) < \mathfrak{p}$, then there also exists a skeletal function from X into $\mathbb{R}$.*

In particular, the above theorem shows that if a compact metric space has a nowhere constant mapping into the reals, then it has also a skeletal mapping into reals. Also, Burke [2] asked whether there exists (in ZFC) a Tychonoff space of π-weight $\mathfrak{p}$ which has a nowhere constant mapping onto $\mathbb{R}$ but does not have a skeletal mapping onto $\mathbb{R}$. We give a partial answer to this question.

Theorem 8. *If $\mathfrak{F} = (\mathcal{F}_s : s \in \mathrm{Seq})$ is a collection of P-filters of character $\aleph_1$, then the space $\mathrm{Seq}(\mathfrak{F})$ is a space of the π-weight equal to $\mathfrak{d}$ which has a nowhere constant mapping into $\mathbb{R}$ but does not have a skeletal mapping into $\mathbb{R}$.*

The above theorem also shows that nowhere constant mapping does not have to be skeletal.

References

[1] J. Baumgartner *Ultrafilters on ω*, *Journal of Symbolic logic* **66** (2001), 792–800.

[2] A. Błaszczyk, *Defining topologies on trees*, *Topology Proceedings Volume* **41** (2013) (E-Published on April 2012), 1–20.

[3] A. Błaszczyk and A. Brzeska *P_λ-sets and skeletal mappings*, *Colloquium Mathematicum* **131** (2013), 89–98.

[4] A. Błaszczyk and S. Shelah, *Regular subalgebras of complete Boolean algebras*, *Journal of Symbolic Logic* **66** (2001), 792–800.

[5] A. Błaszczyk and A. Szymański, *Cohen algebras and nowhere dense ultrafilters*, *Bull. Polish Acad. Sci.* **49** (2001), 15–25.

[6] M. R. Burke, *Continuous functions which take a somewhere dense set of values on every open set*, *Topology Appl.* **103** (2000), 95–110.

[7] A. Dow, A. Gubi and A. Szymański, *Rigid Stone space within ZFC*, *Proc. Amer. Math. Soc.* **102** (1988), 745–748.

[8] I. Juhász and A. Szymański, *d-calibers ad d-tightness in compact spaces*, *Topology Appl.* **151** (2005), 66–76.

[9] P. Simon, *A countable dense-in-itself dense P-sets*, *Topology Appl.* **123** (2002), 193–198.

[10] A. Szymański, *Products and measurable cardinals*, *Rend. Circ. Mat. Palermo*, (1985).

[11] V.Trnková, *Homeomorphisms of products of Boolean algebras*, *Fund. Math.* **126**, (1985) 46–61.

Institute of Mathematics, University of Silesia, Bankowa 14, 40-007 Katowice, Poland
E-mail address: ablaszcz@ux2.math.us.edu.pl