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A SEMICLASSICAL MEASURE APPROACH TO THE AHARONOV-BOHM EFFECT

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ABSTRACT. We examine the Aharonov-Bohm effect on the torus through the light of semiclassical measures. We show how the theory developed in [AM10] adapts to the case of magnetic potentials with vanishing magnetic field and characterise the high-frequency dynamics of positions densities corresponding to solutions to the magnetic Schrödinger equation on the torus. This allows us to give a characterisation of the highly-oscillating sequences of initial data whose corresponding solutions are affected by the magnetic potential in the high-frequency limit.

1. INTRODUCTION

Let $\mathbb{T}^d := \mathbb{R}^d/2\pi\mathbb{Z}^d$ denote the torus equipped with the standard flat metric. Consider a smooth one-form $\theta \in \Omega^1(\mathbb{T}^d)$ and a smooth real potential $V \in C^\infty(\mathbb{T}^d;\mathbb{R})$. The Schrödinger operator corresponding to a particle of mass 1 and charge -1 moving on $\mathbb{T}^d$ under the influence of the magnetic potential θ and the electric potential V is:

$$\widehat{H}_{\theta,V} := \frac{1}{2}\|D_x+\theta\|^2 + V = \frac{1}{2}\sum_{j=1}^{d}\left(D_{x_j}+\theta_j\right)^2 + V, \tag{1}$$

where $D_x := (D_{x_1},...,D_{x_d})$ with $D_{x_j} = -i\partial_{x_j}$ and $\theta = \sum_{j=1}^d \theta_j dx_j$.

The probability density of finding the particle in an infinitesimal neighborhood of x at a given time t is $|u(t,x)|^2$ where u solves the time-dependent Schrödinger equation:

$$\begin{cases} i\partial_t u(t,x) + \widehat{H}_{\theta,V} u(t,x) = 0, & (t,x) \in \mathbb{R}\times\mathbb{T}^d, \\ u(0,x) = u^0(x), & x \in \mathbb{T}^d. \end{cases} \tag{2}$$

In order to simplify the discussion that follows, we have replaced in equation (2) Planck's constant $\hbar$ by one. This will not affect any of the results that will follow.

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When the magnetic field associated to the magnetic potential θ is zero, *i.e.* if the differential form θ is closed:

$$\mathrm{d}\theta = 0,$$

then $\theta(x) = \theta_0 + \mathrm{d}\varphi(x)$ for some $\varphi \in C^\infty(\mathbb{T}^d)$ and $\theta_0 \in (\mathbb{R}^d)^*$ is constant. Write $\theta_0 = \sum_{j=1}^{d} \theta_{0,j} dx_j$, then $\theta_{0,j}$ are the magnetic fluxes corresponding to closed curves forming a basis of the homology group of $\mathbb{T}^d$. Of course, θ_0 is the only constant representative in the cohomology class of θ. In this case, $\widehat{H}_{\theta,V}$ can be unitarily conjugated to $\widehat{H}_{\theta_0,V}$ via a gauge transformation:

$$\widehat{H}_{\theta,V} = e^{-i\varphi} \widehat{H}_{\theta_0,V} e^{i\varphi}. \tag{3}$$

In spite of the fact that the magnetic field vanishes, Aharonov and Bohm discovered [AB59] that the magnetic potential affects the dynamics of the electron, provided $\theta_0 \notin 2\pi\mathbb{Z}^d$. Rather than the torus, they focused on the Euclidean plane with a point obstacle removed $\mathbb{R}^2 \setminus \{(0,0)\}$, which destroys the simple connectivity, and showed that the scattering cross-section is influenced by the flux modulo $2\pi\mathbb{Z}$, $[\theta_0] \in \mathbb{R}/2\pi\mathbb{Z}$. This prediction was confirmed experimentally by Tonomura *et al.* [TOM$^+$86].

Further understanding of the Aharonov-Bohm effect as well as its extension to more general settings than that initially studied in [AB59] has been the object of intense research in recent years, see [RY02, BW09b, BW09a, EIO10, PR11, BW11, Esk13, ER13] among many others. In [Esk13], Eskin considered the the time-dependent Schrödinger equation with vanishing magnetic field on the exterior of a bounded obstacle in the plane. He constructed a highly oscillating sequence of solutions (u_ε) to that equation such that

$$|u_\varepsilon(t,x)|^2 = 2\sin^2(\theta_0/2) + O(\varepsilon),$$

as $\varepsilon \to 0^+$ in an ε-neighborhood of a point. Therefore, $[\theta_0]$ affects the dynamics of $|u_\varepsilon(t,\cdot)|^2$ in the high-frequency regime for a particular family of oscillating solutions.

It is natural to ask how general this behavior can be, or, how is the general structure of the solutions affected by θ_0. Motivated by Eskin's article [Esk13] we address this issue in the case of the torus $\mathbb{T}^d$ presented above.

2. Results

We next proceed to describe the main result of this note. As mentioned in the previous section, we are interested in characterising the high frequency behavior of position densities $|u_\varepsilon(t,x)|^2$ associated to highly oscillating solutions to (2).

We first state precisely we problem we are interested in. Consider a sequence (u_ε^0) in $L^2(\mathbb{T}^d)$ satisfying $\|u_\varepsilon^0\|_{L^2(\mathbb{T}^d)} = 1$. Let u_ε denote the corresponding solutions to (2). We want to describe the behavior of

$$|u_\varepsilon(t,x)|^2, \qquad \text{as } \varepsilon \longrightarrow 0^+.$$

Two remarks are in order:

- Due to the gauge equivalence (3), $v_\varepsilon := e^{i\varphi}u_\varepsilon$ is a solution to:

$$\begin{cases} i\partial_t v_\varepsilon + \widehat{H}_{\theta_0,V} v_\varepsilon = 0, \\ v|_{t=0} = u_\varepsilon^0. \end{cases} \tag{4}$$

  Since $|v_\varepsilon|^2 = |u_\varepsilon|^2$, we can replace, without loss of generality, the dynamics of (2) by those of (4).
- Since (u_ε^0) is highly oscillating, there is no hope in general to describe the pointwise behavior of $|u_\varepsilon(t,x)|^2$. Therefore, we are going to analyse averages of $|u_\varepsilon(t,x)|^2$ both in t and x.

Notice that for each $t \in \mathbb{R}$, the density $|u_\varepsilon(t,\cdot)|^2$ can be identified to an element of $\mathcal{P}(\mathbb{T}^d)$, the set of probability measure on $\mathbb{T}^d$. Moreover,

$$\mathbb{R} \ni t \longmapsto |u_\varepsilon(t,\cdot)|^2 \in \mathcal{P}(\mathbb{T}^d),$$

can be viewed as an element of $L^\infty\left(\mathbb{R};\mathcal{P}(\mathbb{T}^d)\right)$. Since $\mathbb{T}^d$ is compact, we can apply Helly's theorem to ensure that (u_ε) is relatively compact for the weak-$*$ topology on $L^\infty\left(\mathbb{R};\mathcal{P}(\mathbb{T}^d)\right)$.

This means that a subsequence (u_{ε_n}) and a probability measure $\nu \in L^\infty\left(\mathbb{R};\mathcal{P}(\mathbb{T}^d)\right)$ exist such that, for every $a \in C(\mathbb{T}^d)$ and every $\alpha < \beta$ the following convergence takes place:

$$\lim_{n\to\infty} \int_\alpha^\beta \int_{\mathbb{T}^d} a(x)\, |u_{\varepsilon_n}(t,x)|^2\, dx dt = \int_\alpha^\beta \int_{\mathbb{T}^d} a(x)\, \nu(t,dx)\, dt. \tag{5}$$

Our main result, Theorem 1, describes how ν is obtained in terms of the sequence of initial data (u_{ε_n}) and how $\nu(t,\cdot)$ depends on t. In order to state it we need some notations.

We denote by $\mathcal{L}$ the set of all primitive submodules of $\mathbb{Z}^d$. In other words, $\Lambda \in \mathcal{L}$ whenever the lattice Λ satisfies $\operatorname{span}_{\mathbb{R}}\Lambda \cap \mathbb{Z}^d = \Lambda$.

Let

$$e_k(x) := \frac{e^{ik\cdot x}}{(2\pi)^{d/2}};$$

given $u \in L^2(\mathbb{T}^d)$ we write the Fourier series representation of u as:

$$u(x) = \sum_{k\in\mathbb{Z}^d} \widehat{u}_k e_k(x), \qquad \widehat{u}_k := \int_{\mathbb{T}^d} u(x)\, e_{-k}(x)\, dx.$$

Given $\Lambda \in \mathcal{L}$, denote by $L^2(\mathbb{T}^d, \Lambda)$ the subspace of $L^2(\mathbb{T}^d)$ consisting of those u satisfying $\widehat{u}_k = 0$ if $k \notin \Lambda$. Note that such an u satisfies:

$$u(x+v) = u(x), \qquad \text{for every } v \in \Lambda^\perp,$$

where $\Lambda^\perp$ is the orthogonal space to Λ in $\mathbb{R}^d$.

Let $a \in L^\infty(\mathbb{T}^d)$; we denote by $\langle a \rangle_\Lambda$ the average of a along the directions in $\Lambda^\perp$. If $a = \sum_{k\in\mathbb{Z}^d} \widehat{a}_k e_k$ this amounts to:

$$\langle a \rangle_\Lambda (x) := \sum_{k\in\Lambda} \widehat{a}_k e_k(x).$$

We denote by $m_{\langle a \rangle_\Lambda}$ the operator acting on $L^2(\mathbb{T}^d, \Lambda)$ by multiplication by $\langle a \rangle_\Lambda$.

Finally, P_Λ will denote the orthogonal projection onto $\langle \Lambda \rangle$. Note that the operator

$$\widehat{H}_{\theta_0,V,\Lambda} := \frac{1}{2} \left\| P_\Lambda (D_x + \theta_0) \right\|^2 + \langle V \rangle_\Lambda, \tag{6}$$

has a well-defined action on $L^2(\mathbb{T}^d, \Lambda)$.

As a straightforward adaptation of the proof of Theorem 3 of [AM10] we obtain the following result.

Theorem 1. *Let $\nu \in L^\infty(\mathbb{R}; \mathcal{P}(\mathbb{T}^d))$ be a measure obtained as a weak-$*$ limit (5) for some sequence (u_{ε_n}) of solutions to (4). Then for every $\Lambda \in \mathcal{L}$ there exist a continuous one-parameter family $\sigma_\Lambda(t)$, $t \in \mathbb{R}$, of positive, self-adjoint, trace-class operators on $L^2(\mathbb{T}^d, \Lambda)$ such that:*

$$\int_{\mathbb{T}^d} a(x)\, \nu(t, dx) = \sum_{\Lambda\in\mathcal{L}} \operatorname{tr}_{L^2(\mathbb{T}^d,\Lambda)} \left(m_{\langle a \rangle_\Lambda} \sigma_\Lambda(t) \right). \tag{7}$$

In addition, each $\sigma_\Lambda(t)$ satisfies a Heisenberg equation:

$$i\partial_t \sigma_\Lambda(t) = \left[\widehat{H}_{\theta_0,V,\Lambda}, \sigma_\Lambda(t) \right], \tag{8}$$

whose initial datum $\sigma_\Lambda|_{t=0} = \sigma_\Lambda^0$ is completely and uniquely determined by the sequence of initial data $(u_{\varepsilon_n}^0)$.

The operators σ_Λ^0 are obtained from the sequence of initial data $(u_{\varepsilon_n}^0)$ as weak limits of two-microlocal semiclassical measures, see Section 3.1 in [AM10] for a definition. These objects quantify how the mass of the sequence $(u_{\varepsilon_n}^0)$ concentrates on the linear subspace $\Lambda^\perp$, and have their origin in a construction developed independently by Nier [Nie96] and Fermanian-Kammerer [FK00a, FK00b].

The reader interested on general aspects of the study of limits of the type (5) on a general compact Riemannian manifold (M, g) (for the non-magnetic case) can consult [Mac09], the survey papers [Mac11, AM12], and the references therein.

This problem is very hard to attack in its full generality; but progress has been made when the dynamics of the geodesic flow of (M, g) is (Liouville) completely integrable. When $\theta_0 = 0$, Theorem 1 was proved for $d = 2$ and $V = 0$ in [Mac10]; and for arbitrary d and V continuous outside a set of zero Lebesgue measure in [AM10]. As already mentionend, the proof of Theorem 1 is completely identical to that of Theorem 3 in [AM10].

Finally, the case of quantum completely integrable systems was analysed in [AFKM14]. Equation (4) fits in the framework of that article; it should be noted though that if one applies directly the results of [AFKM14] to the present context, on would get a different, but equivalent, statement than Theorem 1, involving a different propagation law as well as slightly different two-microlocal measures.

3. Semiclassical measures and the Aharonov-Bohm effect

In order to obtain a better understanding of equation (7) and (8), and connect it to the discussion presented in the introduction, let us state some remarks.

First, in order to clarify the nature of (8), write the compact self-adjoint operator σ_Λ^0 as a superposition of orthogonal projectors onto its eigenspaces. Let (ϕ_n^Λ) denote an orthonormal basis of $L^2(\mathbb{T}^d, \Lambda)$ consisting of eigenfunctions of σ_Λ^0:

$$\sigma_\Lambda^0 \phi_n^\Lambda = \lambda_n^\Lambda \phi_n^\Lambda,$$

since in addition, σ_Λ^0 is positive and trace-class,

$$\lambda_n^\Lambda \geq 0, \qquad \operatorname{tr}_{L^2(\mathbb{T}^d,\Lambda)} \sigma_\Lambda^0 = \sum_{n\in\mathbb{N}} \lambda_n^\Lambda \leq 1.$$

If $\left|\phi_n^\Lambda\right\rangle \left\langle\phi_n^\Lambda\right|$ denotes the orthogonal projector of $L^2(\mathbb{T}^d, \Lambda)$ onto $\mathbb{C}\phi_n^\Lambda$ we have:

$$\sigma_\Lambda^0 = \sum_{n\in\mathbb{N}} \lambda_n^\Lambda \left|\phi_n^\Lambda\right\rangle \left\langle\phi_n^\Lambda\right|.$$

It turns out that $\sigma_\Lambda(t)$ is then given by:

$$\sigma_\Lambda(t) = \sum_{n\in\mathbb{N}} \lambda_n^\Lambda \left|v_n^\Lambda(t,\cdot)\right\rangle \left\langle v_n^\Lambda(t,\cdot)\right|,$$

where v_n^Λ solves the averaged Schrödinger equation:

$$\begin{cases} i\partial_t v_n^\Lambda + \widehat{H}_{\theta_0,V,\Lambda} v_n^\Lambda = 0, \\ v_n^\Lambda|_{t=0} = \phi_n^\Lambda. \end{cases} \tag{9}$$

Remark 2. *Equation (9) is invariant by translations along directions in $\Lambda^\perp$. Therefore, it can be identified to an equation on a lower dimensional torus, of dimension* $\operatorname{rk}\Lambda$.

Remark 3. *The magnetic potential affects the propagation law in equation (9) if and only if $P_\Lambda\theta_0 \neq 0$,* i.e. *whenever $\theta_0 \notin \Lambda^\perp$.*

Identity (7) can now be rewritten in terms of a superposition of position densities associated to averaged, lower dimensional, Schrödinger evolutions:

$$\int_{\mathbb{T}^d} a(x)\,\nu(t,dx) = \sum_{\Lambda\in\mathcal{L}}\sum_{n\in\mathbb{N}} \int_{\mathbb{T}^d} \langle a\rangle_\Lambda(x)\,\lambda_n^\Lambda \left|v_n^\Lambda(t,x)\right|^2 dx, \tag{10}$$

where v_n^Λ solves (9).

Remark 4. *It can be easily seen from (6) that $\widehat{H}_{\theta_0,V,\{0\}} = \widehat{V}_0$; and by definition, $L^2(\mathbb{T}^d,\{0\}) = \mathbb{C}$. Therefore, the term corresponding to $\Lambda = \{0\}$ in (10) is a constant that does not propagate with respect to t. In particular, it is not affected by θ_0.*

We obtain the following consequence of Theorem 1 that clarifies the structure of those sequences for which the magnetic potential does not affect the high-frequency propagation of the position densities.

Corollary 5. *Let $\nu \in L^\infty(\mathbb{R};\mathcal{P}(\mathbb{T}^d))$ be obtained from a sequence of solutions (u_{ε_n}) as a weak-$*$ limit (5). Let $(\sigma_\Lambda^0)_{\Lambda\in\mathcal{L}}$ be as in Theorem 1. Then ν is is not affected by the magnetic potential θ_0 if and only if, for every $\Lambda \in \mathcal{L}$, $\Lambda \neq \{0\}$:*

$$\sigma_\Lambda^0 \neq 0 \Longrightarrow \theta_0 \in \Lambda^\perp. \tag{11}$$

Therefore, the influence θ_0 on the dynamics is related to the vanishing of certain operators σ_Λ^0. A sufficient condition for σ_Λ^0 to vanish is the following (see Proposition 7 in [Mac10]).

Lemma 6. *If the sequence of initial data $\left(u_{\varepsilon_n}^0\right)$ satisfies*

$$\lim_{n\to\infty} \sum_{\operatorname{dist}\left(k,\Lambda^\perp\right)<R} \left|\widehat{u_{\varepsilon_n,k}^0}\right|^2 = 0, \qquad \textit{for every } R > 0, \tag{12}$$

then $\sigma_\Lambda^0 = 0$.

When $\Lambda = \mathbb{Z}^d$ (resp. $\Lambda = \{0\}$), condition (12) merely states that $(u^0_{\varepsilon_n})$ converges weakly (resp. strongly) to zero in $L^2(\mathbb{T}^d)$.

Corollary 5 admits the following reinterpretation. Let (u^0_ε) a sequence of initial data satisfying the hypotheses of Corollary 5 and such that (12) holds for every $\Lambda \in \mathcal{L}$, $\Lambda \neq \{0\}$, such that $P_\Lambda \theta_0 \neq 0$. Consider the solution of:

$$\begin{cases} i\partial_t w_\varepsilon + \left(\frac{1}{2}\Delta_x - V\right) w_\varepsilon = 0, \\ w_\varepsilon|_{t=0} = u^0_\varepsilon. \end{cases}$$

Then the weak-$*$ limit (5) of $|w_\varepsilon|^2$ exists and equals ν. In other words, $|u_\varepsilon|^2$ and $|w_\varepsilon|^2$ behave identically in the high-frequency limit.

Remark 7. *If $\theta_0 \in \left(\mathbb{R}^d\right)^*$ satisfies $\theta_0 \cdot k \neq 0$ for every $k \in \mathbb{Z}^d \setminus \{0\}$ then $P_\Lambda \theta_0 \neq 0$ for every $\Lambda \in \mathcal{L}$ such that $\Lambda \neq \{0\}$. Therefore, as soon as $\sigma^0_\Lambda \neq 0$ for some $\Lambda \neq \{0\}$, the weak-$*$ limits of $|u_\varepsilon|^2$ and $|w_\varepsilon|^2$ must differ. In other words, the propagation law of weak-$*$ limit of the position densities is affected by the magnetic potential in this case.*

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