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Existence of IIC measure for 2D Ising percolation at high temperatures

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1 Ising percolation

We consider the percolation problem for Ising model on the two-dimensional square lattice $\mathbb{Z}^2$. For $\beta < \beta_c$ and $h \in \mathbb{R}$, there exists a unique Gibbs measure $\mu_{\beta,h}$. The $(+)$-cluster containing the origin is denoted by $\mathbf{C}_0^+$. For each $\beta > 0$, the critical external field is defined by

$$h_c(\beta) := \inf\{h : \mu_{\beta,h}(\#\mathbf{C}_0^+ = \infty) > 0\}.$$

It is known that $h_c(\beta) > 0$ whenever $\beta < \beta_c$, and $\mu_{\beta,h_c(\beta)}(\#\mathbf{C}_0 = \infty) = 0$. (See e.g. Higuchi (1997).)

Let $S(n) = [-n,n]^2$ and $S^c(n) = \mathbb{Z}^2 \setminus S(n)$. Our main result is the following:

Theorem 1. Let $\beta < \beta_c$. For every cylinder event E, the limits

$$\lim_{n\to\infty} \mu_{\beta,h_c(\beta)}\big(E \,\big|\, \mathbf{O} \rightsquigarrow S^c(n)\big) \quad \text{and} \quad \lim_{h\searrow h_c(\beta)} \mu_{\beta,h}\big(E \,\big|\, \#\mathbf{C}_0^+ = \infty\big)$$

exist and are equal. If we denote their common value by $\nu(E)$, then ν extends uniquely to a probability measure on Ω, and

$$\nu\begin{pmatrix} \text{there exists exactly one infinite } (+)\text{-cluster } \tilde{\mathbf{C}}_0^+, \\ \text{and } \tilde{\mathbf{C}}_0^+ \text{ contains the origin } \mathbf{O} \end{pmatrix} = 1.$$

The measure ν is called the incipient infinite cluster (IIC) measure in the sense of Kesten (1986). The expectation with respect to ν is denoted by E_ν. The following theorem is obtained by a similar method as in Higuchi, Takei and Zhang (2012).

Theorem 2. For any $t \geqq 1$,

$$E_\nu \left[\{\#(\tilde{\mathbf{C}}_0^+ \cap S(n))\}^t \right] \asymp \{n^2 \mu_{\beta,h_c(\beta)}(\mathbf{O} \rightsquigarrow S(n))\}^t,$$

where $f(n) \asymp g(n)$ means that $C_1 g(n) \leq f(n) \leq C_2 g(n)$.

Our proof also works for Ising model on the two-dimensional triangular lattice with $\beta < \beta_c$, where $h = h_c(\beta) \equiv 0$ (see Grimmett and Janson (2009)).

2 Idea of the proof of Theorem 1

We list the basic facts:

- In the high temperature case $\beta < \beta_c$, $\mu_{\beta,h}$ has a good mixing property.
- The Markov property: For any A measurable inside $S(k)$,

$$\mu_{\beta,h}\left(A \mid \{\omega(x) : x \in S^c(k)\}\right) = \mu_{\beta,h}\left(A \mid \{\omega(x) : x \in \partial S(k)\}\right),$$

where $\partial S(k) = S(k+1) \setminus S(k)$.

- The FKG inequality: If both A and B are increasing, then

$$\mu_{\beta,h}(A \cap B) \geq \mu_{\beta,h}(A)\mu_{\beta,h}(B).$$

- For any $k \geq 1$ and any $\varepsilon > 0$, there exists $m > k$ such that

$$\mu_{\beta,h}\begin{pmatrix} \exists\ (+)\text{-circuit} \\ \text{around } S(k) \text{ in } S(m) \end{pmatrix} > 1 - \varepsilon \quad \text{for all } h \geq h_c(\beta).$$

Hereafter we adopt an abbreviation $\mu = \mu_{\beta,h_c(\beta)}$. We fix a cylinder event E, which depends only on the spins in $S(n_0)$: We look at $\mu\big(E \cap \{\mathbf{O} \overset{+}{\leftrightarrow} S^c(n)\}\big)$ with $n_0 < n$.

▶ $n_0 < \exists \tilde{n}_1 < \exists n_1 < n$ such that

$$\begin{aligned} &\mu(E \cap \{\mathbf{O} \overset{+}{\leftrightarrow} S^c(n)\}) \\ &\asymp \mu\left(E \cap \left\{ \begin{matrix} \exists \text{ innermost } (+)\text{-circuit } \mathscr{C}_1 \\ \text{in } A(n_1) := S(n_1) \setminus S(\tilde{n}_1) \end{matrix} \right\} \cap \{\mathbf{O} \overset{+}{\leftrightarrow} \mathscr{C}_1\} \cap \{\mathscr{C}_1 \overset{+}{\leftrightarrow} S^c(n)\}\right) \end{aligned}$$

The Markov property separates inside $\mathscr{C}_1$ and outside $\mathscr{C}_1$:

$$= \sum_{C_1 \subset A(n_1)} \mu(E \cap \{\mathscr{C}_1 = C_1\} \cap \{\mathbf{O} \overset{+}{\leftrightarrow} C_1\})\mu(C_1 \overset{+}{\leftrightarrow} S^c(n) \mid C_1 \text{ is } +).$$

▶ Let $\gamma(C_1, n) := \mu(C_1 \overset{+}{\leftrightarrow} S^c(n) \mid C_1 \text{ is } +)$. We have

$$\frac{\mu(E \cap \{\mathbf{O} \overset{+}{\leftrightarrow} S^c(n)\})}{\mu(\mathbf{O} \overset{+}{\leftrightarrow} S^c(n))} \asymp \frac{\sum_{C_1 \subset A(n_1)} \mu(E \cap \{\mathscr{C}_1 = C_1\} \cap \{\mathbf{O} \overset{+}{\leftrightarrow} C_1\})\gamma(C_1, n)}{\sum_{\tilde{C}_1 \subset A(n_1)} \mu(\{\mathscr{C}_1 = \tilde{C}_1\} \cap \{\mathbf{O} \overset{+}{\leftrightarrow} \tilde{C}_1\})\gamma(\tilde{C}_1, n)}.$$

So we want to show the existence of $\lim_{n\to\infty} \dfrac{\gamma(C',n)}{\gamma(C,n)}$ for $C, C' \subset A(n_1)$.
► In the same way as above, we can show: $n_1 < \exists \tilde{n}_2 < \exists n_2 < n$ such that

$$\gamma(C_1,n) = \mu(C_1 \overset{+}{\leftrightarrow} S^c(n) \,|\, C_1 \text{ is } +) \asymp \sum_{C_2 \subset A(n_2)} \mu(\{\mathscr{C}_2 = C_2\} \cap \{C_1 \overset{+}{\leftrightarrow} C_2\})\gamma(C_2,n).$$

► Iterating this, we have the following: For $n_0 < \tilde{n}_1 < n_1 < \tilde{n}_2 < n_2 < \cdots < \tilde{n}_k < n_k < n$, we put

$$\gamma(C_i,n) := \mu(C_i \overset{+}{\leftrightarrow} S^c(n) \,|\, C_i \text{ is } +),$$
$$M_i(C_i, C_{i+1}) := \mu(\{\mathscr{C}_i = C_i\} \cap \{C_i \overset{+}{\leftrightarrow} C_{i+1}\}),$$

where $\mathscr{C}_i$ is the innermost (+)-circuit in $A(n_i) := S(n_i) \setminus S(\tilde{n}_i)$. For a suitable choice of the sequence $n_0 < \tilde{n}_1 < n_1 < \tilde{n}_2 < n_2 < \cdots < \tilde{n}_k < n_k < n$, we have

$$\begin{aligned}\gamma(C_1,n) &\asymp \sum_{C_2,\ldots,C_{k+1}} M_1(C_1,C_2)\cdots M_k(C_k,C_{k+1})\gamma(C_{k+1},n) \\ &=: \left(\prod_{i=1}^{k} M_i\right)(C_1,C_{k+1})\gamma(C_{k+1},n).\end{aligned}$$

► A result of Hopf (1963) implies: If there exists $\lambda > 1$ such that for any $i \geq 1$, C_i, $C_i' \in A(n_i)$, and D_{i+1}, $D_{i+1}' \in A(n_{i+1})$,

$$\frac{M_i(C_i,D_{i+1})M_i(C_i',D_{i+1}')}{M_i(C_i,D_{i+1}')M_i(C_i',D_{i+1})} \leq \lambda, \tag{\#}$$

then

$$\lim_{k\to\infty} \frac{(\prod_{i=1}^k M_i)(C_1,C_{k+1})}{(\prod_{i=1}^k M_i)(C_1',C_{k+1})} = \alpha(C_1,C_1'),$$

uniformly in C_{k+1}; whence

$$\lim_{n\to\infty} \frac{\gamma(C',n)}{\gamma(C,n)} = \alpha(C_1,C_1').$$

This shows the convergence of $\mu(E \mid \mathbf{O} \overset{+}{\leftrightarrow} S^c(n))$ as $n \to \infty$.
► (#) is the key inequality; it is true if for some $\lambda_1 > 1$,

$$(\lambda_1)^{-1}N_i(C_i)L_i(D_{i+1}) \leqq M_i(C_i,D_{i+1}) \leqq \lambda_1 N_i(C_i)L_i(D_{i+1}).$$

For Ising percolation, this is the most delicate part. We have to choose further subsequence carefully, to use the mixing property. Details are found in Higuchi, Kinoshita, Takei, and Zhang (2013).

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